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Some New Sequence Spaces Defined by Bounded Variation in 2-Normed Spaces

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Abstract. In this paper, by using Orlicz function and almost lacunary bounded variation we introduce and examine a new sequence space in 2- normed spaces. We also study some basic topological and algebraic properties of these spaces. In addition, we shall established inclusion theorems between these sequence spaces.

1. Introduction

Let w denote the set of all real and complex sequences $x = (x_k)$. By l_∞ and c, we denote the Banach spaces of bounded and convergent sequences $x = (x_k)$ normed by $\|x\| = \sup_k |x_k|$, respectively. A linear functional L on l_∞ is said to be a Banach limit [1] if it has the following properties:

1. $L(x) \geq 0$ if $n \geq 0$ (i.e. $x_n \geq 0$ for all n),
2. $L(e) = 1$ where $e = (1, 1, \ldots)$,
3. $L(Dx) = L(x)$, where D denotes the sift operator on ℓ_∞, that is $D : \ell_\infty \to \ell_\infty$ defined by $D(x) = D(x_n) = \{x_{n+1}\}$.

Let B be the set of all Banach limits on l_∞. A sequence $x \in \ell_\infty$ is said to be almost convergent if all Banach limits of x coincide. Let $\hat{c}$ denote the space of the almost convergent sequences .

It is natural question to expect that almost convergence must be related to some concept $\widehat{BV}$ in the same vein as convergence is related to the concept of BV. BV is denotes the set of all sequences of bounded variation and a sequence in $\widehat{BV}$ will mean a sequence of almost bounded variation. Nanda and Nayak [10] studied this new concept in some detail.

Also a new sequence space $\widehat{\widehat{BV}}$ which is apparently more general than $\widehat{BV}$ naturally comes up for investigation and is considered along with $\widehat{BV}$.

Consider the sequences of bounded linear transformations $d_{mn}(x) : l_\infty \to l_\infty$ defined by

$$d_{mn}(x) = \frac{1}{m+1} \sum_{i=0}^{m} D^i x_n$$

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with $D^0 = 1$. It is evident that

$$d_{0n}(x) = x_n = D^0 x_n \tag{1}$$

Lorentz [4] proved that

$$\hat{c} = \left\{ x : \lim_m d_{mn}(x) \text{ exists uniformly in } n \right\}$$

Now define

$$d_{-1n}(x) = x_{n-1} = D^{-1} x_n \tag{2}$$

and then write for $m, n \geq 0$,

$$t_{mn}(x) = d_{mn}(x) - d_{m-1,n}(x) \tag{3}$$

So that (1.1), (1.2) and (1.3), we write

$$t_{0n}(x) = D^0 x_n - D^{-1} x_n = x_n - x_{n-1}. \tag{4}$$

When $m \geq 1$ a straightforward calculation shows that

$$t_{mn}(x) = \frac{1}{m(m+1)} \sum_{v=1}^{m} v\left(x_{n+v} - x_{n+v-1}\right)$$

Now we write

$$\widehat{BV} = \left\{ x : \sum_m |t_{mn}(x)| \text{ converges uniformly in } n \right\}$$

and

$$\widehat{\widehat{BV}} = \left\{ x : \sup_n \sum_m |t_{mn}(x)| < \infty \right\}.$$

A lacunary sequence is an increasing integer sequence $\theta = \{k_r\}_{r \in \mathbb{N} \cup \{0\}}$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$, as $r \to \infty$. Let $I_r = (k_{r-1}, k_r]$ and $q_r = \frac{k_r}{k_{r-1}}$.

In another direction, a new type of convergence called almost lacunary convergence was introduced as follows by Das and Mishra, [2].

$$M_\theta = \left\{ x : \text{ there exists } L \text{ such that uniformly in } i \geq 0 \lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} \left(x_{k+i} - L\right) = 0 \right\}.$$

Again it is quite natural to think that lacunary almost convergence must be related to some concept $\widehat{BV_\theta}$ in the same view as almost convergence is related to the concept of $\widehat{BV}$. A sequence in $\widehat{BV_\theta}$ will mean a sequence of lacunary almost bounded variation.

Savas and Karakaya [20] studied this new concept in some details.

