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Variations on the Strongly Lacunary Quasi Cauchy Sequences

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Abstract. In this paper,we introduce concepts of a strongly lacunary p-quasi-Cauchy sequence and strongly lacunary p-ward continuity. We prove that a subset of $\mathbb{R}$ is bounded if and only if it is strongly lacunary p-ward compact. It is obtained that any strongly lacunary p-ward continuous function on a subset A of $\mathbb{R}$ is continuous in the ordinary sense. We also prove that the uniform limit of strongly lacunary p-ward continuous functions on a subset A of $\mathbb{R}$ is strongly lacunary p-ward continuous.

Introduction

In this paper, $\mathbb{N}$, and $\mathbb{R}$ will denote the set of positive integers, and the set of real numbers, respectively. p will always be a fixed element of $\mathbb{N}$. Using the idea of sequential continuity of a real function, many kinds of continuity have been introduced and investigated, we recall some of them in the following: ward continuity ([1, 5]), p-ward continuity ([10]), statistical ward continuity ([6]), slowly oscillating continuity ([3, 21]), quasi-slowly oscillating continuity ([14]), upward and downward statistical continuities ([11]), δ^2 lacunary statistical ward continuity ([22]), N_θ-ward continuity ([9, 13, 18]), which enabled some authors to obtain interesting results.

The notion of strongly lacunary convergence (or N_θ convergence) was introduced, and studied by Freedman, Sember, and M. Raphael in [16] in the sense that a sequence (α_k) of points in $\mathbb{R}$ is strongly lacunary convergent or N_θ convergent to an $L \in \mathbb{R}$, which is denoted by $N_\theta - \lim \alpha_k = L$, if $\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} |\alpha_k - L| = 0$, where $I_r = (k_{r-1}, k_r]$, and $k_0 \neq 0$, $h_r := k_r - k_{r-1} \to \infty$ when $r \to \infty$ and $\theta = (k_r)$ is an increasing sequence of positive integers. In the sequel, it is assumed that $\liminf_r \frac{k_r}{k_{r-1}} > 1$. The sums of the form $\sum_{k_{r-1}+1}^{k_r} |\alpha_k|$ frequently occur, and will often be written for convenience as $\sum_{k\in I_r} |\alpha_k|$.

1. Variations on Strongly Lacunary Ward Compactness

A sequence (α_n) is called quasi Cauchy if $\lim_{n\to\infty} \Delta\alpha_n = 0$, where $\Delta\alpha_n = \alpha_{n+1} - \alpha_n$ for each $n \in \mathbb{N}$ ([1, 5, 12, 20, 23]). The set of all bounded quasi-Cauchy sequences is a closed subspace of the space of all bounded sequences with respect to the norm defined for bounded sequences. A sequence (α_k) of points in $\mathbb{R}$ is slowly oscillating if $\lim_{\lambda\to 1^+} \overline{\lim}_n \max_{n+1\leqslant k\leqslant[\lambda n]} |\alpha_k - \alpha_n| = 0$, where $[\lambda n]$ denotes the integer part of λn ([3]). A sequence (α_k) is quasi-slowly oscillating if $(\Delta\alpha_k)$ is slowly oscillating. A sequence (α_n) is called statistically convergent to a real number L if $\lim_{n\to\infty} \frac{1}{n}|\{k \leqslant n : |\alpha_k - L| \geqslant \varepsilon\}| = 0$ for each $\varepsilon > 0$ ([17]).

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A sequence (α_n) is called strongly lacunary quasi Cauchy if $\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}|\Delta\alpha_k| = 0$ ([9]). Recently in [10] it was proved that a real valued function is uniformly continuous on a bounded subset of $\mathbb{R}$ if it is p-ward continuous. Now we introduce the concept of a strongly lacunary p-quasi-Cauchy sequence.

Definition 1.1. A sequence (α_k) of points in $\mathbb{R}$ is called strongly lacunary p-quasi-Cauchy if $N_\theta - \lim_{k\to\infty}\Delta_p\alpha_k = 0$, i.e. $\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}|\Delta_p\alpha_k| = 0$, where $\Delta_p\alpha_k = \alpha_{k+p} - \alpha_k$ for every $k \in \mathbb{N}$.

