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On the Difference Method for Approximation of Second Order Derivatives of a Solution of Laplace's Equation in a Rectangular Parallelepiped

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Abstract.
We present and justify finite difference schemes with the 14-point averaging operator for the second derivatives of the solution of the Dirichlet problem for Laplace's equations on a rectangular parallelepiped. The boundary functions φ_j on the faces Γ_j, $j = 1, 2, ..., 6$ of the parallelepiped are supposed to have fifth derivatives belonging to the Hölder classes $C^{5,\lambda}$, $0 < \lambda < 1$. On the edges, the boundary functions as a whole are continuous, and their second and fourth order derivatives satisfy the compatibility conditions which result from the Laplace equation.

It is proved that the proposed difference schemes for the approximation of the pure and mixed second derivatives converge uniformly with order $O(h^{3+\lambda})$, $0 < \lambda < 1$ and $O(h^3)$, respectively.

Numerical experiments are illustrated to support the theoretical results.

1. Introduction

For the numerical solution of PDEs, a highly accurate method becomes a powerful tool in reducing the number of unknowns, which is the main problem in the numerical solution of differential equations to get reasonable results. This becomes more valuable in 3D problems when we are looking for the derivatives of the unknown solution by the finite difference or finite element methods for a small discretization parameter h.

The derivative problem was investigated in [5], in which it was proved that the high order difference derivatives uniformly converge to the corresponding derivatives of the solution for the 2D Laplace equation in any strictly interior subdomain, with the same order of h as which the difference solution converges on the given domain. In [10], the uniform convergence of the difference derivatives over the whole grid domain to the corresponding derivatives of the solution for the 2D Laplace equation with order $O(h^2)$ was proved. In [3], for the first and pure second derivatives of the solution for the 2D Laplace equation, special finite difference problems were investigated. It is proved that the solution of these problems converge to the exact derivatives with order $O(h^4)$

In [12] the Dirichlet problem for the Laplace equation on a rectangular parallelepiped was considered. The boundary values on the faces of a parallelepiped are supposed to have fourth derivatives satisfying

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the Hölder condition and on the edges the boundary functions as a whole are continuous. Besides, the compatibility condition, which results from the Laplace equation, for the second derivatives of the boundary values given on the adjacent faces is satisfied on the edges. The constructed difference schemes converge with order $O(h^2)$ to the first and pure second derivatives of the exact solution. Further, using by the obtained grid values of the first derivatives for one variable the second order mixed derivatives are approximated by divided differences with respect to remainder variables. By this approach the mixed second derivatives of the solution were found on a grid with accuracy $O(h^2/(\rho + h))$, where ρ is the distance from a current mesh node to the parallelepiped boundary.

In [4], a 26-point averaging operator is used to get $O(h^4)$ order of accurate approximation of first and pure second derivatives of the 3D Laplace's equation. It was assumed that the boundary values on the faces have the sixth order derivatives satisfying the Hölder condition, and the second and fourth order derivatives satisfy the compatibility conditions on the edges.

In this paper, we consider the Dirichlet problem for the Laplace equation on a rectangular parallelepiped. It is assumed that the boundary values on the faces have fifth order derivatives satisfying the Hölder condition, and the second and fourth order derivatives satisfy compatibility conditions on the edges. Four different schemes by using 14-point averaging operator on a cubic grid with mesh size h are constructed. The solution of the first three of these schemes separately approximate the solution of the Dirichlet problem with the order $O(h^4\rho)$, where $\rho = \rho(x_1, x_2, x_3)$ is the distance from the current point $(x_1, x_2, x_3) \in R$ to the boundary of the rectangular parallelepiped R, approximates its first derivatives with the order $O(h^4)$, and pure second order derivatives with the order $O(h^{3+\lambda})$, $0 < \lambda < 1$. By using the results obtained for the solution and its first order derivatives, the finite difference problem for the second order mixed derivatives is constructed. For the uniform error of this scheme $O(h^3)$ order of estimation is obtained.

Finally, the high accuracy of the proposed schemes are illustrated by solving a test problem with the exact solution.

2. Finite Difference Approximation of the Dirichlet Problem on a Rectangular Parallelepiped

Let $R = \{(x_1, x_2, x_3) : 0 < x_i < a_i,\ i = 1, 2, 3\}$ be an open rectangular parallelepiped; Γ_j, $j = 1, 2, ..., 6$ be its faces including the edges; Γ_j for $j = 1, 2, 3$ $(j = 4, 5, 6)$ belongs to the plane $x_j = 0$ ($x_{j-3} = a_{j-3}$), and let $\Gamma = \bigcup_{j=1}^{6} \Gamma_j$ be the boundary of R; $\gamma_{\mu\nu} = \Gamma_\mu \cap \Gamma_\nu$ be the edges of the parallelepiped R. Let $C^{k,\lambda}(E)$ be the class of functions that have continuous kth derivatives satisfying the Hölder condition with an exponent $\lambda \in (0, 1)$.

