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Positive Solutions of Impulsive Time-Scale Boundary Value Problems with p-Laplacian on the Half-Line

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Abstract. In this paper, four functionals fixed point theorem is used to investigate the existence of positive solutions for second-order time-scale boundary value problem of impulsive dynamic equations on the half-line.

1. Introduction

Many evolution processes are characterized by the fact that at certain moments of time they experience a change of state abruptly. These processes are subject to short-term perturbations whose duration is negligible in comparison with the duration of the process. Consequently, it is natural to assume that these perturbations act instantaneously, that is, in the form of impulses. The branch of modern applied analysis known as impulsive differential equations provides a natural framework to mathematically describe the aforementioned jumping processes. The theory of impulsive differential equation has become important in recent years in mathematical model of real processes rising in phenomena studied in physics, chemical technology, population dynamics, biotechnology and economics. For the introduction of the basic theory of impulsive equations see [4, 5, 21, 31]. In the last few years boundary value problems for impulsive differential equations and impulsive differences equations have received much attention ([2, 9, 16, 17, 22–25, 27–30, 32]). On the other hand, recently the theory of dynamic equations on time scales has become a new important branch [1, 6, 7, 14, 18, 20, 33].

The theory of dynamic systems on time scales goes back to its founder Hilger [18] and is undergoing repeat development as it provides a unifying structure for the study of differential equations in the continuous case and the study of finite difference equations in the discrete case. We refer to the books by Bohner and Peterson [6, 7] and Lakshmikantham et al. [20].

Boundary value problems on infinite intervals appear often in applied mathematics and physics. There are many papers concerning the existence of solutions on the half-line for the boundary value problem (see [8, 10–13, 15, 19, 26, 35, 36]). Due to the fact that an infinite interval is noncompact, the discussion about boundary value problem on the half-line is more complicated, in particular, for p- Laplacian impulsive boundary value problem on infinite intervals, few works were done, see [15]. Especially, the corresponding theory for m-point impulsive boundary value problem on infinite interval on time scale is not investigated till now. Hence, these results can be considered as a contribution to this field.

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Guo, Yu, Wang [13] studied the existence of positive solutions for m-point boundary value problem

$$\begin{cases} [\varphi_p(x'(t))]' + \phi(t)f(t,x(t),x'(t)) = 0, \quad 0 < t < \infty, \\ x(0) = \displaystyle\sum_{i=1}^{m-2} \alpha_i x'(\eta_i), \quad \lim_{t\to+\infty} x'(t) = 0, \end{cases}$$

where $\varphi_p(s) = |s|^{p-2}s$, $p > 1$, $\phi : [0,\infty) \to [0,\infty)$, $0 < \eta_1 < \eta_2 < \cdots < \eta_{m-2} < \infty$, and $\alpha_i \geq 0$, for $i = 1, 2, \cdots, m-2$. They established the existence of three positive solutions by using Avery-Peterson fixed point theorem.

Zhao, Ge [34] considered the second-order boundary value problem on the half line

$$\begin{cases} [\varphi_p(u^\triangle(t))]^\nabla + h(t)f(t,u(t),u^\triangle(t)) = 0, \quad t \in (0,\infty), \\ u(0) = \displaystyle\sum_{i=1}^{m-2} \alpha_i u(\eta_i), \quad u^\triangle(\infty) = \sum_{i=1}^{m-2} \beta_i u^\triangle(\eta_i), \end{cases}$$

where $u^\triangle(\infty) = \lim_{t\in\mathbb{T},t\to\infty} u^\triangle(t)$, $\varphi_p(s) = |s|^{p-2}s$, $p > 1$, $\eta_1, \eta_2, \cdots, \eta_{m-2} \in \mathbb{T}$, $\sigma(0) < \eta_1 < \eta_2 < \cdots < \eta_{m-2} < \infty$, $\alpha_i \geq 0$, $\beta_i \geq 0$, for $i = 1, 2, \cdots, m-2$. They established the sufficient conditions for the existence of positive solutions by using Avery-Peterson theorem.

Zhang, Yang, Feng [37] investigated the minimal nonnegative solution of nonlinear impulsive differential equation

$$\begin{cases} -x'' = f(t,x(t),x'(t)), \quad t \in J, \quad t \neq t_k, \\ \Delta x|_{t=t_k} = I_k(x(t_k)), \quad k \in \mathbb{N}, \\ \Delta x'|_{t=t_k} = \bar{I}_k(x(t_k)), \quad k \in \mathbb{N}, \\ x(0) = \displaystyle\int_0^\infty g(t)x(t)dt, \quad x'(\infty) = 0, \end{cases}$$

where $I_k \in C[\mathbb{R}^+,\mathbb{R}^+]$, $\bar{I}_k \in C[\mathbb{R}^+,\mathbb{R}^+]$, $\mathbb{R}^+ = [0,\infty)$, $g \in C(\mathbb{R}^+,\mathbb{R}^+)$, with $\int_0^\infty g(t)dt < 1$. They established the existence of positive solution by using monoton iterative technique.