We denote the set of all strongly lacunary p-quasi-Cauchy sequences by $\Delta_p^{N_\theta}$ for each $p \in \mathbb{N}$. The sum of two strongly lacunary p-quasi-Cauchy sequences is strongly lacunary p-quasi-Cauchy, the product of a strongly lacunary p-quasi-Cauchy sequence with a constant is strongly lacunary p-quasi-Cauchy, so that the set of all strongly lacunary p-quasi-Cauchy sequences $\Delta_p^{N_\theta}$ is a vector subspace of the space of all sequences. We note that a sequence is strongly lacunary quasi-Cauchy when $p = 1$, i.e. strongly lacunary 1-quasi-Cauchy sequences are strongly lacunary quasi-Cauchy sequences. From the following inequality

$$|\alpha_{k+p} - \alpha_k| \leqslant |\alpha_{k+p} - \alpha_{k+p-1}| + |\alpha_{k+p-1} - \alpha_{k+p-2}| + ... + |\alpha_{k+2} - \alpha_{k+1}| + |\alpha_{k+1} - \alpha_k|$$

that any strongly lacunary quasi-Cauchy sequence is also strongly lacunary p-quasi-Cauchy, for any $p \in \mathbb{N}$, but the converse of this fact is not always true as it can be seen by considering the sequence (α_k) defined by $(\alpha_k) = (0, 1, 0, 1, ..., 0, 1, ...)$ is strongly lacunary 2-quasi Cauchy which is not strongly lacunary quasi Cauchy.

Definition 1.2. A subset A of $\mathbb{R}$ is called strongly lacunary p-ward compact if any sequence of points in A has a strongly lacunary p-quasi-Cauchy subsequence.

After this definition, we have following remark:

Remark 1.3. Let A be a subset of $\mathbb{R}$. Then, A is strongly lacunary p-ward compact if and only if $-A := \{-a : a \in A\}$ is strongly lacunary p-ward compact.

We note that this definition of strongly lacunary p-ward compactness cannot be obtained by any summability matrix in the sense of [4].

Since any strongly lacunary quasi-Cauchy sequence is strongly lacunary p-quasi-Cauchy then, it can be said that any strongly lacunary ward compact subset of $\mathbb{R}$ is strongly lacunary p-ward compact for any $p \in \mathbb{N}$. A finite subset of $\mathbb{R}$ is strongly lacunary p-ward compact, the union of finite number of strongly lacunary p-ward compact subsets of $\mathbb{R}$ is strongly lacunary p-ward compact, and the intersection of any number of strongly lacunary p-ward compact subsets of $\mathbb{R}$ is strongly lacunary p-ward compact. Furthermore any subset of a strongly lacunary p-ward compact set of $\mathbb{R}$ is strongly lacunary p-ward compact and any bounded subset of $\mathbb{R}$ is strongly lacunary p-ward compact. From these observations the following result is obtained:

Theorem 1.4. *A subset A of $\mathbb{R}$ is bounded if and only if it is strongly lacunary p-ward compact.*

Proof. Any bounded subset of $\mathbb{R}$ is strongly lacunary p-ward compact, because of a sequence in a bounded set is bounded. Therefore, it has a convergent subsequence which is also strongly lacunary p-quasi Cauchy. To prove the converse (sufficiency part) suppose that A is an unbounded subset of real numbers. Especially assume that A is unbounded above. Then, it can be constructed a sequence $\boldsymbol{\alpha} = (\alpha_n)$ which satisfies

(i) $\boldsymbol{\alpha} = (\alpha_n)$ is strictly increasing,

(ii) For any $p \in \mathbb{N}$, $\alpha_{n+1} \geqslant p + \alpha_n$ holds for all $n \in \mathbb{N}$.

It is clear from (ii) that $\alpha_n \geqslant (n-1)p + \alpha_1$ holds for all $n \in \mathbb{N}$.