Consider the boundary value problem

$$\Delta u = 0 \text{ on } R,\ u = \varphi_j \text{ on } \Gamma_j,\ j = 1, 2, ..., 6 \tag{1}$$

where $\Delta \equiv \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$, φ_j are given functions.

Assume that

$$\varphi_j \in C^{5,\lambda}\left(\Gamma_j\right),\ 0 < \lambda < 1,\ j = 1, 2, ..., 6, \tag{2}$$

$$\varphi_\mu = \varphi_\nu \text{ on } \gamma_{\mu\nu}, \tag{3}$$

$$\frac{\partial^2 \varphi_\mu}{\partial t_\mu^2} + \frac{\partial^2 \varphi_\nu}{\partial t_\nu^2} + \frac{\partial^2 \varphi_\mu}{\partial t_{\mu\nu}^2} = 0 \text{ on } \gamma_{\mu\nu}, \tag{4}$$

$$\frac{\partial^4 \varphi_\mu}{\partial t_\mu^4} + \frac{\partial^4 \varphi_\mu}{\partial t_\mu^2 \partial t_{\mu\nu}^2} = \frac{\partial^4 \varphi_\nu}{\partial t_\nu^4} + \frac{\partial^4 \varphi_\nu}{\partial t_\nu^2 \partial t_{\nu\mu}^2} \text{ on } \gamma_{\mu\nu}, \tag{5}$$

where $1 \le \mu < \nu \le 6$, $\nu - \mu \ne 3$, $t_{\mu\nu}$ is an element in $\gamma_{\mu\nu}$, and t_μ and t_ν is an element of the normal to $\gamma_{\mu\nu}$ on the face Γ_μ and Γ_ν, respectively.

We introduce a cubic grid with a step $h > 0$ defined by the planes $x_i = 0, h, 2h, ...,\ i = 1, 2, 3$. It is assumed that the edge lengths of R and h are such that $\frac{a_i}{h} \ge 8$ $(i = 1, 2, 3)$ are integers.

Let D_h be the set of nodes of the grid constructed, $\overline{R}_h = \overline{R} \cap D_h$, let $R_h = R \cap D_h$, $R_h^k \subset R_h$ be the set of nodes of R_h lying at a distance of kh away from the boundary Γ of R, and let $\Gamma_h = \Gamma \cap D_h$.

Let S be a 14-point difference operator defined by (see [13])

$$Su(x_1, x_2, x_3) = \frac{1}{56}\left(8\sum_{p=1^{(1)}}^{6} u_p + \sum_{q=7^{(3)}}^{14} u_q\right), \quad (x_1, x_2, x_3) \in R_h, \tag{6}$$

where $\sum_{(m)}$ is the sum extending over the nodes lying at a distance of $m^{1/2}h$ away from the point (x_1, x_2, x_3) and u_p and u_q are the values of u at the corresponding nodes.

On the boundary Γ of R, we define continuous on the entire boundary including the edges of R, the function φ as follows

$$\varphi = \begin{cases} \varphi_1 & \text{on } \Gamma_1 \\ \varphi_j & \text{on } \Gamma_j \backslash \left(\bigcup\limits_{i=1}^{j-1} \Gamma_i\right), \quad j = 2, 3, ..., 6. \end{cases} \tag{7}$$

Obviously,

$$\varphi = \varphi_j \text{ on } \Gamma_j, \quad j = 1, 2, ..., 6.$$

We consider the finite difference problem approximating Dirichlet problem (1):

$$u_h = Su_h \text{ on } R_h, \; u_h = \varphi \text{ on } \Gamma_h, \tag{8}$$

where S is the difference operator given by (6) and φ is the function defined by (7). By the maximum principle, system (8) has a unique solution (see [7], Chap. 4).

In what follows and for simplicity, we denote by $c, c_1, c_2, ...$ constants, which are independent of h and the nearest factors, identical notation will be used for various constants.

Define

$$N(h) = \left[\frac{\min\{a_1, a_2, a_3\}}{2h}\right], \tag{9}$$

where $[a]$ is the integer part of a.

