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Geometric Inequalities for CR-Warped Product Submanifolds of Locally Conformal Almost Cosymplectic Manifolds

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Abstract. In this paper, we establish some inequalities for the squared norm of the second fundamental form and the warping function of warped product submanifolds in locally conformal almost cosymplectic manifolds with pointwise φ-sectional curvature. The equality cases are also considered. Moreover, we prove a triviality result for CR-warped product submanifold by using the integration theory on a compact orientate manifold without boundary.

1. Introduction

The concept of warped product manifolds was first introduced by Bishop and O'Neil (cf. [7]) manifolds of negative curvature. Let us consider that N_1 and N_2 are two Riemannian manifolds of dimensions n_1 and n_2 endowed with Riemannian matrices $g_1$and g_2 such that $f : N_1 \to (0, \infty)$ be a positive differentiable function on N_1. Thus the warped product manifold $M = N_1 \times_f N_2$ is defined as the product manifold $N_1 \times N_2$ with an equipped metric $g = g_1 + f^2.g_2$. Moreover, If $f = 1$ and $f \neq 1$, then M is called a simply Riemannian product manifold and non-trivial warped product manifold respectively. Let $M = N_1 \times_f N_2$ be a non-trivial warped product manifold of an arbitrary Riemannian manifold $\widetilde{M}$. Then,

$$\nabla_X Z = \nabla_Z X = (X \ln f)Z,$$

for any vector fields $X, Y \in \Gamma(TN_1)$ and $Z \in \Gamma(TN_2)$. Further, ∇ is a Levi-Civitas connection of the induced Riemannian manifold M.

Let $\phi : M = N_1 \times_f N_2 \to \widetilde{M}$ be an isometric immersion of a warped product manifold $N_1 \times_f N_2$ into a Riemannian manifold of $\widetilde{M}$ of constant section curvature c. Assume that $n_1,\ n_2$ and n be the dimensions of $N_1,\ N_2$, and $N_1 \times_f N_2$, respectively. Then for unit vector fields X, Z tangent to $N_1,\ N_2$, respectively, ones have

$$\begin{aligned} K(X \wedge Z) &= g(\nabla_Z \nabla_X X - \nabla_X \nabla_Z X, Z) \\ &= \frac{1}{f}\{(\nabla_X X) f - X^2 f\}. \end{aligned} \tag{1.1}$$

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If we consider a local orthonormal frame $\{e_1, e_2 \cdots e_n\}$ such that $e_1, e_2, \cdots e_{n_1}$ tangent to M_1 and $e_{n_1+1}, \cdots e_n$ are tangent to M_2, we have

$$\sum_{1\leq i\leq n_1}\sum_{n_1+1\leq j\leq n} K(e_i \wedge e_j) = \frac{n_2\Delta f}{f}. \tag{1.2}$$

In [9] Chen obtained the sharp relationship between the norm of the squared mean curvature and the warping function f of CR-warped product manifold $N_T \times_f M_\perp$ isometrically immersed in a complex space form, i.e.,

Theorem 1.1. *[9] $M = N_T^h \times_f N_\perp^p$ be a CR-warped product submanifold into complex space form $\widetilde{M}(4c)$ with constant sectional curvature c. Then*

$$||\sigma||^2 \geq 2p\{||\nabla \ln f||^2 + \Delta(\ln f) + 2hc\}, \tag{1.3}$$

where Δ is the Lapalcian operator of N_T. Moreover, the equality hold in equation (1.3) *then N_T is totally geodesic and $N_\perp$ is totally umbilical submanifolds in $\widetilde{M}(4c)$.*

Furthermore, the Munteanu recalls some of the basic problems of CR-warped productsubmanifolds in Sasakian space forms as to a simple relationship between the second fundamental form and the main intrinsic invariants. In [16], a sharp inequality is established for the sectional curvature of warped product manifold in a locally conformal almost cosymplectic manifold in terms of the warping functions and the squared norm of mean curvature vector field. Afterward several geometers [1–5, 11, 15] obtained similar inequalities for different type of warped product manifold in different kind of structures. In [13], Shukla et.al proved the existence of contact CR-warped product submanifolds in a locally conformal almost cosymplectic manifold and also obtained an inequality for the second fundamental form without constant sectional curvature in terms of the warping function. In this article, we establish a Chen type inequality for CR-warped product submanifolds in a locally conformal almost cosymplectic manifold. We also find some applications of the inequality in a compact Riemannian manifold by using integration theory on manifolds.

