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Recurrent Equiaffine Projective Euclidean Spaces

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Abstract. In this paper, we study n-dimensional recurrent equiaffine projective Euclidean manifolds, i.e. manifolds with absolute recurrent curvature tensor, which admit geodesic mappings onto Euclidean space, and they are equiaffine (where was obtained the symmetric Ricci tensor). We obtained main conditions of recurrent projective Euclidean spaces and constructed their examples.

1. Introduction

This paper is devoted to n-dimensional recurrent projective Euclidean equiaffine manifolds A_n.

Let $A_n = (M, \nabla)$ be n-dimensional manifold M with affine connection ∇ without torsion. *Symmetric*, *semisymmetric* and *recurrent space*, respectively, is manifold A_n in which the curvature tensor R satisfies, respectively, one of the following condition

$$(a) \quad \nabla R = 0, \qquad (b) \quad R \circ R = 0, \qquad (c) \quad \nabla R = \varphi \cdot R, \tag{1}$$

where φ is a linear form which is called *recurrence tensor*.

It is known, that P.A. Shirokov (see [18]) began to study symmetric and semisymmetric spaces. They implicitly started to study the conditions $\nabla R = 0$ and $R \circ R = 0$ (as integrability conditions of $\nabla R = 0$). The names *symmetric* and *semisymmetric* were explicitly introduced by É. Cartan and N.S. Sinyukov, respectively, see [3, 8, 9, 20]. *Recurrent spaces* were introduced by H.S. Ruse [14, 24]. These spaces play an important role in the theory of relativity, because they describe spaces with gravitational waves.

Symmetric and recurrent (with gradient-like field φ) spaces are semisymmetric. The geometry of symmetric, recurrent and semisymmetric spaces play an important role in the theory of Riemannian manifolds and their generalizations, as well as applications in theoretical physics, especially, general theory of relativity. The great interest in semisymmetric spaces had Nomizu hypothesis [10], which was out casted later [23], see also papers by Szabó [21, 22]. Nowadays, study of the symmetric and recurrent spaces and their generalization is devoted to many works, for example [1, 2, 4, 5, 9].

Diffeomorphism between manifolds with affine connection is called a *geodesic mapping* if it preserves geodesics. Geodesic mappigs of symmetric, recurrent and semisymmetric manifolds and their generalizations were studied by Sinyukov, Prvanović, Mikeš, and others, see [6–9, 13, 19, 20].

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Projective Euclidean spaces were investigated in many different ways. These spaces are geodesically equivalent to Euclidean spaces. Components of affine connection of symmetric projective Euclidean spaces were obtained by P.A. Shirokov, see [16, 18]. The study of these spaces is devoted to dissertation work by Sabykanov [15]. We continued in that way and obtained components of linear connection ∇ of semisymmetric projective Euclidean spaces [11]. There were proved that semisymmetric projective Euclidean spaces are necessary equiaffine, for which Ricci tensor is symmetric.

In this paper, we continued to study of the recurrent equiaffine projective Euclidean manifolds. We obtained main conditions of recurrent projective Euclidean spaces and showed their examples.

2. Equiaffine projective Euclidean spaces

Let A_n and $\bar{A}_n$ be **equiaffine** spaces with affine connection ∇ and $\bar{\nabla}$, respectively, without torsion. In an equiaffine space, the Ricci tensor is symmetric, i.e. $R_{ik} = R_{ki} = R^{\alpha}_{i\alpha k}$.

Bellow, we will remind a well-know facts about geodesic mappings, see [7–9, 12, 20].

A diffeomorphism $f\colon A_n \to \bar{A}_n$ is called a *geodesic mapping* if any geodesic curve in A_n is mapped onto geodesic curve in $\bar{A}_n$. The necessary and sufficient condition of geodesic mapping $f\colon A_n \to \bar{A}_n$ is the Levi-Civita equation

$$\bar{\Gamma}^h_{ij}(x) = \Gamma^h_{ij}(x) + \delta^h_i \psi_j(x) + \delta^h_j \psi_i(x), \tag{2}$$

where Γ^h_{ij} and $\bar{\Gamma}^h_{ij}$ are components of ∇ and $\bar{\nabla}$, $x = (x^1, x^2, \dots, x^n)$ is a common coordinate respective f, and ψ_i are components of a linear form, which are gradient-like, i.e. $\psi_i = \partial_i \Psi,\ \partial_i = \partial/\partial x^i$.

