Filomat 33:4 (2019), 1179–1184
https://doi.org/10.2298/FIL1904179V

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On Semisymmetric Connection

Ana M. Velimirović[a], Milan L. Zlatanović[a]

[a]*Faculty of Sciences and Mathematics, University of Niš, Serbia*

Abstract. Using the non-symmetry of a connection, it is possible to introduce four types of covariant derivatives. Based on these derivatives, several types of Ricci's identities and twelve curvature tensors are obtained. Five of them are linearly independent but the other curvature tensors can be expressed as linear combinations of these five linearly independent curvature tensors and the curvature tensor of the corresponding associated symmetric space.

The semisymmetric connection is defined and the properties of two of the five independent curvature tensors are analyzed. In the same manner, the properties for three others curvature tensors may be derived.

1. Introduction

Although the notion of non-symmetric affine connection is used in several works before A. Einstein, for example in [2, 4], the use of non-symmetric connection became especially actual after appearance the works of Einstein, relating to create the Unified Field Theory (UFT).

Einstein was not satisfied with his General Theory of Relativity (GTR, 1916), and from 1923. to the end of his life (1955), he worked on various variants of UFT. This theory had the aim to unite the gravitation theory, to which is related GTR, and the theory of electromagnetism.

Let $L_N = (\mathcal{M}_N, L^i_{jk} \neq L^i_{kj})$ be a **non-symmetric affine connection** space. Based on the non-symmetry of the connection, we can introduce four types of the covariant derivatives. For instance, for the tensor a^i_j, we have

$$a^i_{\underset{1}{j|m}} = a^i_{j,m} + L^i_{pm}a^p_j - L^p_{jm}a^i_p, \quad a^i_{\underset{2}{j|m}} = a^i_{j,m} + L^i_{mp}a^p_j - L^p_{mj}a^i_p, \tag{1.1}$$

$$a^i_{\underset{3}{j|m}} = a^i_{j,m} + L^i_{pm}a^p_j - L^p_{mj}a^i_p, \quad a^i_{\underset{4}{j|m}} = a^i_{j,m} + L^i_{mp}a^p_j - L^p_{jm}a^i_p, \tag{1.2}$$

where by comma $(,)$ is denoted partial derivative. Based on the covariant derivatives, we can derive Ricci's type identities and twelve curvature tensors, and among them five are linearly independent, while the rest of them can be expressed through this five curvature tensors [6, 8]. Linearly independent curvature tensors are:

$$\underset{1}{R}{}^i_{jmn} = L^i_{jm,n} - L^i_{jn,m} + L^p_{jm}L^i_{pn} - L^p_{jn}L^i_{pm}, \tag{1.3}$$

2010 *Mathematics Subject Classification*. 53A45; 53B05
Keywords. non-symmetric affine connection, semisymmetric connection, torsion tensor, curvature tensors
Received: 01 October 2018; Accepted: 22 October 2018
Communicated by Mića S. Stanković
The authors were supported from the research project 174012 of the Serbian Ministry of Science
Email addresses: velimirovic018@yahoo.com (Ana M. Velimirović), zlatmilan@yahoo.com (Milan L. Zlatanović)

$$\underset{2}{R}{}^i_{jmn} = L^i_{mj,n} - L^i_{nj,m} + L^p_{mj}L^i_{np} - L^p_{nj}L^i_{mp}, \tag{1.4}$$

$$\underset{3}{R}{}^i_{jmn} = L^i_{jm,n} - L^i_{nj,m} + L^p_{jm}L^i_{np} - L^p_{nj}L^i_{pm} + L^p_{nm}T^i_{pj}, \tag{1.5}$$

$$\underset{4}{R}{}^i_{jmn} = L^i_{jm,n} - L^i_{nj,m} + L^p_{jm}L^i_{np} - L^p_{nj}L^i_{pm} + L^p_{mn}T^i_{pj}, \tag{1.6}$$

$$\underset{5}{R}{}^i_{jmn} = \frac{1}{2}(L^i_{jm,n} - L^i_{jn,m} + L^p_{jm}L^i_{pn} - L^p_{jn}L^i_{mp} + L^i_{mj,n} - L^i_{nj,m} + L^p_{mj}L^i_{np} - L^p_{nj}L^i_{pm}), \tag{1.7}$$