2. Preliminaries

An $(2m+1)$-dimensional smooth manifold $\widetilde{M}$ is called *a locally conformal almost cosymplectic* manifold, if it is consisting an endomorphism φ of its tangent bundle $T\widetilde{M}$, a structure vector field ξ and a 1-form η, which is satisfying the following:

$$\varphi^2 = -I + \eta \oplus \xi, \quad \eta(\xi) = 1, \quad \eta o\varphi = 0, \tag{2.4}$$

$$g(\varphi U, \varphi V) = g(U, V) - \eta(U)\eta(V), \quad \eta(U) = g(U, \xi), \tag{2.5}$$

$$(\widetilde{\nabla}_U\varphi)V = \alpha\{g(\varphi U, V) - \eta(V)\varphi U\}, \tag{2.6}$$

$$\widetilde{\nabla}_U\xi = \alpha\{U - \eta(U)\xi\}, \tag{2.7}$$

for any U, V tangent to $\widetilde{M}$ and $\omega = \alpha\eta$ (see [12]). A plane section σ in $T_p\widetilde{M}$ of an almost contact manifold$\widetilde{M}$, is said to be a φ-section if $\sigma \perp \xi$ and $\varphi(\sigma) = \sigma$. A manifold $\widetilde{M}$ is called pointwise constant φ-sectional curvature if the sectional curvature $\widetilde{K}(\sigma)$ does not depend on the choice of the φ-section tangent space σ of $T_p\widetilde{M}$ at each point $p \in M$. In this case, for $p \in M$ and for $\varphi-$*scetion* σ of $T_p\widetilde{M}$, the function c defined by $c(p) = \widetilde{K}(p)$ is said to be φ-sectional curvature of $\widetilde{M}$. Then the curvature tensor $\widetilde{R}$ of a locally conformal almost cosymplectic manifold $\widetilde{M}(c)$ of *dimension* ≥ 5 is given by,

$$\begin{aligned}\widetilde{R}(X,Y,Z,W) =&\Big(\frac{c-3\alpha^2}{4}\Big)\Big\{g(X,W)g(Y,Z) - g(X,Z)g(Y,W)\Big\}\\&+\Big(\frac{c+\alpha^2}{4}\Big)\Big\{g(X,\varphi W)g(Y,\varphi Z) - g(X,\varphi Z)g(Y,\varphi W)\\&-2g(X,\varphi Y)g(Z,\varphi W)\Big\}\\&-\Big(\frac{c+\alpha^2}{4}+\alpha'\Big)\Big\{g(X,W)\eta(Y)\eta(Z) - g(X,Z)\eta(X)\eta(W)\\&+g(Y,Z)\eta(X)\eta(W)g(Y,W)(X)\eta(Z)\Big\}\\&+g(h(X,W),h(Y,Z) - g(h(X,Z),h(Y,W)),\end{aligned} \tag{2.8}$$

where α is the conformal function such that $\omega = \alpha\eta$ and $\alpha' = \xi\alpha$. Moreover, c is a function of constant φ−sectional curvature of $\widetilde{M}$.