Put

$$t_{rn} = t_{rn}(x) = \frac{1}{h_r + 1} \sum_{j=k_{r-1}+1}^{k_r+1} x_{j+n}.$$

Then write $r, n > 0$

$$\varphi_{r_n}(x) = t_{rn}(x) - t_{r-1,n}(x).$$

When $r > 1$, straightforward calculation shows that

$$\varphi_{rn}(x) = \varphi_{rn} = \frac{1}{h_r(h_r+1)} \sum_{u=1}^{h_r} u\left(x_{k_{r-1}+u+1+n} - x_{k_{r-1}+u+n}\right).$$

Now we write

$$\widehat{BV}_\theta = \left\{ x : \sum_m \left|\varphi_{rn}(x)\right| \text{ converges uniformly in } n \right\}$$

and

$$\widehat{\widehat{BV}}_\theta = \left\{ x : \sup_n \sum_m \left|\varphi_{rn}(x)\right| < \infty \right\}.$$

The following theorem was proved in [20]:

Theorem 1.1. *$\widehat{BV}_\theta \subset \widehat{\widehat{BV}}_\theta$ for every θ.*

But we do not know whether $\widehat{\widehat{BV}}_\theta \subset \widehat{BV}_\theta$, that is $\widehat{\widehat{BV}}_\theta = \widehat{BV}_\theta$. It is still open problem.

It is quite natural to ask whether $\widehat{BV}_\theta$ can be extended to $\widehat{BV}_\theta(p)$, the set of all sequences of lacunary almost $p-$bounded variation just as $\widehat{BV}$, is extended to $\widehat{BV}(p)$, the set of all sequences of almost $p-$ bounded variation (see [15]).

In [6], the lacunary almost bounded variation sequence spaces to lacunary almost $p-$ bounded variation sequence spaces was generalized as follows:.

$$\widehat{BV}_\theta(p) = \left\{ x : \sum_r \left|\varphi_{rn}(x)\right|^{p_r} \text{ converges uniformly in } n \right\}$$

and

$$\widehat{\widehat{BV}}_\theta(p) = \left\{ x : \sup_n \sum_r \left|\varphi_{rn}(x)\right|^{p_r} < \infty \right\}.$$

where $p = (p_r)$ be a sequence of positive real numbers.

It is clear that $\widehat{BV}_\theta(p) = \widehat{BV}_\theta$ and $\widehat{\widehat{BV}}_\theta(p) = \widehat{\widehat{BV}}_\theta$ if $p_r = 1$ for all $r \in N$. Here and afterwards summation without limits sum from 1 to ∞.

Let X be a linear space. A function $g : X \longrightarrow R$ is called paranorm, if

(P1) $g(x) \geq 0$, for all $x \in X$,

(P2) $g(-x) = g(x)$, for all $x \in X$,

(P3) $g(x+y) \leq g(x) + g(y)$, for all $x, y \in X$,

(P4) if λ_n is a sequence of scalar with $\lambda_n \longrightarrow \lambda\,(n \longrightarrow \infty)$ and (x_n) is a sequence of vector with $g(x_n - x)\,(n \longrightarrow \infty)$ then $g(\lambda_n x_n - \lambda x)\,(n \longrightarrow \infty)$.

A pair (X, g) is called a paranormed space, (see [27]).

Write $M = \max(1, H)$, $H = \sup_r p_r$. For $x \in \widehat{BV}_\theta(p)$ define

$$h(x) = \sup_n \left(\sum_r \left| \varphi_{rn} \right|^{p_r} \right)^{\frac{1}{M}};$$

Also in [6]) the following theorem proved.

Theorem 1.2. *Let $1 \leq p_r < \infty$. The space $\widehat{BV}_\theta(p)$ is a complete linear topological space paranormed by h.*

2. Some Topological Results

In this section we study the local boundedness and s-convexity for $\widehat{B\hat{V}}(p)$. We first quote some definitions (see, for example, Maddox and Roles [9] and Simons [26]).

For this we first quote some definitions:

For $0 < s \leq 1$ a nonvoid subset U of linear space is said to be absolutely $s-$convex if $x, y \in U$ and $|\lambda|^s + |\mu|^s \leq 1$ together imply that $\lambda x + \mu y \in U$. A linear topological space X is said to be $s-$convex if every neighbourhodd of $0 \in X$ contains an absolutely $s-$convex neighbourhood of 0. A subset B of X is said to be bounded if for each neighbourhood U of $0 \in X$, there exists an integer $N > 1$ such that $B \subset NU$. X is called locally bounded if there is a bounded neighbourhood of $0 \in X$.