The following theorem holds

Theorem 1.1. *[6, 8] If the following notation are introduced (with an omission of the indices on the left side),*

$$\mathcal{A} = \frac{1}{2}T^i_{jm;n} \quad \mathcal{A}' = \frac{1}{2}T^i_{jn;m} \tag{1.8}$$

$$\mathcal{B} = \frac{1}{4}T^i_{jm}T^i_{pn} \quad \mathcal{B}' = \frac{1}{4}T^p_{jn}T^i_{pm} \quad C = \frac{1}{4}T^p_{mn}T^i_{pj} \tag{1.9}$$

where the covariant derivative with respect to the symmetric connection $L^i_{\underline{jk}} = \frac{1}{2}(L^i_{jk} + L^i_{kj})$ *is denoted by semicolon,* $T^i_{jk} = L^i_{jk} - L^i_{kj}$ *is the torsion tensor of the connection, then the equations* (1.3) − (1.7) *can be expressed as follows*

$$\underset{1}{R} = R + \mathcal{A} - \mathcal{A}' + \mathcal{B} - \mathcal{B}', \tag{1.10}$$

$$\underset{2}{R} = R - \mathcal{A} + \mathcal{A}' + \mathcal{B} - \mathcal{B}', \tag{1.11}$$

$$\underset{3}{R} = R + \mathcal{A} + \mathcal{A}' - \mathcal{B} + \mathcal{B}' - 2C, \tag{1.12}$$

$$\underset{4}{R} = R + \mathcal{A} + \mathcal{A}' - \mathcal{B} + \mathcal{B}' + -2C, \tag{1.13}$$

$$\underset{5}{R} = R + \mathcal{B} + \mathcal{B}', \tag{1.14}$$

where $R \equiv R^i_{jmn}$ *is the curvature tensor with respect to the symmetric connection* $L^i_{\underline{jk}}$.

A lot of research papers and monographs [1] - [21] are dedicated to the theory of Riemannian spaces, affine connected ones and their generalizations.

2. Semisymmetric affine connection space

Particularly interesting cases can be observed at the non-symmetry of the connection. One such case is the semisymmetric affine connection.

Definition 2.1. *Connection* L^i_{jk} *of the space* L_N *is* **semisymmetric**, *if*

$$L^i_{jk} = L^i_{\underline{jk}} + \delta^i_j\tau_k - \delta^i_k\tau_j, \tag{2.1}$$

where $L^i_{\underline{jk}}$ *symmetrical part of* L^i_{jk}, *and* **torsion tensor**

$$T^i_{jk} = L^i_{jk} - L^i_{kj} = 2(\delta^i_j\tau_k - \delta^i_k\tau_j), \tag{2.2}$$

where τ_k *it is a vector.*

Namely, it holds the following theorem

Theorem 2.1. *If $T^i_{jk} = L^i_{jk} - L^i_{kj}$ is a torsion tensor of L_N, then*

$$\tau_i = \frac{1}{2(N-1)} T^p_{pi}, \tag{2.3}$$

for τ_i from the equation (2.1)

Proof. After contracting (2.1) with respect to the index k and i, we get

$$T^i_{ji} = 2(\delta^i_j \tau_i - \delta^i_i \tau_j) = 2(\tau_j - N\tau_j) = 2(1-N)\tau_j,$$

i.e. $-T^i_{ij} = 2(1-N)\tau_j$. Thus, $\tau_j = \frac{1}{2(N-1)} T^i_{ij}$. By substituting $i \to p$, $j \to i$, we get (2.1). □

Let us start from the equation (1.10), i.e.

$$\underset{1}{R}{}^i_{jmn} = R^i_{jmn} + \frac{1}{2} T^i_{jm;n} - \frac{1}{2} T^i_{jn;m} + \frac{1}{4} T^p_{jm} T^i_{pn} - \frac{1}{4} T^p_{jn} T^i_{pm}. \tag{2.4}$$