Let M be a submanifold of an almost contact metric manifold $\widetilde{M}$ with an induced metric g. If ∇ and $\nabla^{\perp}$ are the induced Riemannian connections on the tangent bundle TM and the normal bundle $T^{\perp}M$ of M, respectively. Then Gauss and Weingarten formulas are given by

$$(i)\ \widetilde{\nabla}_U V = \nabla_U V + h(U,V), \quad (ii)\ \widetilde{\nabla}_U N = -A_N U + \nabla^{\perp}_U N, \tag{2.9}$$

for each $U, V \in \Gamma(TM)$ and $N \in \Gamma(T^{\perp}M)$, where h and A_N are the second fundamental form and the shape operator (corresponding to the normal vector field N), respectively, for the immersion of M into $\widetilde{M}$. They are related as;

$$g(h(U,V),N) = g(A_N U, V), \tag{2.10}$$

where g denote the Riemannian metric on $\widetilde{M}$ as well as the metric induced on M. Now for any $U \in \Gamma(TM)$ and $N \in \Gamma(T^{\perp}M)$, we have

$$(i)\ \varphi U = TU + FU, \quad (ii)\ \varphi N = tN + fN, \tag{2.11}$$

where $TU(tN)$ and $FU(fN)$ are tangential and normal components of $\varphi U(\varphi N)$, respectively. If T is identically zero, then a submanifold M is called totally real submanifold. For a subamnifold M, the Gauss equation is defined as;

$$\begin{aligned}\widetilde{R}(U,V,Z,W) =& R(U,V,Z,W) + g(h(U,Z),h(V,W))\\&-g(h(U,W),h(V,Z)),\end{aligned} \tag{2.12}$$

for any $U, V, Z, W \in \Gamma(TM)$, where $\widetilde{R}$ and R are the curvature tensors on $\widetilde{M}$ and M, respectively. The mean curvature vector H for an orthonormal frame $\{e_1, e_2, \cdots, e_n\}$ of tangent space TM on M is defined by

$$H = \frac{1}{n} trace(h) = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i), \tag{2.13}$$

where $n = dimM$. Also we set

$$h^r_{ij} = g(h(e_i,e_j),e_r) \ \ and\ \ \|h\|^2 = \sum_{i,j=1}^{n} g(h(e_i,e_j),h(e_i,e_j)). \tag{2.14}$$

The scalar curvature ρ for a submanifold M of an almost complex manifolds $\widetilde{M}$ is given by

$$\rho = \sum_{1\leq i\neq j\leq n} K(e_i \wedge e_j), \tag{2.15}$$

where $K(e_i \wedge e_j)$ is the sectional curvature of plane section spanned by e_i and e_j. Let G_r be a r-plane section on TM and $\{e_1, e_2, \cdots, e_r\}$ any orthonormal basis of G_r. Then the scalar curvature $\rho(G_r)$ of G_r is given by

$$\rho(G_r) = \sum_{1\leq i\neq j\leq r} K(e_i \wedge e_j). \tag{2.16}$$

Assume that $\widetilde{M}$ is an almost contact metric manifold and M to be a submanifold of $\widetilde{M}$ is called *totally umbilical* if $h(U,V) = g(U,V)H$ and *totally geodesic* $h(U,V) = 0$, for all $U,V \in \Gamma(TM)$ where H is the mean curvature vector field of M. Furthermore, if $H = 0$, them M is *minimal* in $\widetilde{M}$. If φ preserves any tangent space of M which is tangent to structure vector field ξ, i.e., $\varphi(T_pM) \subseteq T_pM$, for each $p \in M$, then M is called *invariant* submanifold. Similarly, the *anti-invariant* submanifold tangent to structure vector field ξ is defined as, i.e., φ maps any tangent space of M into normal space, that is $\varphi(T_pM) \subseteq T^\perp M$, for each $p \in M$. Now we give the following definition;

Definition 2.1. *A submanifold M tangent to a structure vector field ξ of an almost contact metric manifold $\widetilde{M}$ is said to be a CR-submanifold if there exists a pair of orthogonal distributions $\mathcal{D}$ and $\mathcal{D}^\perp$ such that*

(i) $TM = \mathcal{D} \oplus \mathcal{D}^\perp \oplus <\xi>$, where $<\xi>$ is 1-dimensional distribution spanned by ξ.
(ii) The distribution $\mathcal{D}$ is invariant, i.e., $\varphi(\mathcal{D}) \subseteq \mathcal{D}$,
(iii) The distribution $\mathcal{D}^\perp$ is anti-invariant, i.e., $\varphi\mathcal{D}^\perp \subseteq (T^\perp M)$.