We have

Theorem 2.1. *$\widehat{\widehat{BV}}_\theta(p)$ is locally bounded if* $\inf p_r > 0$.

Proof. Let $\inf p_r = \gamma > 0$. If $a \in \widehat{\widehat{BV}}_\theta(p)$, then $\exists$ a constant $C > 0$ such that

$$\sum_r \left| \varphi_{rn}(x) \right|^{p_r} \leq C$$

for all n.

For this C and $\delta > 0$, choose an integer $N > 1$ such that

$$N^\gamma > C/\delta.$$

Since $(1/N) < 1$ and $p_r \geq \gamma\,(\forall r)$, we have

$$\frac{1}{N^{p_r}} \leq \frac{1}{N^\gamma}.$$

Therefore, we have for all n,

$$\begin{aligned} \sum_r \left| \varphi_{mn}\left(\frac{x}{N}\right) \right|^{p_r} &= \sum_r \left| \frac{\varphi_{rn}(x)}{N} \right|^{p_r} \\ &\leq (C/N^\gamma) \leq \delta. \end{aligned}$$

Thus

$$\{x : g(x) \leq C\} \subset N\{x : g(x) \leq \delta\}.$$

For every $\delta > 0, \exists N > 1$ such that the above inclusion holds. Therefore $\{x : g(x) \leq C\}$ is bounded and this completes the proof. □

It is known that every locally bounded space is $s-$convex for some s such that $0 < s \leq 1$. But the following theorem gives a sufficient condition for $s-$convexity.

Theorem 2.2. *Let $0 < p_r < 1$. Then $\widehat{\widehat{BV}}_\theta(p)$ is $s-$convex for all s where $0 < s < \liminf p_r$. Further, if $p_r = p\,(\forall r)$, then $\widehat{\widehat{BV}}_\theta(p)$ is $p-$convex.*

Proof. Let $x \in \widehat{\widehat{BV}}_\theta(p)$ and $s \in (0, \liminf p_r)$. Then $\exists r_0$ such that $s < p_r \forall r > r_0$. Define

$$\overline{g}(x) = \sup_n \left[\sum_{r=1}^{r_0} \left|\varphi_{rn}(x)\right|^s + \sum_{r=r_0+1}^{\infty} \left|\varphi_{rn}(x)\right|^{p_r} \right].$$

Since $p_r \leq 1$ and $s < p_r \forall r > r_0$,

$\overline{g}$ is sub-additive and further for $r > r_0$

$$|\lambda|^{p_r} \leq |\lambda|^r \quad (0 < |\lambda| \leq 1)$$

Therefore, for such λ,

$$\overline{g}(\lambda x) \leq |\lambda|^s \overline{g}(x).$$

Now for $0 < \delta < 1, U = \{x : \bar{g}(x) \leq \delta\}$ is an absolutely $s-$convex set, for $|\lambda|^s + \left|\mu\right|^s \leq 1$ and $a, b \in U$ imply that

$$\begin{aligned} \overline{g}(\lambda x + \mu y) &\leq \overline{g}(\lambda x) + \overline{g}(\mu y) \\ &\leq |\lambda|^s \overline{g}(x) + |\lambda|^s \overline{g}(y) \\ &\leq \left(|\lambda|^s + \left|\mu\right|^s\right)\delta \\ &\leq \delta. \end{aligned}$$

Therefore $\lambda x + \mu y \in U$. if $p_r = p\,(\forall r)$, then $V = \{x : g(x) \leq \delta\}$ is an absolutely $p-$ convex set. The proof is similar and omitted. □

3. New Sequence Spaces

In this section we introduce a new sequence spaces by using almost lacunary bounded variation and Orlicz function in 2- normed spaces. Also various algebraic and topological properties and certain inclusion relations involving this space have been discussed.

Before continuing with this section we present some definitions and preliminaries.

Recall in [7] that an Orlicz function $M : [0, \infty) \to [0, \infty)$ is continuous, convex, non-decreasing function such that $M(0) = 0$ and $M(x) > 0$ for $x > 0$, and $M(x) \to \infty$ as $x \to \infty$. If convexity of Orlicz function is replaced by $M(x + y) \leq M(x) + M(y)$ then this function is called the modulus function and characterized by Ruckle [13]. An Orlicz function M is said to satisfy Δ_2-condition for all values of u, if there exists $K > 0$ such that $M(2u) \leq KM(u), u \geq 0$.