If we replace the torsion tensor (2.2) in the previous equation, we obtain the curvature tensor $\underset{1}{R}$ of the semisymmetric affine connection

$$\begin{aligned} \underset{1}{R}{}^i_{jmn} &= R^i_{jmn} + (\delta^i_j \tau_m - \delta^i_m \tau_j)_{;n} - (\delta^i_j \tau_n - \delta^i_n \tau_j)_{;m} \\ &\quad + (\delta^p_j \tau_m - \delta^p_m \tau_j)(\delta^i_p \tau_n - \delta^i_n \tau_p) - (\delta^p_j \tau_n - \delta^p_m \tau_j)(\delta^i_p \tau_m - \delta^i_m \tau_p). \end{aligned} \tag{2.5}$$

After simple calculation, we get

$$\underset{1}{R}{}^i_{jmn} = R^i_{jmn} + \delta^i_j(\tau_{m;n} - \tau_{n;m}) - \delta^i_m(\tau_{j;n} + \tau_j \tau_n) + \delta^i_n(\tau_{j;m} + \tau_j \tau_m). \tag{2.6}$$

Theorem 2.2. *The curvature tensor $\underset{1}{R}{}^i_{jmn}$ of the semisymmetric affine space connection* (2.1*) is given by the equation* (2.6)*, where R^i_{jmn} is curvature tensor of the symmetric connection $L^i_{\underline{jk}}$, τ_i is the vector expressed in the function of the torsion T^i_{jk} by the equation* (2.3)*, and covariant derivative* (;) *is given by $L^i_{\underline{jk}}$.*

2.1. Properties of the curvature tensors of the semisymmetric connection

Based on the general case, and according to the equation (2.6) we get

$$\underset{1}{R}{}^i_{jmn} = -\underset{1}{R}{}^i_{jnm}. \tag{2.7}$$

If we take into account that $\underset{jmn}{Cycl} R^i_{jmn} = 0$, the first Bianchi identity for $\underset{1}{R}{}^i_{jmn}$ is

$$\underset{jmn}{Cycl} \underset{1}{R}{}^i_{jmn} = \underset{jmn}{Cycl}[R^i_{jmn} + \delta^i_j(\tau_{m;n} - \tau_{n;m}) - \delta^i_m(\tau_{j;n} + \tau_j \tau_n) + \delta^i_n(\tau_{j;m} + \tau_j \tau_m)]. \tag{2.8}$$

i.e.

$$\underset{jmn}{Cycl} \underset{1}{R}{}^i_{jmn} = 2\underset{jmn}{Cycl} \delta^i_j(\tau_{m;n} - \tau_{n;m}). \tag{2.9}$$

Therefore, the following theorem is valid

Theorem 2.3. *The curvature tensor $\underset{1}{R}{}^i_{jmn}$ of the semisymmetric affine connection given by the equation (2.6) satisfies antisymmetry with respect to the last two lower indices (2.7), and the first Bianchi's identity* (2.9).

Let us consider the second Bianchi identity. In this case, we use (19) from [21].

$$\begin{aligned}\underset{mnv}{Cycl}\underset{1}{R}{}^i_{jmn|v} &= \underset{mnv}{Cycl}T^p_{mn}\underset{1}{R}{}^i_{jpv} = \underset{mnv}{Cycl}2(\delta^p_m\tau_n - \delta^p_n\tau_m)\underset{1}{R}{}^i_{jpv} \\ &= 2\underset{mnv}{Cycl}(\tau_n\underset{1}{R}{}^i_{jmv} - \tau_m\underset{1}{R}{}^i_{jnv}).\end{aligned} \tag{2.10}$$

Since $\underset{1}{R}$ is antisymmetric by n and v, we obtain

$$\underset{mnv}{Cycl}\underset{1}{R}{}^i_{jmn|v} = 2\underset{mnv}{Cycl}\tau_n\underset{1}{R}{}^i_{jmv} + 2\underset{mnv}{Cycl}\tau_m\overset{1}{R}{}^i_{jvn}. \tag{2.11}$$

For the addend $\underset{mnv}{Cycl}\tau_m\underset{1}{R}{}^i_{jvn}$, we have

$$2\underset{mnv}{Cycl}\tau_m\underset{1}{R}{}^i_{jvn} = 2\underset{mnv}{Cycl}\tau_n\underset{1}{R}{}^i_{jvn}. \tag{2.12}$$

By substituting (2.12) in the equation (2.11), we get

$$\underset{mnv}{Cycl}\underset{1}{R}{}^i{}_{jmn\underset{1}{|}v} = 4\underset{mnv}{Cycl}\tau_m\underset{1}{R}{}^i{}_{jmn}. \tag{2.13}$$

Therefore, we have proved the following theorem:

Theorem 2.4. *The equation* (2.13) *is the second Bianchi identity for the curvature tensor* $\underset{1}{R}$ *of the semisymmetric affine connection, where* τ_m *is determined by* (2.3).