Consider for a fixed k, $1 \le k \le N(h)$ the system of grid equations

$$v_h^k = Sv_h^k + g_h^k \text{ on } R_h^k, \;\; v_h^k = 0 \text{ on } \Gamma_h, \tag{10}$$

where

$$g_h^k = \begin{cases} 1, & \rho(x_1, x_2, x_3) = kh, \\ 0, & \rho(x_1, x_2, x_3) \neq kh. \end{cases}$$

Lemma 2.1. *The solution v_h^k of the system* (10) *satisfies the inequality*

$$v_h^k(x_1, x_2, x_3) \le T_h^k, \;\; 1 \le k \le N(h), \tag{11}$$

where T_h^k is defined as

$$T_h^k = T_h^k(x_1, x_2, x_3) = \begin{cases} \frac{5\rho}{h}, & 0 \le \rho(x_1, x_2, x_3) \le kh, \\ 5k, & \rho(x_1, x_2, x_3) > kh. \end{cases} \tag{12}$$

Proof. By the direct calculation of the expression ST_h^k, we obtain

$$T_h^k - ST_h^k \ge \begin{cases} 1, & \rho(x_1, x_2, x_3) = kh, \\ 0, & \rho(x_1, x_2, x_3) \neq kh, \end{cases} \tag{13}$$

on R_h. On the basis of (10), inequalities (13) and taking the boundary condition $T_h^k = 0$ on Γ_h into account, by the comparison theorem [7], we get (11). □

Lemma 2.2. *The following estimation holds*

$$\max_{(x_1,x_2,x_3)\in R_h^k} |Su-u| \le c\frac{h^{5+\lambda}}{k^{1-\lambda}},\ k=1,2,...,N(h), \tag{14}$$

where u is the solution of the Dirichlet problem (1) *.*

Proof. The proof is carried out by analogy with the proof of Lemma 13 and Lemma 14 in [13]. □

Theorem 2.3. *Assume that the boundary functions φ_j satisfy conditions* (2)−(5)*. Then at each point $(x_1,x_2,x_3)\in R_h$*

$$|u_h-u| \le ch^4\rho, \tag{15}$$

where u_h is the solution of the finite difference problem (8)*, u is the exact solution of problem* (1) *, and $\rho=\rho(x_1,x_2,x_3)$ is the distance from the current point $(x_1,x_2,x_3)\in R_h$ to the boundary of the parallelepiped R.*

Proof. Let

$$\varepsilon_h = u_h - u \ on\ \overline{R}_h. \tag{16}$$

By (8) and (16) the error function ε_h satisfies the system of equations

$$\varepsilon_h = S\varepsilon_h + (Su-u) \text{ on } R_h,\ \varepsilon_h = 0 \text{ on } \Gamma_h. \tag{17}$$

We represent a solution of the system (17) as follows

$$\varepsilon_h = \sum_{k=1}^{N(h)} \varepsilon_h^k, \tag{18}$$

where $N(h)$ defined by (9) , ε_h^k, $1\le k\le N(h)$, is a solution of the system

$$\varepsilon_h^k = S\varepsilon_h^k + \sigma_h^k \text{ on } R_h,\ \varepsilon_h^k = 0 \text{ on } \Gamma_h, \tag{19}$$

when

$$\sigma_h^k = \begin{cases} Su-u \text{ on } R_h^k \\ 0 \qquad \text{ on } R_h\backslash R_h^k. \end{cases} \tag{20}$$

On the basis of (18), (19), (20), Lemma 2.1 and Lemma 2.2, for the solution of (17), we have

$$\begin{aligned} |\varepsilon_h| &\le \sum_{k=1}^{N(h)} \left|\varepsilon_h^k\right| \le 5c_1h^{5+\lambda}\sum_{k=1}^{\rho/h-1} k^{\lambda} + 5c_2h^{4+\lambda}\rho\sum_{k=\rho/h}^{N(h)} k^{-1+\lambda} \\ &\le c_3h^4\rho^{1+\lambda} + c_4h^4\rho \le ch^4\rho. \end{aligned} \tag{21}$$

From (16) and (21) , for any point $(x_1,x_2,x_3)\in R_h$, we obtain the inequality (15). □

3. Approximation of the Pure Second Derivatives

We denote by $\omega = \frac{\partial^2 u}{\partial x_1^2}$. The function ω is harmonic on R, on the basis of Theorem 2.1 in [8] is continuous on $\overline{R}$, and is solution of the following Dirichlet problem

$$\Delta\omega = 0 \text{ on } R,\ \omega = \chi_j \text{ on } \Gamma_j,\ j=1,2,...,6, \tag{22}$$

where

$$\chi_\tau = \frac{\partial^2 \varphi_\tau}{\partial x_1^2},\ \tau = 2,3,5,6, \tag{23}$$

$$\chi_\upsilon = -\left(\frac{\partial^2 \varphi_\nu}{\partial x_2^2} + \frac{\partial^2 \varphi_\upsilon}{\partial x_3^2}\right),\ \upsilon = 1,4. \tag{24}$$

Let ω_h be the solution of the finite difference problem

$$\omega_h = S\omega_h \text{ on } R_h,\ \omega_h = \chi_j \text{ on } \Gamma_j^h,\ j = 1,2,...,6, \tag{25}$$

where χ_j, $j = 1,2,...,6$ are the functions defined by (23) and (24).