W. Orlicz [11] used the idea of Orlicz function to construct the space $\left(L^M\right)$. Lindentrauss and Tzafriri [8] use the idea of Orlicz function and defined the sequence space ℓ_M such as

$$\ell_M = \left\{ x = (x_i) : \sum_{i=1}^{\infty} M\left(\frac{|x_i|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\}$$

The space ℓ_M with the norm $\|x\| = \inf\left\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1\right\}$ becomes a Banach space which is called an Orlicz sequence space. The space ℓ_M is closely related to the sequence space ℓ_p, which is an Orlicz sequence space with $M(x) = x^p$ for $1 \leq p \leq \infty$.

Note that an Orlicz function satisfies the inequality

$$M(\lambda x) = \lambda M(x) \text{ for all } \lambda \text{ with } 0 < \lambda < 1.$$

In the later stage different classes of Orlicz sequence spaces were introduced and studied by Parashar and Choudhary [12], Savas [16, 17, 21–25], and many others.

The following well-known inequality will be used throughout the article. Let $p = (p_k)$ be any sequence of positive real numbers with $0 \leq p_k \leq \sup_k p_k = H, C = \max\{1, 2^{H-1}\}$ then

$$|a_k + b_k|^{p_k} \leq C(|a_k|^{p_k} + |b_k|^{p_k}) \tag{5}$$

for all $k \in \mathbf{N}$ and $a_k, b_k \in \mathbf{C}$. Also

$$|a_k|^{p_k} \leq \max\{1, |a|^H\} \tag{6}$$

for all $a \in \mathbf{C}$.

Definition 3.1. ([3]) Let X be a real vector space of dimension d, where $2 \leq d < \infty$. A 2-norm in on X is a function $\|.,.\| : X \times X \to \mathbb{R}$ which satisfies (i) $\|x, y\|=0$ if and only if x and y are linearly independent; (ii) $\|x, y\| = \|y, x\|$; (iii) $\|\alpha x, y\| = |\alpha|\|x, y\|$, $\alpha \in \mathbb{R}$; (iv) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$. The ordered pair $(X, \|.,.\|)$ is then called a 2-normed space.

As an example we may take $X = \mathbb{R}^2$ being equipped with the 2-norm $\|x, y\|$= the area of the parallelogram spanned by the vectors x and y, which may be given explicitly by the formula $\|x, y\| = |x_1 y_2 - x_2 y_1|$, $x = (x_1, x_2)$, $y = (y_1, y_2)$. Recall that $(X, \|.,.\|)$ is a 2-Banach space if every Cauchy sequence in X is convergent to some x in X. Let $(X, \|.,.\|)$ be any 2-normed space.
More recent work in this line can be found in [14, 18, 19] where many references can be found.

Let E be a sequence space. Then E is called solid (or normal), if $(\alpha_n x_n) \in E$, whenever $(x_n) \in E$ for all sequences of scalar (α_n) with $|\alpha_n| \leq 1$ for all $k \in \mathbb{N}$.

Lemma 3.2. ([5]) *A sequence space E is solid implies E is monotone.*

Let M be an Orlicz function, $(X, \|.,.\|)$ be a 2-normed space and $p = (p_r)$ be any sequence of strictly positive real numbers. Now we define the following sequence spaces,

$$\widehat{BV}_\theta (M, p, \|.,.\|) = \{x = (x_k) : \sum_{r=1}^{\infty} \left[M\left(\left\| \frac{\phi_{r,n}(x)}{\rho}, z \right\| \right) \right]^{p_r} < \infty \text{ uniformly in } n$$
$$\text{and for some } \rho > 0, \text{ and each } z \in X\}$$

For $M(x) = x$, we get (see [6])

$$\widehat{BV}_\theta(p, \|.,.\|) = \left\{ x = (x_k) : \sum_{r=1}^{\infty} \left(\left\| \phi_{r,n}(x), z \right\| \right)^{p_r} < \infty \text{ converges uniformly in } n, \text{ and each } z \in X \right\}.$$

For $p_r = 1$ for all r, we get

$$\widehat{BV}_\theta (M, \|.,.\|) = \{x = (x_k) : \sum_{r=1}^{\infty} \left[M\left(\left\| \frac{\phi_{r,n}(x)}{\rho}, z \right\| \right) \right] < \infty \text{ uniformly in } n$$
$$\text{and for some } \rho > 0, \text{ and each } z \in X\}.$$