For curvature, Ricci and Weyl projective tensor in A_n and $\bar{A}_n$ the following formulas hold:

$$(a)\ \bar{R}^h_{ijk} = R^h_{ijk} + \delta^h_k \psi_{ij} - \delta^h_j \psi_{ik}\,, \quad (b)\ \bar{R}_{ij} = R_{ij} - (n-1)\,\psi_{ij}\,, \quad (c)\ \bar{W}^h_{ijk} = W^h_{ijk}\,, \tag{3}$$

where $\psi_{ij} = \psi_{i,j} - \psi_i \psi_j$. Here and in the following, comma "," denotes the *covariant derivative* on ∇.

The Weyl tensor of projective curvature in equiaffine A_n has the following form:

$$W^h_{ijk} = R^h_{ijk} + \frac{1}{n-1}\,(\delta^h_k R_{ij} - \delta^h_j R_{ik}).$$

Space A_n is called *flat* (or *affine*), if there exists an affine coordinate system x for which $\Gamma^h_{ij}(x) = 0$. It is known that the tensor criterion for these spaces is that the curvature and torsion tensor are vanished.

In natural way, in flat spaces A_n we can implement Euclidean and pseudo-Euclidean metrics thus we call them *Euclidean spaces* E_n.

Space A_n is *projective Euclidean* if it admits a geodesic mapping onto an Euclidean space. For $n > 2$ the space A_n is projective Euclidean if and only if $W^h_{ijk} = 0$ and equivalently from (3) for equiaffine space A_n, the curvature tensor R has the following form:

$$R^h_{ijk} = \delta^h_j \psi_{ik} - \delta^h_k \psi_{ij}, \tag{4}$$

where ψ_{ij} is a symmetric tensor. The Ricci tensor of this space has form $R_{ij} = (n-1)\,\psi_{ij}$ and from the Bianchi identity it is known:

$$\psi_{ij,k} = \psi_{ji,k}. \tag{5}$$

Since 1925 P.A. Shirokov [17, 18] studied symmetric projective Euclidean space. He proved that in non-flat symmetric projective Euclidean space, there exists a projective coordinate system x in which the components of an affine connection ∇ have form:

$$\Gamma^h_{ij} = \delta^h_i \psi_j + \delta^h_j \psi_i, \quad \psi_i = \partial_i \Psi, \quad \Psi = -\ln \sqrt{\left| a_{\alpha\beta} x^\alpha x^\beta + b_\alpha x^\alpha + c \right|},$$

where a_{ij}, b_i, c are real constants and $a_{ij} = a_{ji} \not\equiv 0$.

From this result, it follows that a set of symmetric projective Euclidean spaces depend on $(n+1)(n+2)/2$ real parameters, which are a_{ij} $(= a_{ji})$, b_i and c.

In [11], we proved that a projective Euclidean space A_n is semisymmetric if and only if it is equiaffine, and in a projective coordinate system x components of an affine connection ∇ take the form of

$$\Gamma^h_{ij} = \delta^h_i \psi_j + \delta^h_j \psi_i, \quad \psi_i = \partial_i \Psi,$$

where Ψ is a function.

3. Recurrent and semisymmetric spaces

Conditions (1c) of absolute recurrence of the curvature tensor R, which characterize recurrent space A_n, are written in a coordinate form in the following way [14, 24]:

$$R^h_{ijk,l} = \varphi_l \, R^h_{ijk}. \tag{6}$$

A.G. Walker [24] proved that a recurrent Riemannian space is semisymmetric. If A_n is a recurrent space with an affine connection, then this property is generally not valid. Now, we covariantly differentiate (6) with respect to x^m, and after that, we alternate the indices l and m. We get

$$R^h_{ijk,[lm]} = \varphi_{[l,m]} \, R^h_{ijk}, \tag{7}$$

where the square bracket denote alternation of given indices.

We remind that conditions of semisymmetric spaces (1b) has in common coordinate system the form $R^h_{ijk,[lm]} = 0$. Therefore, from (7) follows that *recurrent space A_n is semisymmetric if and only if the form φ is locally gradient-like*, i.e. locally there exists a function Φ for which

$$\varphi_i = \partial_i \Phi$$

holds. This follows from the following term:

$$\varphi_i = \partial_i \Phi \ \text{ if and only if } \ \varphi_{l,m} = \varphi_{m,l} \ \ (\Leftrightarrow \partial_m \varphi_l = \partial_l \varphi_m).$$

In the paper [11], it was proved that a *recurrent projective Euclidean space A_n is semisymmetric if and only if it is equiaffine.*

4. Main condition of recurrent equiaffine projective Euclidean spaces

Let A_n be a recurrent equiaffine projective Euclidean space. Then formula (6) with $\varphi_l = \partial_l \Phi$ holds, and the curvature tensor R has form (4).