Theorem 3.1. *The estimation holds*

$$\max_{\overline{R}_h} |\omega_h - \omega| \leq ch^{3+\lambda}, \tag{26}$$

where $\omega = \frac{\partial^2 u}{\partial x_1^2}$, *u is the solution of problem* (1) *and* ω_h *is the solution of the finite difference problem* (25).

Proof. From the continuity of the function ω on $\overline{R}$, and from (2) − (5) and (23), (24) it follows that

$$\chi_j \in C^{3,\lambda}\left(\Gamma_j\right),\ 0 < \lambda < 1,\ j = 1,2,...,6, \tag{27}$$

$$\chi_\mu = \chi_\nu \text{ on } \gamma_{\mu\nu}, \tag{28}$$

$$\frac{\partial^2 \chi_\mu}{\partial t_\mu^2} + \frac{\partial^2 \chi_\nu}{\partial t_\nu^2} + \frac{\partial^2 \chi_\mu}{\partial t_{\mu\nu}^2} = 0 \text{ on } \gamma_{\mu\nu}. \tag{29}$$

The boundary functions χ_j, $j = 1,2,...,6$, in (22) on the basis of (27) − (29) satisfy all conditions of Theorem 2 in [13] in which follows the proof of the error estimation (26). □

4. Approximation of the First Derivative

Let u be a solution of the problem (1) with the conditions (2)-(5).

We put $v = \frac{\partial u}{\partial x_1}$, and $\Phi_j = \frac{\partial u}{\partial x_1}$ on Γ_j, $j = 1,2,...,6$. It is obvious that the function v is a solution of the following boundary value problem

$$\Delta v = 0 \text{ on } R,\ v = \Phi_j \text{ on } \Gamma_j,\ j = 1,2,...,6, \tag{30}$$

We define an approximate solution of problem (30) as a solution of the following finite difference problem

$$\nu_h = S\nu_h \text{ on } R_h,\ \nu_h = \Phi_{jh}(u_h) \text{ on } \Gamma_j^h,\ j = 1,2,...,6, \tag{31}$$

where u_h is the solution of problem (8), Φ_{1h} (Φ_{4h}) is the fourth order forward (backward) numerical differentiation operator (see [1], [2]). On the nodes Γ_p^h, $p = 2,3,5,6$ the boundary values are defined as $\Phi_{ph}(u_h) = \frac{\partial \varphi_p}{\partial x_1}$.

Theorem 4.1. *The estimation is true*

$$\max_{(x_1,x_2,x_3)\in\overline{R}_h} \left|\nu_h - \frac{\partial u}{\partial x_1}\right| \leq ch^4, \tag{32}$$

where u is the solution of the problem (1), ν_h *is the solution of the finite difference problem* (31).

Proof. Let

$$\epsilon_h = \nu_h - \nu \text{ on } \overline{R}_h, \tag{33}$$

where $\nu = \frac{\partial u}{\partial x_1}$. From (31) and (33), we have

$$\begin{aligned}\epsilon_h &= S\epsilon_h + (S\nu - \nu) \text{ on } R_h,\\ \epsilon_h &= \Phi_{kh}(u_h) - \nu \text{ on } \Gamma_k^h,\ k = 1, 4,\ \epsilon_h = 0 \text{ on } \Gamma_p^h,\ p = 2, 3, 5, 6.\end{aligned}$$

We represent

$$\epsilon_h = \epsilon_h^1 + \epsilon_h^2, \tag{34}$$

where

$$\epsilon_h^1 = S\epsilon_h^1 \text{ on } R_h, \tag{35}$$

$$\epsilon_h^1 = \Phi_{kh}(u_h) - \nu \text{ on } \Gamma_k^h,\ k = 1, 4,\ \epsilon_h^1 = 0 \text{ on } \Gamma_q^h,\ q = 2, 3, 5, 6, \tag{36}$$

$$\epsilon_h^2 = S\epsilon_h^2 + (S\nu - \nu) \text{ on } R_h,\ \epsilon_h^2 = 0 \text{ on } \Gamma_j^h,\ j = 1, 2, ..., 6. \tag{37}$$