Therefore, for any $r \in \mathbb{N}$ we have

$$\sum_{k_{r-1}+1}^{k_r}|\Delta_p\alpha_j| = \sum_{k_{r-1}+1}^{k_r}|\alpha_{j+p} - \alpha_j| \geqslant p^2(k_r - k_{r-1}).$$

So the following inequality

$$\frac{1}{h_r}\sum_{k_{r-1}+1}^{k_r}|\Delta_p\alpha_j| \geqslant p^2$$

holds and this gives that the sequence $\boldsymbol{\alpha} = (\alpha_n)$ is not strongly lacunary p-quasi Cauchy. From the construction of the sequence $\boldsymbol{\alpha} = (\alpha_n)$, it has no strongly lacunary p-quasi Cauchy subsequence.

Now assume that A is unbounded below. $-A$ is unbounded above, and $-A$ is not strongly lacunary p-ward compact by the above proved fact. Hence, from Remark 1.3 given above, A is strongly lacunary p-ward compact. □

It follows from Theorem 1.4 that strongly lacunary p-ward compactness of a subset of A of $\mathbb{R}$ coincides with either of the following kinds of compactness: p-ward compactness ([10, Theorem 2.3]), statistical ward compactness ([6, Lemma 2]), strongly lacunary ward compactness ([9, Theorem 3.3]).

Corollary 1.5. *A subset of* $\mathbb{R}$ *is strongly lacunary p ward compact if and only if:*
(i) A is strongly lacunary q ward compact for any $q \in \mathbb{N}$,
(ii) A is both statistically upward half compact and statistically downward half compact,
(iii) A is both lacunary statistically upward half compact and lacunary statistically downward half compact.

Proof. The proof of (ii), and (iii) follow from [11, Corollary 3.9], and [2, Theorem 1.3 and Theorem 1.9], respectively, whereas the proof of (i) follows from Theorem 1.4 straightforwardly. □

2. Variations on Strongly Lacunary Ward Continuity

In this section, we investigate strongly lacunary p-ward continuity of functions. A function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous if and only if it preserves convergent sequences.

Definition 2.1. A function f is called strongly lacunary p-ward continuous on a subset A of $\mathbb{R}$ if it preserves strongly lacunary p-quasi-Cauchy sequences, i.e. the sequence $(f(\alpha_n))$ is strongly lacunary p-quasi-Cauchy whenever (α_n) is a strongly lacunary p-quasi-Cauchy sequence of points in A.

We see that this definition of strongly lacunary p-ward continuity can not be obtained by any summability matrix A (see [15]).

We note that the sum of two strongly lacunary p-ward continuous functions is strongly lacunary p-ward continuous, and for any constant $c \in \mathbb{R}$, cf is strongly lacunary p-ward continuous whenever f is a strongly lacunary p-ward continuous function, so that the set of all strongly lacunary p ward continuous functions is a vector subspace of the space of all continuous functions. The composite of two strongly lacunary p-ward continuous functions is strongly lacunary p-ward continuous, but the product of two strongly lacunary p-ward continuous functions need not be strongly lacunary p-ward continuous as it can be seen by considering product of the strongly lacunary p-ward continuous function $f(x) = x$ with itself. If f is a strongly lacunary p-ward continuous function, then $|f|$ is also strongly lacunary p-ward continuous since

$$||f(\alpha_{k+p})| - |f(\alpha_k)|| \leqslant |f(\alpha_{k+p}) - f(\alpha_k)|.$$

If f and g are strongly lacunary p-ward continuous, then $max\{f, g\}$ is also strongly lacunary p-ward continuous, which follows from the equality $max\{f, g\} = \frac{1}{2}\{|f - g| + |f + g|\}$. Also, $min\{f, g\}$ is a strongly lacunary p-ward continuous function, if f and g are strongly lacunary p-ward continuous.

In connection with strongly lacunary p-quasi-Cauchy sequences, slowly oscillating sequences, and convergent sequences the problem arises to investigate the following types of continuity of a function on $\mathbb{R}$.