If d_1 and d_2 be the dimensions of *invariant* distribution $\mathcal{D}$ and anti-invariant distribution $\mathcal{D}^\perp$ of a contact CR-submanifold of an almost contact metric manifold $\widetilde{M}$. Then M is invariant if $d_2 = 0$, and anti-invariant if $d_1 = 0$. It is called a proper contact CR-subamnifold if neither $d_1 = 0$ nor $d_2 = 0$. Moreover, if μ is an *invariant* subspace under φ of normal bundle $T^\perp M$. Then, in case of a contact CR-submanifold, the normal bundle $T^\perp M$ can be decomposed as

$$T^\perp M = \varphi\mathcal{D}^\perp \oplus \mu.$$

If M is a compact orientate Riemannian submanifold without boundary, then, the following satisfies by using the integration theory on manifold, i.e.,

$$\int_M \Delta(f)dV = 0, \tag{2.17}$$

where dV is the volume element of M [17].

3. Main inequalities of CR-warped product submanifolds

We study the contact CR-warped product submanifolds tangent to ξ in a locally conformal almost cosymplectic manifold.

Proposition 3.1. *Let $M = N_T \times_f N_\perp$ be a CR-warped product submanifold in a locally conformal almost cosmplectic manifold $\widetilde{M}$ such that the structure vector field ξ is tangent to N_T. Then*

$$g(h(\varphi X, Y), \varphi Z) = g(h(X,Y), \varphi Z) = 0, \tag{3.18}$$
$$g(h(X,X), \tau) = -g(h(\varphi X, \varphi X), \tau), \tag{3.19}$$

for any $X, Y \in \Gamma(TN_T)$ and $Z, W \in \Gamma(TN_\perp)$. Moreover, $\tau \in \Gamma(\mu)$ where μ is an invariant subspace.

Proof. From (2.9)(i), (2.6) and (2.5), we obtain

$$g(h(\varphi X, Y), \varphi Z) = g(\bar{\nabla}_Y \varphi X, \varphi Z) = g(\bar{\nabla}_Y X, Z) - \eta(\bar{\nabla}_Y X)\eta(Z).$$

Thus from the facts that N_T is tangent to ξ which is totally geodesic in M, then using (2.7), we get required result (3.18). In similar way, we obtain (3.19). This completes the proof of the proposition.
□

Proposition 3.2. *On a CR-warped product submanifold $M = N_T \times_f N_\perp$ of a locally conformal almost cosymplectic manifold $\widetilde{M}$ such that N_T is invariant submanifold of dimension n_1 tangent to ξ. Then N_T is φ-minimal submanifold of $\widetilde{M}$.*

Proof. The mean curvature vector of N_T is defined as

$$\|H_T\|^2 = \frac{1}{n_1^2} \sum_{r=n_1+1}^{2m+1} \left[h_{11}^r + h_{22}^r +, \cdots + h_{n_1 n_1}^r\right]^2,$$

where n_1 is the dimension of N_T such that $n_1 = 2d_1 + 1$. Thus from adapted frame, we can expressed the above equation as

$$\|H_T\|^2 = \frac{1}{n_1^2} \sum_{r=n_1+1}^{2m+1} \left(h_{11}^r +, \cdots + h_{d_1 d_1}^r + h_{d_1+1 d_1+1}^r +, \cdots + h_{2d_1 2d_1}^r + h_{\xi\xi}^r\right)^2.$$