Since $\Phi_{kh}(u), k = 1, 4$ are the fourth order approximation of $\partial u/\partial x_1$ on Γ_k and the fifth order partial derivatives of the solution u are bounded in $\overline{R}$ using the pointwise estimation (15) in Theorem 2.3, we have

$$\max_{k=1,4} |\Phi_{kh}(u_h) - v| \leq c_1 h^4. \tag{38}$$

On the basis of (38), and the maximum principle, for the solution ϵ_h^1 of system (35), (36), we have

$$\max_{(x_1,x_2,x_3)\in\overline{R}_h} \left|\epsilon_h^1\right| \leq c_2 h^4. \tag{39}$$

The solution ε_h^2 of system (37) is the error function of the finite difference solution for problem (30), when the boundary functions $\Phi_j = \partial u/\partial x_1$, $j = 1, 2, ..., 6$, as follows from (2) – (5) satisfy the conditions

$$\Phi_j \in C^{4,\lambda}\left(\Gamma_j\right),\ 0 < \lambda < 1,\ j = 1, 2, ..., 6,$$

$$\Phi_\mu = \Phi_\nu \text{ on } \gamma_{\mu\nu},$$

$$\frac{\partial^2 \Phi_\mu}{\partial t_\mu^2} + \frac{\partial^2 \Phi_\nu}{\partial t_\nu^2} + \frac{\partial^2 \Phi_\mu}{\partial t_{\mu\nu}^2} = 0 \text{ on } \gamma_{\mu\nu}$$

Then, on the basis of Theorem 4 in [13] for the error ε_h^2, we have

$$\max_{(x_1,x_2,x_3)\in\overline{R}_h} \left|\epsilon_h^2\right| \leq c_3 h^4. \tag{40}$$

By virtue of (34), (39), and (40) follows the inequality (32). □

Remark 4.2. We have investigated the method of high order approximations of the first derivative $\partial u/\partial x_1$. The same results are obtained for the derivatives $\partial u/\partial x_l$, $l = 2, 3$, by using the same order forward or backward formulae in the corresponding faces of the parallelepiped.

5. Approximation of the Mixed Second Derivatives

Let $\varpi = \frac{\partial^2 u}{\partial x_1 \partial x_2} = \frac{\partial v}{\partial x_2}$ and let $\Psi_j = \frac{\partial^2 u}{\partial x_1 \partial x_2} = \frac{\partial v}{\partial x_2}$ on Γ_j, $j = 1, 2, ..., 6$, where u is a solution of the boundary value problem (1) and v is a solution of the boundary value problem (30).

Consider the boundary value problem:

$$\Delta\varpi = 0 \text{ on } R, \ \varpi = \Psi_j \text{ on } \Gamma_j, \ j = 1, 2, ..., 6. \tag{41}$$

We define the sets

$$\Gamma_k^{h+} = \left\{ 0 \le x_2 \le \frac{a_2}{2}, \ x_1 = t_k \right\} \cap \Gamma_k^h, \quad k = 1, 4, \tag{42}$$

and

$$\Gamma_k^{h-} = \left\{ \frac{a_2}{2} + h \le x_2 \le a_2, \ x_1 = t_k \right\} \cap \Gamma_k^h, \quad k = 1, 4, \tag{43}$$

where $t_1 = 0$ and $t_4 = a_1$.

We introduce the following operators $\Psi_{\varpi h}$, $\varpi = 1, 2, ..., 6$,

$$\begin{aligned} \Psi_{kh}(v_h) &= \frac{1}{6h}[-11v_h(t_k, x_2, x_3) + 18v_h(t_k, x_2 + h, x_3) - 9v_h(t_k, x_2 + 2h, x_3) \\ &\quad +2v_h(t_k, x_2 + 3h, x_3)], \ (t_k, x_2, x_3) \in \Gamma_k^{h+}, \end{aligned} \tag{44}$$

$$\begin{aligned} \Psi_{kh}(v_h) &= \frac{1}{6h}[11v_h(t_k, x_2, x_3) - 18v_h(t_k, x_2 - h, x_3) + 9v_h(t_k, x_2 - 2h, x_3) \\ &\quad -2v_h(t_k, x_2 - 3h, x_3)], \ (t_k, x_2, x_3) \in \Gamma_k^{h-}, \end{aligned} \tag{45}$$

where $k = 1, 4$.

$$\begin{aligned} \Psi_{2h}(v_h) &= \frac{1}{6h}[-11\Phi_2(x_1, x_3) + 18v_h(x_1, h, x_3) - 9v_h(x_1, 2h, x_3) \\ &\quad +2v_h(x_1, 3h, x_3)] \text{ on } \Gamma_2^h, \end{aligned} \tag{46}$$

$$\begin{aligned} \Psi_{5h}(v_h) &= \frac{1}{6h}[11\Phi_5(x_1, x_3) - 18v_h(x_1, a_2 - h, x_3) + 9v_h(x_1, a_2 - 2h, x_3) \\ &\quad -2u_h(x_1, a_2 - 3h, x_3)] \text{ on } \Gamma_5^h, \end{aligned} \tag{47}$$

$$\Psi_{ph}(v_h) = \frac{\partial^2 \varphi_p}{\partial x_1 \partial x_2} \text{ on } \Gamma_p^h, \ p = 3, 6, \tag{48}$$

where Φ_2 and Φ_5 are the functions defined in (30), φ_3 and φ_6 are the given functions in (1), v_h is the solution of the finite difference problem (31).