For $M(x) = x$, and $p_r = 1$ for all r, we get

$$\widehat{BV}_\theta(\|.,.\|) = \left\{ x = (x_k) : \sum_{r=1}^{\infty} \left\| \phi_{r,n}(x), z \right\| < \infty \text{ uniformly in } n, \text{ and each } z \in X \right\}.$$

Theorem 3.3. *The sequence space $B\hat{V}_\theta (M, p, \|.,.\|)$ is a linear space over the field $\mathbb{C}$ of complex numbers.*

Proof. Let $x, y \in \widehat{BV}_\theta(M, p, \|., .\|)$ and $\alpha, \beta \in \mathbb{C}$. Then there exist positive numbers ρ_1 and ρ_2 such that

$$\sum_{r=1}^{\infty}\left[M\left(\left\|\frac{\phi_{r,n}(x)}{\rho_1}, z\right\|\right)\right]^{p_r} < \infty$$

and

$$\sum_{r=1}^{\infty}\left[M\left(\left\|\frac{\phi_{r,n}(x)}{\rho_2}, z\right\|\right)\right]^{p_r} < \infty$$

uniformly in n. Define $\rho_3 = \max\left(2|\alpha|\rho_1, 2\left|\beta\right|\rho_2\right)$. Since M is non-decreasing and convex we have

$$\begin{aligned}
&\sum_{r=1}^{\infty}\left[M\left(\left\|\frac{\alpha\phi_{r,n}(x)+\beta\phi_{r,n}(y)}{\rho_3}, z\right\|\right)\right]^{p_r}\\
\leq\; &\sum_{r=1}^{\infty}\left[M\left(\left\|\frac{\alpha\phi_{r,n}(x)}{\rho_3}, z\right\| + \left\|\frac{\beta\phi_{r,n}(y)}{\rho_3}, z\right\|\right)\right]^{p_r}\\
\leq\; &\sum_{r=1}^{\infty}\frac{1}{2}\left[M\left(\left\|\frac{\alpha\phi_{r,n}(x)}{\rho_1}, z\right\|\right)\right]^{p_r} + \left[M\left(\left\|\frac{\beta\phi_{r,n}(y)}{\rho_2}, z\right\|\right)\right]^{p_r} < \infty
\end{aligned}$$

uniformly in n.This proves that $\widehat{BV}_\theta(M, p, \|., .\|)$ is linear space over the field $\mathbb{C}$ of complex numbers. □

Theorem 3.4. *For any Orlicz function M and a bounded sequence $p = (p_r)$ of strictly positive real numbers, $\widehat{BV}_\theta(M, p, \|., .\|)$ is a paranormed space with*

$$g(x) = \inf_{n\geq 1}\left\{\rho^{p_n/k} : \left(\sum_{r=1}^{\infty}\left[M\left(\left\|\frac{\phi_{r,n}(x)}{\rho}, z\right\|\right)\right]^{p_r}\right)^{1/K} \leq 1, \textit{uniformly in } n\right\},$$

where $K = \max(1, \sup p_r)$.

Proof. It is clear that $g(x) = g(-x)$. Since $M(0) = 0$, we get

$$\inf\left\{\rho^{p_n/K}\right\} = 0 \text{ for } x = 0$$

By using Theorem 3.3, for $\alpha = \beta = 1$, we get

$$g(x+y) \leq g(x) + g(y)$$

For continuity of scalar multiplication let $\lambda \neq 0$ be any complex number. Then by definition we have

$$g(\lambda x) = \inf_{n\geq 1}\left\{\rho^{p_n/k} : \left(\sum_{r=1}^{\infty}\left[M\left(\left\|\frac{\phi_{r,n}(\lambda x)}{\rho}, z\right\|\right)\right]^{p_r}\right)^{1/K} \leq 1, \text{uniformly in } n\right\}$$

$$g(\lambda x) = \inf_{n\geq 1}\left\{(s|\lambda|)^{\rho^{p_n/k}} : \left(\sum_{r=1}^{\infty}\left[M\left(\left\|\frac{\phi_{r,n}(\lambda x)}{(s|\lambda|)}, z\right\|\right)\right]^{p_r}\right)^{1/K} \leq 1, \text{ uniformly in } n\right\},$$