Thus from the fact $h_{\xi\xi}^r = 0$, we derive

$$\|H_T\|^2 = \frac{1}{n_1^2} \sum_{r=n_1+1}^{2m+1} \left(h_{11}^r +, \cdots + h_{d_1 d_1}^r + h_{d_1+1 d_1+1}^r +, \cdots + h_{2d_1 2d_1}^r\right)^2. \tag{3.20}$$

Hence, there are two cases, i.e., $e_r \in \Gamma(\varphi\mathcal{D}^\perp)$ or $e_r \in \Gamma(\mu)$. If $e_r \in \Gamma(\varphi\mathcal{D}^\perp)$, then equation (3.20) can be written as

$$\|H_T\|^2 = \frac{1}{n_1^2} \sum_{r=1}^{2m+1} \Big(g(h(\tilde{e}_1, \tilde{e}_1), \varphi\bar{e}_r) + \cdots + g(h(\tilde{e}_{d_1}, \tilde{e}_{d_1}), \varphi\bar{e}_r)$$
$$+ g(h(\tilde{e}_{d_1+1}, \tilde{e}_{d_1+1}), \varphi\bar{e}_r) + \cdots + g(h(\tilde{e}_{2d_1}, \tilde{e}_{2d_1}), \varphi\bar{e}_r)\Big)^2.$$

Thus from (3.18), we get $H_T = 0$, i.e., N_T is a minimal submanifold. On the other case, if $e_r \in \Gamma(\mu)$, then (3.20) can be written by using (3.19), i.e.,

$$\|H_T\|^2 = \frac{1}{n_1^2} \sum_{r=1}^{2m+1} \Big(g(h(\tilde{e}_1, \tilde{e}_1), e_r) + \cdots + g(h(\tilde{e}_{d_1}, \tilde{e}_{d_1}), e_r)$$
$$- g(h(\tilde{e}_1, \tilde{e}_1), e_r) - \cdots - g(h(\tilde{e}_{d_1}, \tilde{e}_{d_1}), e_r)\Big)^2,$$

which implies that $H_T = 0$. Hence in both the cases it easy to conclude that N_T is φ-minimal submanifold of $\widetilde{M}$. This completes the proof of proposition. □

Proposition 3.3. *Assume that $\phi : M = N_T \times_f N_\perp \longrightarrow \widetilde{M}$ be an isometric immersion of an n-dimensional CR-warped product submanifold M into a locally conformal almost cosymplectic manifold $\widetilde{M}$. Thus* *(i) The squared norm of the second fundamental form of M is satisfied*

$$||h||^2 \geq 2\Big(n_2||\nabla \ln f||^2 + \widetilde{\tau}(TM) - \widetilde{\tau}(TN_T) - \widetilde{\tau}(TN_\perp) - n_2\Delta(\ln f)\Big), \tag{3.21}$$

where n_2 is the dimension of an anti-invariant subamnifold $N_\perp$ and Δ is the Laplacian operator of N_T.

(ii) The equality holds in (3.21) *if and only if N_T is totally geodesic and $N_\perp$ is totally umbilical in $\widetilde{M}$. Moreover M is a minimal submanifold of $\widetilde{M}$.*

Proof. The above Proposition 3.3 can be easily proved as similar as the proof of Theorem 4.4[11] if we consider Riemannian submanifold as a CR-warped product submanifold. Further base manifold is a locally conformal almost cosymplectic manifold instead of a Kenmotsu manifold. □

Now we are able to prove our important theorem using the Proposition 3.3 for locally conformal almost cosymplectic manifolds.