Let ϖ_h be the solution of the following finite difference problem

$$\varpi_h = S\varpi_h \text{ on } R_h, \ \varpi_h = \Psi_{jh} \text{ on } \Gamma_j^h, \ j = 1, 2, ..., 6, \tag{49}$$

where Ψ_{jh}, $j = 1, 2, ..., 6$, are defined by (44) – (48).

Lemma 5.1. *The inequality is true*

$$|\Psi_{kh}(v_h) - \Psi_{kh}(v)| \le ch^3, \ k = 1, 2, ..., 6 \tag{50}$$

where v_h is the solution of the finite difference problem (31), *v is the solution of problem* (30).

Proof. It is obvious that $\Psi_{ph}(v_h) - \Psi_{ph}(v) = 0$ for $p = 3, 6$.

For $k = 1, 4$, by using equation (44) and apply the Theorem 4.1, we have

$$\begin{aligned}
|\Psi_{kh}(v_h) - \Psi_{kh}(v)| &\leq \frac{1}{6h}[11\,|v_h\,(t_k, x_2, x_3) - v\,(t_k, x_2, x_3)| \\
&\quad +18\,|v_h\,(t_k, x_2 + h, x_3) - v\,(t_k, x_2 + h, x_3)| \\
&\quad +9\,|v_h\,(t_k, x_2 + 2h, x_3) - v\,(t_k, x_2 + 2h, x_3)| \\
&\quad +2\,|v_h\,(t_k, x_2 + 3h, x_3) - v\,(t_k, x_2 + 3h, x_3)|] \\
&\leq \frac{1}{6h}\left[40c_1h^4\right] \\
&\leq c_2h^3 \text{ on } \Gamma_k^{h+}.
\end{aligned}$$

The same inequality is obtain on Γ_k^{h-} by using (45) . Since $\Gamma_k^h = \Gamma_k^{h+} \cup \Gamma_k^{h-}$, we have

$$|\Psi_{kh}(v_h) - \Psi_{kh}(v)| \leq c_3h^3,\ k = 1, 4 \text{ on } \Gamma_k^h. \tag{51}$$

Similarly, from (46), (47), and Theorem 4.1, we have

$$|\Psi_{mh}(v_h) - \Psi_{mh}(v)| \leq c_4h^3,\ \text{ on } \Gamma_m^h,\ m = 2, 5.$$

□

Lemma 5.2. *The inequality holds*

$$\max_{(x_1,x_2,x_3)\in\Gamma_k^h} |\Psi_{kh}(v_h) - \Psi_k| \leq ch^3,\ k = 1, 2, 4, 5 \tag{52}$$

where Ψ_{kh}, $k = 1, 2, 4, 5$ *are defined by* (44) , (48) , *and* $\Psi_k = \frac{\partial^2 u}{\partial x_1 \partial x_2}$ *on* Γ_k, $k = 1, 2, 4, 5$.

Proof. From (44)-(47) follows that $\Psi_{qh}(v)$, $q = 1, 2, 4, 5$ are the third order forward ($q = 1, 2$) and backward ($q = 4, 5$) formulae for the approximation of $\frac{\partial^2 u}{\partial x_1 \partial x_2}$. Since the solution u of problem (1)-(5) is from $C^{5,\lambda}(\overline{R})$, from the truncation error formulas [2], we have

$$\max_{(x_1,x_2,x_3)\in\Gamma_h^k} |\Psi_{kh}(v) - \Psi_k| \leq c_1h^3, \quad k = 1, 2, 4, 5. \tag{53}$$

On the basis of Lemma 5.1, (48) and estimation (53) follows (52). □

From the conditions (2)-(5) follows that the solution of the problem (1) belongs to the class $C^{5,1-\varepsilon}(\overline{R})$ for any $\varepsilon \in (0, 1)$ (see Theorem 2.1 in [8]). Then, its any second order derivatives belongs to $C^{3,1-\varepsilon_1}(\overline{R})$, for any $\varepsilon_1 \in (0, 1)$. Since any order derivatives of harmonic functions are harmonic, the following Lemma follows from Lemma 1.2 in [9] (see Lemma 3, Chap. 4, Sec. 3 in [6], also).