In the same manner, we can derive the first and second Bianchi identities for the curvature tensors $\underset{2}{R}$, $\underset{3}{R}$, $\underset{4}{R}$ and $\underset{5}{R}$. Let us consider, for example, the curvature tensor $\underset{5}{R}$ of the semisymmetric affine connection. According to (1.14) and (1.9), (2.2) we have

$$\underset{5}{R}{}^i{}_{jmn} = R^i{}_{jmn} + \frac{1}{4}(T^p_{jm}T^i_{pn} + T^p_{jn}T^i_{pm}), \tag{2.14}$$

wherefrom

$$\underset{5}{R}{}^i{}_{jmn} = R^i{}_{jmn} + (\delta^p_j\tau_m - \delta^p_m\tau_j)(\delta^i_p\tau_n - \delta^i_n\tau_p) + (\delta^p_j\tau_n - \delta^p_n\tau_j)(\delta^i_p\tau_m - \delta^i_m\tau_p), \tag{2.15}$$

i.e.

$$\underset{5}{R}{}^i{}_{jmn} = R^i{}_{jmn} + 2\delta^i_j\tau_m\tau_n - \delta^i_m\tau_j\tau_n - \delta^i_n\tau_j\tau_m. \tag{2.16}$$

Thus, from the last equation, we can conclude that

$$\underset{5}{R}{}^i{}_{j\underline{mn}} = 2\delta^i_j\tau_m\tau_n - \delta^i_m\tau_j\tau_n - \delta^i_n\tau_j\tau_m, \tag{2.17}$$

$$\underset{5}{R}{}^i{}_{j\underset{\vee}{mn}} = R^i{}_{jmn}, \text{(because } R^i{}_{jmn} = -R^i{}_{jnm}). \tag{2.18}$$

Finally, it holds that

$$\underset{5}{R}{}^i{}_{jmn} \neq \underset{5}{R}{}^i{}_{jnm}, \quad \underset{5}{R}{}^i{}_{jmn} \neq -\underset{5}{R}{}^i{}_{jnm}. \tag{2.19}$$

We are also checking the first Bianchi's identity for the curvature tensor $\underset{5}{R}$:

$$\underset{jmn}{Cycl}\underset{5}{R}{}^i{}_{jmn} \underset{(2.14)}{=} 0 + \underset{jmn}{Cycl}(2\delta^i_j\tau_m\tau_n - \delta^i_m\tau_j\tau_n - \delta^i_n\tau_j\tau_m),$$

wherefrom

$$\underset{jmn}{Cycl}\underset{5}{R}^{i}{}_{jmn} = 0. \tag{2.20}$$

Regarding of the second Bianchi identity, all types of covariant derivatives should be examined. After some calculation, we conclude that

$$\underset{5}{R}^{i}{}_{jmn\underset{\theta}{|}v} \neq 0 \quad \text{for} \quad \theta = 1, \ldots, 4.$$

So, the next theorem is valid:

Theorem 2.5. *The curvature tensor* $\underset{5}{R}^{i}_{jmn}$ *of the semisymmetric affine connection given by the equation (2.16) satisfies the following properties:*

1. $\underset{5}{R}^{i}{}_{jmn} \neq \underset{5}{R}^{i}{}_{jnm}, \quad \underset{5}{R}^{i}{}_{jmn} \neq -\underset{5}{R}^{i}{}_{jnm},$
2. $\underset{jmn}{Cycl}\underset{5}{R}^{i}{}_{jmn} = 0,$
3. $\underset{5}{R}^{i}{}_{jmn\underset{\theta}{|}v} \neq 0$ *for* $\theta = 1, \ldots, 4.$

3. Conclusion

In this paper we consider one class of the space L_N with the non-symmetric connection. We obtained five linearly independent curvature tensors and find some their properties. This relationship will be of significance for the further development of semisymmetric affine connection spaces. Especially, these results are of interest for studying geodesic mappings of these spaces.