Theorem 3.1. *Let $\widetilde{M}(c)$ be a $(2m+1)-$dimensional locally conformal almost cosymplectic manifold and $\phi : N_T \times_f N_\perp \longrightarrow \widetilde{M}(c)$ to be a CR-warped product submanifolds into a $\widetilde{M}(c)$ such that c is a pointwise constant $\varphi-$sectional curvature and ξ is tangent to M_T. Then the second fundamental form is given by*

$$||h||^2 \geq 2n_2\Big\{||\nabla \ln f||^2 + \Big(\frac{c-3\alpha^2}{4}\Big)n_1 + \Big(\frac{c+\alpha^2}{8}\Big)n_2 - \alpha' - \Delta(\ln f)\Big\}, \tag{3.22}$$

where $n_i = dimN_i$, $i = 1, 2$ and Δ is the Laplacian operator on N_T. The equality holds in (3.22) *if and only if N_T and $N_\perp$ are totally geodesic and totally umbilical submanifolds in $\widetilde{M}(c)$, respectively. Moreover, M is a minimal submanifold of $\widetilde{M}(c)$.*

Proof. Substituting $X = W = e_i$ and $Y = Z = e_j$ in a equation (2.8), we get

$$\begin{aligned}
\widetilde{R}(e_i, e_j, e_j, e_i) =& \Big(\frac{c-3\alpha^2}{4}\Big)\Big\{g(e_i, e_i)g(e_j, e_j) - g(e_i, e_j)g(e_i, e_j)\Big\} \\
&+ \Big(\frac{c+\alpha^2}{4}\Big)\Big\{g(e_i, \varphi e_j)g(\varphi e_j, e_i) - g(e_i, \varphi e_i)g(e_j, \varphi e_j) + 2g^2(\varphi e_j, e_i)\Big\}. \\
&- \Big(\frac{c+\alpha^2}{4} + \alpha'\Big)\Big\{g(e_i, e_i)\eta(e_j)\eta(e_j) - g(e_i, e_j)\eta(e_i)\eta(e_i) \\
&+ g(e_j, e_j)\eta(e_i)\eta(e_i)g(e_j, e_i)(e_i)\eta(e_j)\Big\} \\
&+ g(h(e_i, e_i), h(e_j, e_j)) - g(h(e_i, e_j), h(e_j, e_i)),
\end{aligned} \tag{3.23}$$

Taking summation over the basis vector fields of TM such that $1 \leq i \neq j \leq n$, then it is easy to obtain that

$$2\widetilde{\tau}(TM) = \frac{c-3\alpha^2}{4}n(n-1) + 3\frac{c+\alpha^2}{4}\sum_{1\leq i\neq j\leq n} g^2(\varphi e_i, e_j) - \Big(\frac{c+\alpha^2}{4} + \alpha'\Big)2(n-1).$$

As, for an n-dimensional CR-warped product submanifold tangent ξ, we derive $||T||^2 = n - 1$, then

$$2\widetilde{\tau}(TM) = \Big(\frac{c-3\alpha^2}{4}\Big)n(n-1) + \Big(\frac{c+\alpha^2}{4}\Big)3n - \Big(\frac{c+\alpha^2}{4} + \alpha'\Big)2(n-1) \tag{3.24}$$

On the other hand, by helping the frame field of $TN_\perp$ and Proposition 3.2, we have

$$2\widetilde{\tau}(TN_\perp) = \Big(\frac{c-3\alpha^2}{4}\Big)n_2(n_2-1). \tag{3.25}$$

Similarly, we consider that ξ is tangent to invariant submanifold N_T. Then using the frame vector fields of TN_T, we get from (3.23)

$$2\widetilde{\tau}(TN_T) = \Big(\frac{c-3\alpha^2}{4}\Big)n_1(n_1-1) + \Big(\frac{c+\alpha^2}{4}\Big)3n_1 - \Big(\frac{c+\alpha^2}{4} + \alpha'\Big)2(n_1-1). \tag{3.26}$$

Therefore, using (3.24), (3.25) and (3.26) in Proposition 3.3, we get the required result. The equality case follows from the Proposition 3.3. Thus, the proof is completed. □

4. Applications

Remark 4.1. *If we substitute $\alpha = 1$ in Theorem 3.1, then Theorem 3.1 is generalized the result for a contact CR-warped product submanifold in Kenmotsu space forms.*

Remark 4.2. *If we put $\alpha = 0$ in Theorem 3.1, It becomes same as the Theorem 1.2 in [15].*