Lemma 5.3. *Let* $\rho(x_1, x_2, x_3)$ *be the distance from the current point of the open parallelepiped* R *to its boundary. Then for any derivative* ϖ *of the solution of the problem (41) of order* m, ($m > 3$) *with respect to* x_1, x_2, x_3 *satisfy the inequality*

$$\left|\varpi^{(m)}(x_1, x_2, x_3)\right| \leq M_m\rho^{-m+3}, \tag{54}$$

where $M_m > 0$ *is constant dependent on* m *only.*

Let $\Psi_j(h)$ be the trace of $\varpi = \frac{\partial^2 u}{\partial x_1 \partial x_2}$ on Γ_j^h, and let $\varpi_h^{'}$ be the solution of the following problem

$$\varpi_h^{'} = S\varpi_h^{'} \text{ on } R,\ \varpi_h^{'} = \Psi_j(h) \text{ on } \Gamma_j^h,\ j = 1, 2, ..., 6. \tag{55}$$

Lemma 5.4. *The estimation holds*

$$\max_{(x_1,x_2,x_3)\in\overline{R}_h}\left|\varpi_h' - \frac{\partial^2 u}{\partial x_1 \partial x_2}\right| \leq ch^3, \tag{56}$$

where ϖ_h' is a solution of the finite difference problem (55).

Proof. From the definition of the boundary grid function $\Psi_j(h)$ and from (55) for the error function

$$\overline{\varepsilon}_h = \varpi_h' - \varpi \tag{57}$$

we have

$$\overline{\varepsilon}_h = S\overline{\varepsilon}_h + (S\varpi - \varpi) \text{ on } R_h,\ \overline{\varepsilon}_h = 0 \text{ on } \Gamma_j^h,\ j = 1,2,...,6 \tag{58}$$

where ϖ is the solution of problem (41). By virtue of (54), by analogy with the proof of Lemma 3 [14] it follows, that

$$\max_{(x_1,x_2,x_3)\in\overline{R}_h} |S\varpi - \varpi| \leq c_1\frac{h^3}{k^3},\ \ k = 1,2,...,N(h). \tag{59}$$

On the basis of Lemma 2.1 and (59) for the solution of problem (58), we obtain

$$|\overline{\varepsilon}_h| \leq c_1 h^3 \sum_{k=1}^{N(h)} \frac{1}{k^2} \leq c_2 h^3.$$

□

Theorem 5.5. *The estimation is true*

$$\max_{(x_1,x_2,x_3)\in\overline{R}_h}\left|\varpi_h - \frac{\partial^2 u}{\partial x_1 \partial x_2}\right| \leq ch^3, \tag{60}$$

where u is the solution of the problem (1) , *ϖ_h is the solution of the finite difference problem* (49) .

Proof. Let

$$\varepsilon_h = \varpi_h - \varpi \text{ on } \overline{R}_h, \tag{61}$$

where $\varpi = \frac{\partial^2 u}{\partial x_1 \partial x_2}$. From (49) and (61) , we have

$$\begin{aligned} \varepsilon_h &= S\varepsilon_h + (S\varpi - \varpi) \text{ on } R_h, \\ \varepsilon_h &= \Psi_{kh}(v_h) - \varpi \text{ on } \Gamma_k^h,\ k = 1,2,4,5\ \varepsilon_h = 0 \text{ on } \Gamma_p^h,\ p = 3,6. \end{aligned}$$

We represent

$$\varepsilon_h = \varepsilon_h^1 + \varepsilon_h^2, \tag{62}$$

where

$$\varepsilon_h^1 = S\varepsilon_h^1 \text{ on } R_h, \tag{63}$$

$$\varepsilon_h^1 = \Psi_{kh}(v_h) - \varpi \text{ on } \Gamma_k^h,\ k = 1,2,4,5,\ \varepsilon_h^1 = 0 \text{ on } \Gamma_p^h,\ p = 3,6; \tag{64}$$

$$\varepsilon_h^2 = S\varepsilon_h^2 + (S\varpi - \varpi) \text{ on } R_h, \varepsilon_h^2 = 0 \text{ on } \Gamma_j^h,\ j = 1,2,...,6. \tag{65}$$

By Lemma 5.2 and by the maximum principle, for the solution of system (63), (64) , we have

$$\max_{(x_1,x_2,x_3)\in\overline{R}_h}\left|\varepsilon_h^1\right| \leq \max_{q=1,2,...,6}\max_{(x_1,x_2,x_3)\in\Gamma_q^k}\left|\Phi_{qh}(v_h)-\varpi\right| \leq c_1h^3. \tag{66}$$

The solution ε_h^2 of system (65) is the error of the approximate solution obtained by the finite difference method for problem (41) , when it is assumed that on the boundary nodes Γ_j^h, the exact values of the functions Ψ_j, $j = 1,2,...,6$ are used.