By substituting condition (4) to (6), we have $\delta^h_k \psi_{ijl} - \delta^h_j \psi_{ikl} = 0$, where $\psi_{ijl} = \psi_{ij,l} - \varphi_l \psi_{ij}$. From this, it follows that $\psi_{ijk} = 0$, so we have

$$\psi_{ij,l} = \varphi_l \psi_{ij}. \tag{8}$$

Because in a projective Euclidean space A_n formula (5) holds too, then from (8) follows $\varphi_k \psi_{ij} = \varphi_j \psi_{ik}$.

Now, let us suppose that $\varphi_k \neq 0$. Due to this, there exists a vector field a^k, for which $a^k \varphi_k = 1$. Now, we contract the last formula with a^k. We get $\psi_{ij} = \varphi_j \psi_{ik} a^k$ and from the symmetry of the tensor ψ_{ij} it follows

$$\varphi_j \psi_{ik} a^k = \varphi_i \psi_{jk} a^k.$$

Now, we will contract the last formula with respect to a^j and obtain $\psi_{ik} a^k = \psi_i \psi_{jk} a^j a^k$.

Finally, we have:

$$\psi_{ij} = \varkappa\, \varphi_i \varphi_j, \tag{9}$$

where $\varkappa$ is a function.

Because A_n is non-flat space, the function $\varkappa$ and φ_i are non-vanishing. By substituting (9) to (8), we obtain:

$$\varkappa_{,l}\, \varphi_i \varphi_j + \varkappa\, \varphi_{i,l} \varphi_j + \varkappa\, \varphi_i \varphi_{j,l} = \varkappa\, \varphi_l \varphi_i \varphi_j.$$

Now, we can rewrite this formula into following form:

$$\varphi_i(\varkappa\varphi_{j,l} + \varphi_j \varkappa_{,l} - 1/2\, \varkappa\varphi_j\varphi_l) + \varphi_j(\varkappa\varphi_{i,l} + \varphi_i \varkappa_{,l} - 1/2\, \varkappa\varphi_i\varphi_l) = 0.$$

Because φ_i and $\varkappa$ does not vanish, from the last formula we get

$$\varphi_{i,l} = \frac{1}{2}\, \varphi_i \varphi_l - \varphi_i\, \frac{\varkappa_{,l}}{\varkappa}.$$

The vector field φ_i is locally gradient-like, i.e. $\varphi_i = \partial_i \Phi$. Therefore from $\varphi_{i,l} = \varphi_{l,i}$ follows $\varphi_i \varkappa_{,l} = \varphi_l \varkappa_{,i}$. It is clear to see that function $\varkappa$ is a function of the argument Φ, i.e., we can write $\varkappa = \varkappa(\Phi)$. This function is differentiable, i.e. $\varkappa \in C^1$.

Because $\varkappa_{,i} = \varkappa' \cdot \varphi_i$ we have

$$\varphi_{i,j} = \frac{1}{2}\left(1 - \frac{\varkappa'}{\varkappa}\right) \varphi_i \varphi_j. \tag{10}$$

On the other hand, we can see that if the curvature tensor R has the form (4) and the conditions (9) and (10) hold, the space A_n is recurrent eqiuaffine projective Euclidean. Finally, we proved the following

Theorem 4.1. *Space A_n with affine connection is a recurrent equiaffine projective Euclidean space if and only if its components of the curvature tensor R have the following form*

$$R^h_{ijk} = \delta^h_k \psi_{ij} - \delta^h_j \psi_{ik},$$

where $\psi_{ij} = \varkappa(\Phi)\, \varphi_i \varphi_j,\ \varphi_{i,j} = \dfrac{1}{2}\left(1 - \dfrac{\varkappa'}{\varkappa}\right) \varphi_i \varphi_j,\ \varphi_i = \partial_i \Phi,\ \varkappa \in C^1$, *symbol " , " is a covariant derivative.*

5. On the existence of recurrent projective Euclidean spaces

Theorem 4.1 does not give us answer to the questions: *Does there exist any recurrent projective Euclidean space? How many such spaces are there?* Answers on these questions are in the set of recurrent equiaffine projective Euclidean spaces.