References

[1] A. Einstein, Generalization of theory of gravitation, Appendix II in the book:*The meaning of relativity,* Fourth edit., Princeton, 1953.

[2] L. Eisenhart, *Non-Riemannian geometry*, AMS, New York, 37 (1964).

[3] L. Eisenhart, *Generalized Riemann spaces* , Proc. Nat. Acad. Sci. USA, 37 (1951).

[4] H. A. Hayden, , *Subspaces of a space with torsion*, Proc. London math. soc., 34, (2) (1932), pp. 27–50.

[5] S. Ivanov, M. Zlatanović, *Connections on non-symmetric (generalized) Riemannian manifold and gravity*, Classical and Quantum Gravity, Volume 33, Number 7, 075016, (2016).

[6] S. Minčić, *Independent curvature tensors and pseudotensors of spaces with non-symmetric affine connexion,* Coll. math. soc. Janos Bolyai, 31 Dif. geom., Budapest (Hungary), (1979), 445–460.

[7] S. Minčić, *New Bianchi type identities in the space of nonsymmetric affine connexion,* Facta universitatis (Niš). Ser. Math. Inform 10 (1995), 35–43.

[8] S. Minčić, M. Stanković, Lj. Velimirović, *Generalized Riemannian Spaces and Spaces of Non-symmetric Affine Connection*, Faculty of Science and Mathematics, University of Niš, (2013), Niš.

[9] M. Najdanović, M. Zlatanović, I. Hinterleitner, , *Conformal and geodesic mappings of generalized equidistant spaces*, Publications de l'institut mathematique, 98(112) (2015), 71-84.

[10] S. Saxena, R. Behari, *Some properties of generalized Riemann spaces,* Proc. nat. Inst. of sci. India, 426 Vol No 2, 1960, pp. 95–103.

[11] S. Saxena, R. Behari, *Some properties and applications of Eisenhart's generalized Riemann spaces,* Proc. nat. Inst. of sci. India, Part A (26), 1960, pp. 48–57.

[12] M. S. Stankovic, *First type almost geodesic mappings of general affine connection spaces*, Novi Sad J. Math. 29, No. 3 (1999), 313–323.

[13] M. S. Stanković, *On a canonic almost geodesic mappings of the second type of affine spaces*, FILOMAT 13, (1999), 105–114.

[14] M. S. Stanković *On a special almost geodesic mappings of third type of affine spaces*, Novi Sad J. Math. Vol. 31, No. 2, 2001, 125–135.

[15] M. S. Stanković, *Special equitorsion almost geodesic mappings of the third type of non-symmetric affine connection spaces*, Applied Mathematics and Computation, 244, (2014), 695–701.

[16] M. Stanković, M. Zlatanović, N. Vesić, *Some Properties of ETProjective Tensors Obtained from Weyl Projective Tensor,* Filomat 29:3 (2015), 573– 584.

[17] M. Stanković, S. Mincić, M. Zlatanović , *Ricci Coefficients of Rotation of Generalized Finsler Spaces,* Miskolc Mathematical Notes, Vol. 16, (2015), No. 2, pp. 1025–1039.

[18] M. Stanković, N. Vesić, *Some relations in non-symmetric affine connection spaces with regard to a special almost geodesic mappings of the third type,* Filomat Vol. 29, No. 9 (2015), 1941–1951.

[19] V. Stanković, *Certain properties of generalized Einstein spaces*, Filomat 32:13 (2018), 4803–4810.

[20] N. Vesić, M. Stanković, *Invariants of Special Second-Type Almost Geodesic Mappings of Generalized Riemannian Space,* Mediterranean Journal of Mathematics, Vol. 15, No 2, (2018) 15:60.

[21] N. Vesić, Lj. Velimirović, M. Stanković, *Some Invariants of Equitorsion Third Type Almost Geodesic Mappings,* Mediterr. J. Math. 13 (2016), 4581-4590.