From Lemma 5.4 follows that

$$\max_{(x_1,x_2,x_3)\in\overline{R}_h}\left|\varepsilon_h^2\right| \leq c_2h^3. \tag{67}$$

By (61), (62), (66) and (67) follows the estimation (60). □

6. Numerical Results

Let $R = \{(x_1, x_2, x_3) : 0 < x_i < 1,\ i = 1,2,3\}$, and let Γ_j, $j = 1,...,6$ be its faces. We consider the following problem:

$$\Delta u = 0 \text{ on } R,\ \ u = \varphi(x_1,x_2,x_3) \text{ on } \Gamma_j,\ j = 1,...,6, \tag{68}$$

where

$$\varphi(x_1,x_2,x_3) = \left(x_3-\frac{1}{2}\right)^2 - \left(\frac{x_1^2+x_2^2}{2}\right) + \left(x_1^2+x_2^2\right)^{\frac{\left(5+\frac{1}{30}\right)}{2}}.\cos\left(\left(5+\frac{1}{30}\right).\arctan\left(\frac{x_2}{x_1}\right)\right) \tag{69}$$

is the exact solution of this problem, which is in $C^{5,1/30}$.

We solve the systems (8), (25), (31) and (49) to find the approximate solution u_h for u, approximate second pure derivative w_h for $w = \frac{\partial^2 u}{\partial x_1^2}$, approximate first derivative v_h for $v = \frac{\partial u}{\partial x_1}$ and approximate second order mixed derivative ϖ_h for $\varpi = \frac{\partial^2 u}{\partial x_1 \partial x2}$ respectively.

In the following Table the maximum errors are given. Estimation of the solution $\|u_h - u\|_{\overline{R}_h}$ shows that the convergence order more than 4 which is corresponds to the product ρ in Theorem 2.3. Estimation of the second pure derivative $\|w_h - w\|_{\overline{R}_h}$ shows that the convergence order more than 3 which is corresponds to the λ in Theorem 3.1. The presented column for $\|v_h - v\|_{\overline{R}_h}$conformed the estimation obtained in Theorem 4.1, i.e., the fourth order convergence. The error estimation of the second order mixed derivative $\|\varpi_h - \varpi\|_{\overline{R}_h}$, shows the third order convergence, which justifies of Theorem 5.5.

h	$\|u_h-u\|_{\overline{R}_h}$	E_u^m	$\|w_h-w\|_{\overline{R}_h}$	E_ω^m	$\|v_h-v\|_{\overline{R}_h}$	E_v^m	$\|\varpi_h-\varpi\|_{\overline{R}_h}$	E_ϖ^m
2^{-4}	$2,34E-10$	$32,66$	$3,68E-07$	$8,20$	$3.39E-04$	$14,76$	$6,92E-03$	$7,35$
2^{-5}	$7,16E-12$	$32,74$	$4,49E-08$	$8,18$	$2.30E-05$	$15,40$	$9,41E-04$	$7,71$
2^{-6}	$2,19E-13$	$32,75$	$5,49E-09$	$8,19$	$1.49E-06$	$15,70$	$1,22E-04$	$7,92$
2^{-7}	$6,68E-15$		$6,70E-10$		$9.51E-08$		$1,54E-05$	

Table: Errors in maximum norm

In Table we have used the following notations:

$\|U_h - U\|_{\overline{R}_h} = \max_{\overline{R}_h}|U_h - U|$, $E_U^m = \frac{\|U-U_{2^{-m}}\|_{\overline{R}_h}}{\|U-U_{2^{-(m+1)}}\|_{\overline{R}_h}}$, where U be the exact solution of the continuous problem, and U_h be its approximate values on $\overline{R}_h$.

Conclusion

Four different schemes with the 14-point averaging operator are constructed on a cubic grid with mesh size h, whose solutions separately approximate the solution of the Dirichlet problem for 3D Laplace's equation with the order $O(h^4\rho)$, where $\rho = \rho\,(x_1, x_2, x_3)$ is the distance from the current point $(x_1, x_2, x_3) \in R_h$ to the boundary of the rectangular parallelepiped R, its first derivatives with the order $O(h^4)$, and its second order pure and mixed derivatives with the orders $O(h^{3+\lambda})$, $0 < \lambda < 1$ and $O(h^3)$, respectively. The obtained results can be used to obtain a high approximation of the derivatives of the solution of 3D Laplace's boundary value problems on a prism with an arbitrary polygonal base, and on a polyhedron by developing the combined or composite grid methods [9], [11].

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