where $s = \frac{\rho}{|\lambda|}$ Since $|\lambda|^{p_n} \leq \max\left(1, |\lambda|^H\right)$, We have

$$\begin{aligned} g(\lambda x) &\leq \max\left(1,|\lambda|^H\right)\inf_{n\geq 1}\left\{s^{p_n/k}:\left(\sum_{r=1}^{\infty}\left[M\left(\left\|\frac{\phi_{r,n}(x)}{s},z\right\|\right)\right]^{p_r}\right)^{1/K}\leq 1\right\} \\ &= \max\left(1,|\lambda|^H\right)g(x) \end{aligned}$$

and therefore $g(\lambda x)$ converges to zero when $g(x)$ converges to zero in $\widehat{BV}_\theta(M,p,\|,.,\|)$. Now let x be fixed element in $\widehat{BV}_\theta(M,p,\|,.,\|)$. Then there exists $\rho > 0$ such that

$$g(x) = \inf_{n\geq 1}\left\{\rho^{p_n/k}:\left(\sum_{r=1}^{\infty}\left[M\left(\left\|\frac{\phi_{r,n}(x)}{\rho},z\right\|\right)\right]^{p_r}\right)^{1/K}\leq 1, \text{uniformly in } n\right\}.$$

Now

$$g(\lambda x) = \inf_{n\geq 1}\left\{\rho^{p_n/k}:\left(\sum_{r=1}^{\infty}\left[M\left(\left\|\frac{\phi_{r,n}(\lambda x)}{\rho},z\right\|\right)\right]^{p_r}\right)^{1/K}\leq 1, \text{ uniformly in } n\right\}\to 0 \text{ as } \lambda\to 0.$$

This completes the proof. □

Theorem 3.5. *Let M, M_1, M_2 be Orlicz functions. Then*

i) *If there is a positive constant β such that $M(t)\leq \beta t$ for all $t\geq 0$, then $\widehat{BV}_\theta(M_1,p,\|,.,\|)\subseteq \widehat{BV}_\theta(M\circ M_1,p,\|,.,\|)$,*

ii) $\widehat{BV}_\theta(M_1,p)\cap \widehat{BV}_\theta(M_2,p,\|,.,\|)\subseteq \widehat{BV}_\theta(M_1+M_2,p,\|,.,\|)$,

iii) *If $\limsup_t \frac{M_1(t)}{M_2(t)} < \infty$, then $\widehat{BV}_\theta(M_2,p,\|,.,\|)\subseteq \widehat{BV}_\theta(M_1,p,\|,.,\|)$.*

Proof. (i) Let $x\in \widehat{BV}_\theta(M_1,p,\|,.,\|)$ so that $\sum_{r=1}^{\infty}\left[M_1\left(\left\|\frac{\phi_{r,n}(x)}{\rho},z\right\|\right)\right]^{p_r} < \infty$ converges uniformly in n and for some $\rho > 0$. Since $M(t)\leq \beta t$ for all $t\geq 0$, we write

$$\sum_{r=1}^{\infty}[M(y_r)]^{p_r}\leq \max\left(1,\beta^H\right)\sum_{r=1}^{\infty}(y_r)^{p_r},$$

where

$$y_r = M_1\left(\left\|\frac{\phi_{r,n}(x)}{\rho},z\right\|\right)$$

and hence $x\in \widehat{BV}_\theta(M\circ M_1,p,\|,.,\|)$.

(ii) The proof is obvious and hence is omitted.

(iii) We can find $K > 0$ such that $\frac{M_1(t)}{M_2(t)}\leq K$ for all $t\geq 0$, since $\limsup_{t\longrightarrow\infty}\frac{M_1(t)}{M_2(t)} < \infty$. For $x\in \widehat{BV}_\theta(M_2,p,\|,.,\|)$, we write

$$\sum_{r=1}^{\infty}\left[M_1\left(\left\|\frac{\phi_{r,n}(x)}{\rho},z\right\|\right)\right]^{p_r}\leq \max\left(1,K^H\right)\sum_{r=1}^{\infty}\left[M_2\left(\left\|\frac{\phi_{r,n}(x)}{\rho},z\right\|\right)\right]^{p_r}$$

by (5) whence $x\in \widehat{BV}_\theta(M_1,p,\|,.,\|)$. □

Theorem 3.6. *For Orlicz function M, if* $\lim_u \frac{M(u/\rho)}{(u/\rho)} > 0$ *for some* $\rho > 0$ *then* $\widehat{BV}_\theta(M, p, \|, ., \|) \subseteq \widehat{BV}_\theta(p, \|, ., \|)$.