Corollary 4.1. *Let $\ln f$ to be a harmonic function on N_T. Then there does not exist any CR-warped product submanifold $N_T \times_f N_\perp$ into a locally conformal almost cosymplectic manifold $\widetilde{M}(c)$ with $c \leq 3\alpha^2$.*

Corollary 4.2. *Assume that $\ln f$ to be a non-negative eigenfunction on N_T with corresponding non-constant eigenvalue $\lambda > 0$. Then there does not exist any CR-warped product submanifold $N_T \times_f N_\perp$ into a locally conformal almost cosymplectic manifold $\widetilde{M}(c)$ with $c \leq 3\alpha^2$.*

Theorem 4.3. *Let $M = N_T \times_f N_\perp$ be a compact orientate CR-warped product submanifold into a locally conformal almost manifold $\widetilde{M}(c)$. Then M is a trivial warped product submanifold if and only if*

$$\|h\|^2 \geq \Big(\frac{c-3\alpha^2}{2}\Big)n_1n_2 + \Big(\frac{c+\alpha^2}{4} - 2\alpha'\Big)n_2 \tag{4.27}$$

where $n_1 = \dim N_T$ and $n_2 = \dim N_\perp$.

Proof. From Theorem 3.1, we get

$$\|h\|^2 \geq \Big(\frac{c-3\alpha^2}{2}\Big)n_1n_2 + \Big(\frac{c+\alpha^2}{4} - 2\alpha'\Big)n_2 - n_2\Delta(\ln f) + n_2\|\nabla \ln f\|^2,$$

and

$$n_2\|\nabla \ln f\|^2 + \Big(\frac{c-3\alpha^2}{2}\Big)n_1n_2 + \Big(\frac{c+\alpha^2}{4} - 2\alpha'\Big)n_2 - \|h\|^2 \leq n_2\Delta(\ln f). \tag{4.28}$$

From the integration theory on compact orientate Riemannian manifold M without boundary, we obtain

$$\int_M \Big(\Big(\frac{c-3\alpha^2}{2}\Big)n_1n_2 + \Big(\frac{c+\alpha^2}{4} - 2\alpha'\Big)n_2 + n_2\|\nabla \ln f\|^2 - \|h\|^2\Big)\mathrm{d}V \leq n_2\int_M \Delta(\ln f)\mathrm{d}V = 0. \tag{4.29}$$

Now, if

$$\|h\|^2 \geq \Big(\frac{c-3\alpha^2}{2}\Big)n_1n_2 + \Big(\frac{c+\alpha^2}{4} - 2\alpha'\Big)n_2,$$

Then, from (4.29), we find

$$\int_M (\|\nabla \ln f\|^2)dV \leq 0,$$

which is impossible for a positive integrable function, and hence $\nabla \ln f = 0$, i.e., f is a constant function on M. Thus by definition of warped product manifold, M is trivial. The converse part is straightforward. □

Corollary 4.4. *Assume that $M = N_T \times_f N_\perp$ to be a CR-warped product submanifold in a locally conformal almost cosymplectic manifold $\widetilde{M}(c)$. Let N_T is a compact invariant submanifold and λ be non-zero eigenvalue of the Laplacian on N_T. Then*

$$\int_{N_T} ||h||^2 dV_T \geq \int_{N_T} \Big(\Big(\frac{c-3\alpha^2}{2}\Big)n_1 n_2 + \Big(\frac{c+\alpha^2}{4} - \alpha'\Big)n_2\Big) dV_T + 2n_2\lambda \int_{N_T} (\ln f)^2 dV_T. \tag{4.30}$$

Proof. Thus using the minimum principle property, we obtain

$$\int_{N_T} ||\nabla \ln f||^2 dV_T \geq \lambda \int_{N_T} (\ln f)^2 dV_T. \tag{4.31}$$

From (3.22) and (4.31) we get required the result (4.30). It completes the proof of corollary. □

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