Let A_n be a recurrent equiaffine projective Euclidean space and $\bar{\mathcal{E}}_n$ be a projective equiaffine Euclidean space. Components of affine connections of A_n and $\bar{\mathcal{E}}_n$ are connected to the Levi-Civita equation (2):

$$\bar\Gamma^h_{ij} = \Gamma^h_{ij} + \delta^h_i \psi_j + \delta^h_j \psi_i.$$

Because

$$\psi_{ij} = \psi_{i,j} - \psi_i \psi_j,\ \psi_{i,j} = \partial_j \psi_i - \psi_\alpha \Gamma^\alpha_{ij},\ \psi_{i|j} = \partial_j \psi_i - \psi_\alpha \bar\Gamma^\alpha_{ij},\ \varphi_{i,j} = \partial_j \varphi_i - \varphi_\alpha \Gamma^\alpha_{ij} \text{ and } \varphi_{i|j} = \partial_j \varphi_i - \varphi_\alpha \bar\Gamma^\alpha_{ij},$$

we can rewrite the equations in the Theorem 4.1 (i.e. the conditions (9) and (10)) as follows

$$\begin{aligned} \Phi_{|i} &= \varphi_i, \\ \psi_{i|j} &= 2\psi_i \psi_j + \varkappa(\Phi)\, \varphi_i \varphi_j\,, \\ \varphi_{i|j} &= \varphi_i \psi_j + \varphi_j \psi_i + \frac{1}{2}\left(1 - \frac{\varkappa'}{\varkappa}\right) \varphi_i \varphi_j\,, \end{aligned} \tag{11}$$

where symbol " | " denotes a covariant derivative, respective connection $\bar{\nabla}$ of $\bar{\mathcal{E}}_n$.

For apriori defined functions $\varkappa \in C^1$, the conditions (11) are a nonlinear system of partial differential equations of Cauchy type in covariant derivative with respect to unknown function $\Phi(x), \varphi_i(x)$ and $\psi_i(x)$. Therefore, for given function $\varkappa \in C^1$, the system (11) with initial conditions at the point x_0

$$\Phi(x_0) = \overset{0}{\Phi}, \qquad \varphi_i(x_0) = \overset{0}{\varphi}_i, \qquad \psi_i(x_0) = \overset{0}{\psi}_i, \tag{12}$$

can have only one solution.

On the other hand, by checking the integrability conditions of the system (11), we can find out, the mentioned system is absolute integrable (when $\varkappa \in C^2$) and thus it has solution for any initial conditions (12). Thus the set of the solution (11) and also the set of those spaces depend only on one function $\varkappa$ and $2n+1$ parameters.

Theorem 5.1. *The set of recurrent equiaffine projective Euclidean spaces A_n are generalized by the system of partial differential equations* (11) *in covariant derivative. For any function $\varkappa \in C^2$ and initial conditions* (12)*, that system has solution. The set of those spaces depend only on one function $\varkappa \in C^1$ and $2n+1$ real parameters.*

Finally, we remark that if in $\bar{A}_n$ is affine coordinate x, then system (11) has form of partial differential equations

$$\partial_i\Phi = \varphi_i, \quad \partial_j\psi_i = 2\psi_i\psi_j + \varkappa(\Phi)\,\varphi_i\varphi_j\,, \quad \partial_j\varphi_i = \varphi_i\psi_j + \varphi_j\psi_i + \frac{1}{2}\left(1 - \frac{\varkappa'}{\varkappa}\right)\varphi_i\varphi_j\,.$$

6. Example of recurrent equiaffine projective Euclidean spaces

Finding the general solution of system (11) is practically impossible. We will try to find some solutions. We will assume that we have the functions Ψ and Φ, which generate gradient vectors $\psi_i = \partial_i\Psi$ and $\varphi_i = \partial_i\Phi$, depending on variable x^1.

On the base of formula (2), components of an affine connection in the recurrent equiaffine projective spaces have the following form:

$$\Gamma^h_{ij} = s(x^1)\cdot(\delta^h_i\delta^1_j + \delta^h_j\delta^1_i) \tag{13}$$

where s is a function of x^1 variable. By calculation, we convince ourselves that the curvature tensor R ($R^h_{ijk} = \partial_j\Gamma^h_{ik} - \partial_k\Gamma^h_{ij} + \Gamma^\alpha_{ik}\Gamma^h_{\alpha j} - \Gamma^\alpha_{ij}\Gamma^h_{\alpha k}$) takes the form of:

$$R^h_{ijk} = (s' - s^2)\,\delta^1_i(\delta^h_k\delta^1_j - \delta^h_j\delta^1_k). \tag{14}$$

Because $\delta^1_{i,j} = -2s\cdot\delta^1_i\delta^1_j$, then from (14) follows,

$$R^h_{ijk,l} = \varphi_l\,R^h_{ijk},$$

where $\varphi_l = ((\ln|s' - s^2|)' - 4s)\,\delta^1_l$.

The following theorem holds.

Theorem 6.1. *A space A_n with the affine connection* (13) *and with $s(x^1) \in C^1$ is a recurrent equiaffine projective Euclidean space. If the equation* $(\ln|s' - s^2|)' = 4s$ *is fulfilled, then A_n is symmetric.*

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