Proof. If $\lim_u \frac{M(u/\rho)}{(u/\rho)} > 0$ then there exists a number $\alpha > 0$ such that $M(u/\rho) \geq \alpha(u/\rho)$ for all $u > 0$ and some $\rho > 0$. Let $x \in B\hat{V}_\theta(M, p, \|, ., \|)$ so that $\sum_{r=1}^{\infty} \left[M\left(\left\|\frac{\phi_{r,n}(x)}{\rho}, z\right\|\right)\right]^{p_r} < \infty$ for some $\rho > 0$. Then

$$\sum_{r=1}^{\infty} \left[M\left(\left\|\frac{\phi_{r,n}(x)}{\rho}, z\right\|\right)\right]^{p_r} \geq \max\left(1, \left(\frac{\alpha}{\rho}\right)^H\right) \sum_{r=1}^{\infty} \left[\left(\left\|\phi_{r,n}(x), z\right\|\right)\right]^{p_r}.$$

Hence $x \in \widehat{BV}_\theta(p, \|, ., \|)$. □

Theorem 3.7. *If* $p = (p_r)$ *and* $t = (t_r)$ *are bounded sequences of positive real numbers with* $0 < p_r \leq t_r < \infty$ *for each* $r \in \mathbb{N}$, *then for any Orlicz function* M

$$\widehat{BV}_\theta(M, p, \|, ., \|) \subseteq \widehat{BV}_\theta(M, t, \|, ., \|).$$

Proof. Suppose that $x \in \widehat{BV}_\theta(M, p, \|, ., \|)$. Then there exists some $\rho > 0$ such that $\sum_{r=1}^{\infty} \left[M\left(\left\|\frac{\phi_{r,n}(x)}{\rho}, z\right\|\right)\right]^{p_r} < \infty$. This implies that $\left[M\left(\left\|\frac{\phi_{i,n}(x)}{\rho}, z\right\|\right)\right]^{p_r} \leq 1$ for sufficiently large values of i, say that $i \geq r_0$ for some fixed $r_0 \in \mathbb{N}$. Since M is non-decreasing, we have

$$\sum_{r=r_0}^{\infty} \left[M\left(\left\|\frac{\phi_{r,n}(x)}{\rho}, z\right\|\right)\right]^{t_r} \leq \sum_{r=r_0}^{\infty} \left[M\left(\left\|\frac{\phi_{r,n}(x)}{\rho}, z\right\|\right)\right]^{p_r} < \infty.$$

Hence $x \in \widehat{BV}_\theta(M, t, \|, ., \|)$. □

Theorem 3.8. *The sequence space* $B\hat{V}_\theta(M, p, \|, ., \|)$ *is solid.*

Proof. Let $x \in \widehat{BV}_\theta(M, p, \|, ., \|)$. This implies that

$$\sum_{r=1}^{\infty} \left[M\left(\left\|\frac{\phi_{r,n}(x)}{\rho}, z\right\|\right)\right]^{p_r} < \infty.$$

Let $\alpha = (\alpha_r)$ be sequence of scalars such that $|\alpha_r| \leq 1$ for all $r \in \mathbb{N}$. Then the result follows from the following inequality

$$\sum_{r=1}^{\infty} \left[M\left(\left\|\frac{\alpha_r \phi_{r,n}(x)}{\rho} z\right\|\right)\right]^{p_r} \leq \sum_{r=1}^{\infty} \left[M\left(\left\|\frac{\phi_{r,n}(x)}{\rho}, z\right\|\right)\right]^{p_r} < \infty.$$

Hence $\alpha x \in B\hat{V}_\theta(M, p, \|, ., \|)$ for all sequence of scalar (α_r) with $|\alpha_r| \leq 1$ for all $r \in \mathbb{N}$, whenever $x \in \widehat{BV}_\theta(M, t, \|, ., \|)$. □

From Theorem 3.8 and Lemma we have:

Corollary 3.9. *The sequence space* $\widehat{BV}_\theta(M, p, \|, ., \|)$ *is monotone.*

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