$(\Delta_p^{N_\theta})]$ $(\alpha_n) \in \Delta_p^{N_\theta} \Rightarrow (f(\alpha_n)) \in \Delta_p^{N_\theta}$,
$(\Delta_p^{N_\theta}c)$ $(\alpha_n) \in \Delta_p^{N_\theta} \Rightarrow (f(\alpha_n)) \in c$,
(Δ^{N_θ}) $(\alpha_n) \in \Delta^{N_\theta} \Rightarrow (f(\alpha_n)) \in \Delta^{N_\theta}$,

(c) $(\alpha_n) \in c \Rightarrow (f(\alpha_n)) \in c$,
(d) $(\alpha_n) \in c \Rightarrow (f(\alpha_n)) \in \Delta_p^{N_\theta}$,
(e) $(\alpha_n) \in w \Rightarrow (f(\alpha_n)) \in \Delta_p^{N_\theta}$,
where w denotes the set of slowly oscillating sequences, and $\Delta^{N_\theta} = \Delta_1^{N_\theta}$. Meaning of the symbol $(\Delta_p^{N_\theta})$ and c given above are strongly lacunary p-ward continuity, and the ordinary continuity of f, respectively. It is easy to see that $(\Delta_p^{N_\theta} c)$ implies $(\Delta_p^{N_\theta})$, and $(\Delta_p^{N_\theta})$ does not imply $(\Delta_p^{N_\theta} c)$; $\Delta_p^{N_\theta}$ implies (d), and (d) does not imply $(\Delta_p^{N_\theta})$; and $(\Delta_p^{N_\theta})$ implies (e), and (e) does not imply $(\Delta_p^{N_\theta})$; $(\Delta_p^{N_\theta} c)$ implies (c) and (c) does not imply $(\Delta_p^{N_\theta} c)$; and (c) implies (d).

Theorem 2.2. *If f is strongly lacunary p-ward continuous on a subset A of $\mathbb{R}$, then it is strongly lacunary continuous on A.*

Proof. If $p = 1$, then it is obvious. So we would suppose that $p > 1$. Take any strongly lacunary p-ward continuous function f on A. Let (α_k) be any strongly lacunary convergent sequence of points in A with strongly lacunary limit ℓ. Write

$$(\xi_i) = (\alpha_1, \alpha_1, ..., \alpha_1, , \ell, \ell, ..., \ell, \alpha_2, \alpha_2, ..., \alpha_2, ..., \alpha_n, \alpha_n, ..., \alpha_n, ...),$$

where the same term repeats p times. The sequence

$$(\alpha_1, \alpha_1, ..., \alpha_1, , \ell, \ell, ..., \ell, \alpha_2, \alpha_2, ..., \alpha_2, ..., \alpha_n, \alpha_n, ..., \alpha_n, ...)$$

is also strongly lacunary quasi-Cauchy so it is strongly lacunary p-quasi-Cauchy. By the strongly lacunary p-ward continuity of f, the sequence

$$(f(\alpha_1), f(\alpha_1), ..., f(\alpha_1), f(\ell), f(\ell), ..., f(\ell), f(\alpha_2), f(\alpha_2), ..., f(\alpha_2), , f(\ell), f(\ell), ..., f(\ell), ..., , f(\ell), f(\ell), ..., f(\ell),$$
$$f(\alpha_n), f(\alpha_n), ..., f(\alpha_n), ...)$$

is strongly lacunary p-quasi-Cauchy, where the same term repeats p-times. Thus the sequence

$$(f(\alpha_1), f(\alpha_1), ..., f(\alpha_1), f(\ell), f(\ell), ..., f(\ell), f(\alpha_2), f(\alpha_2), ..., f(\alpha_2), , f(\ell), f(\ell), ..., f(\ell), ..., , f(\ell), f(\ell), ..., f(\ell),$$
$$f(\alpha_n), f(\alpha_n), ..., f(\alpha_n), ...)$$

is also strongly lacunary convergent with strongly lacunary limit $f(\ell)$, which completes the proof of the theorem. □

Corollary 2.3. *If f is strongly lacunary p-ward continuous on a subset A of $\mathbb{R}$, then it is continuous on A in the ordinary case.*

Proof. The proof follows from the previous theorem and [7, Theorem 3] since the sequential method method N_θ is regular and subsequential. □

Theorem 2.4. *Strongly lacunary p-ward continuous image of any strongly lacunary p-ward compact subset of $\mathbb{R}$ is strongly lacunary p-ward compact.*

Proof. Let f be a strongly lacunary p-ward continuous function, and A be a strongly lacunary p-ward compact subset of $\mathbb{R}$. Take any sequence $\boldsymbol{\beta} = (\beta_n)$ of terms in $f(E)$. Write $\beta_n = f(\alpha_n)$ where $\alpha_n \in E$ for each $n \in \mathbb{N}$, $\boldsymbol{\alpha} = (\alpha_n)$. strongly lacunary p-ward compactness of A implies that there is a strongly lacunary p-quasi-Cauchy subsequence $\boldsymbol{\xi} = (\xi_k) = (\alpha_{n_k})$ of $\boldsymbol{\alpha}$. Since f is strongly lacunary p-ward continuous, $(t_k) = f(\boldsymbol{\xi}) = (f(\xi_k))$ is strongly lacunary p-quasi-Cauchy. Thus (t_k) is a strongly lacunary p-quasi-Cauchy subsequence of the sequence $f(\boldsymbol{\alpha})$. This completes the proof of the theorem. □

Corollary 2.5. *Strongly lacunary p-ward continuous image of any G-sequentially connected subset of $\mathbb{R}$ is G-sequentially connected for a regular subsequential method G (see [7]).*

Proof. The proof follows from the preceding theorem, so is omitted (see [8] for the definition of G-sequential connectedness and related concepts). □

Theorem 2.6. *If f is uniformly continuous on a subset A of $\mathbb{R}$, then $(f(\alpha_n))$ is strongly lacunary p-quasi-Cauchy whenever (α_n) is a p-quasi-Cauchy sequence of points in A.*

Proof. Let (α_n) be any p-quasi-Cauchy sequence of points in A. Take any $\varepsilon > 0$. Uniform continuity of f on A implies that there exists a $\delta > 0$, depending on ε, such that $|f(x) - f(y)| < \varepsilon$ whenever $|x - y| < \delta$ and $x, y \in A$. Since (α_n) is p-quasi Cauchy, then there exists an $N = N(\delta) \in \mathbb{N}$ such that $|\Delta_p \alpha_n| < \delta$ whenever $n > N$. From the uniform continuity of f we have $|\Delta_p f(\alpha_n)| < \varepsilon$ if $n > N$. Therefore

$$\frac{1}{h_r} \sum_{k \in I_r} |\Delta_p f(\alpha_k)| < \frac{1}{h_r}(k_r - k_{r-1})\varepsilon = \varepsilon.$$

It follows from this that $(f(\alpha_n))$ is a strongly lacunary p-quasi-Cauchy sequence. This completes the proof of the theorem. □

Corollary 2.7. *If f is slowly oscillating continuous on a bounded subset A of $\mathbb{R}$, then $(f(\alpha_n))$ is strongly lacunary p-quasi-Cauchy whenever (α_n) is a p quasi-Cauchy sequence of points in A.*

Proof. If f is a slowly oscillating continuous function on a bounded subset A of $\mathbb{R}$, then it is uniformly continuous on A by [14, Theorem 2.3]. Hence the proof follows from Theorem 2.6. □

3. Conclusion

In this paper, we introduce strongly lacunary p-ward continuity via strongly lacunary p-quasi Cauchy sequences, and prove results related to this kind of continuity and some other kinds of continuities. The results in this paper not only involves the related results in [9] and [13] as a special case for $p = 1$, but also some interesting results which are also new for the special case $p = 1$. The strongly lacunary p-quasi Cauchy concept for $p > 1$ might find more interesting applications than strongly lacunary quasi Cauchy sequences to the cases when strongly lacunary quasi Cauchy does not apply. For a further study, we suggest to investigate strongly lacunary p-quasi-Cauchy sequences of soft points and strongly lacunary p-quasi-Cauchy sequences of fuzzy points. We also suggest to investigate strongly lacunary p-quasi-Cauchy double sequences of points in $\mathbb{R}$. For another further study, we suggest to investigate strongly lacunary p-quasi-Cauchy sequences in abstract normed spaces (see [19]).

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