Motivated by these results mentioned above, in this paper, we consider the existence of positive solution for m-point impulsive boundary value problem (IBVP) on time scales

$$\begin{cases} [\varphi_p(u^\triangle(t))]^\nabla + h(t)f(t,u(t),u^\triangle(t)) = 0, \quad t \in (0,\infty), \quad t \neq t_k, \\ u(0) = \displaystyle\sum_{i=1}^{m-2} \alpha_i u^\triangle(\eta_i), \quad u^\triangle(\infty) = \sum_{i=1}^{m-2} \beta_i u(\eta_i), \\ u(t_k^+) - u(t_k^-) = I_k(u(t_k)), \quad \varphi_p(u^\triangle(t_k^+)) - \varphi_p(u^\triangle(t_k^-)) = -\bar{I}_k(u(t_k)), \quad k \in \mathbb{N}, \end{cases} \tag{1.1}$$

where $\mathbb{T}$ is a time scale, φ_p is the p-Laplacian operator, i.e., $\varphi_p(s) = |s|^{p-2}s$, $p > 1$, $(\varphi_p)^{-1} = \varphi_q$, $\frac{1}{p} + \frac{1}{q} = 1$, $I_k \in C[[0,\infty),[0,\infty)]$, $\bar{I}_k \in C[[0,\infty),[0,\infty)]$, $\eta_1, \eta_2, \cdots, \eta_{m-2} \in \mathbb{T}$, $\sigma(0) < \eta_1 < \eta_2 < \cdots < \eta_{m-2} < \infty$.

Now we list some conditions in this section for convenience.

Let $F(t,u,v) = f(t,(1+t^2)u,(1+t)v)$.

$(H1)$ $\sum_{i=1}^{m-2}\alpha_i < 1,\ \sum_{i=1}^{m-2}\beta_i < 1,\ \max\{1,\ 2^{q-2}\}\sum_{j=1}^{m-2}\beta_j\left(\sum_{i=1}^{m-2}\alpha_i+\eta_j\right) < 1,\ \alpha_i \geq 0,\ \beta_i \geq 0,\ \eta_i \geq 0,\ i = 1,\ 2,\ 3,\cdots,m-2;$

$(H2)$ $h \in C_{ld}([0,\infty),[0,\infty])$ and $\int_0^{\infty} h(r)\nabla r < \infty;$

$(H3)$ $F : [0,\infty)\times[0,\infty)\times[0,\infty) \to [0,\infty);$

$(H4)$ For $t \in [0,\infty)$, $F(t,\cdot,\cdot)$ is continuous;

$(H5)$ For $u, v \in [0,\infty)$, $F(\cdot,u,v)$ is ld-continuous on $\mathbb{T}$;

$(H6)$ When u, v are bounded, $F(t,u,v)$ is bounded;

$(H7)$ $\sum_{k=1}^{\infty}\bar{I}_k(u(t_k)) < \infty.$

In this work, we establish the existence of at least one positive solution for the IBVP (1.1). In fact, our result is new when $\mathbb{T} = \mathbb{R}$ (the differential case) and $\mathbb{T} = \mathbb{Z}$.

The main tool in our approach is the following four functionals fixed point theorem in cone.

Theorem 1.1. ([3]) *If P is a cone in a real Banach space $\mathbb{B}$, α and ψ are nonnegative continuous concave functionals on P, β and ϕ are nonnegative continuous convex functionals on P and there exist positive numbers r, j, l, R, such that*

$$T : Q(\alpha,\beta,r,R) \to P$$

is a completely continuous operator, and $Q(\alpha,\beta,r,R)$ is a bounded set. If

(i) $\{u \in U(\psi, j) : \beta(u) < R\} \cap \{u \in V(\phi,l) : r < \alpha(u)\} \neq \emptyset;$

(ii) $\alpha(T(u)) \geq r$*, for all* $u \in Q(\alpha,\beta,r,R)$*, with* $\alpha(u) = r$ *and* $l < \phi(Tu)$*;*

(iii) $\alpha(T(u)) \geq r$*, for all* $u \in V(\phi,l)$*, with* $\alpha(u) = r$*;*

(iv) $\beta(T(u)) \leq R$*, for all* $u \in Q(\alpha,\beta,r,R)$*, with* $\beta(u) = R$ *and* $\psi(T(u)) < j$*;*

(v) $\beta(T(u)) \leq R$*, for all* $u \in U(\psi, j)$*, with* $\beta(u) = R$*.*

then T has a fixed point u in $Q(\alpha,\beta,r,R)$.

This paper is organized as follows. In Section 2, we provide some definitions and preliminary lemmas which are key tools for our main result. We give and prove our main result in Section 3. Finally, in Section 4, we give an example where the main result can be applied.

2. Preliminaries

In this section, to state the main result of this paper, we need the following lemmas.

Throughout the rest of the paper, we assume that the points of impulses t_k are right dense for $k \in \mathbb{N}$.

Let $J = [0,\infty)$, $J' = J \setminus \{t_1,t_2,t_3,\cdots\}$.

Set

$PC(J) = \Big\{u : [0,\infty) \to \mathbb{R} :\ u \in C(J'),\ u(t_k^+)$ and $u(t_k^-)$ exist, and $u(t_k^-) = u(t_k),\ 1 \leq k < \infty\Big\},$

$$PC^1(J) = \Big\{u \in PC(J):\ \ u^{\triangle} \in C(J'),\ u^{\triangle}(t_k^+) \text{ and } u^{\triangle}(t_k^-) \text{ exist, and } u^{\triangle}(t_k^-) = u^{\triangle}(t_k),\ 1 \le k < \infty\Big\}.$$

Obviously, $PC^1(J)$ is Banach space with the norm $\|u\| = \max\Big\{\|u\|_1, \|u\|_2\Big\}$, where

$$\|u\|_1 = \sup_{t\in[0,\infty)} \left|\frac{u(t)}{1+t^2}\right|, \quad \|u\|_2 = \sup_{t\in[0,\infty)} \left|\frac{u^{\triangle}(t)}{1+t}\right|.$$

A function $u \in PC^1(J) \cap C^2(J')$ is called a solution to (1.1) if it satisfies all equations of (1.1).

Define the cone $P \subset PC^1(J)$ by

$$P = \Big\{u \in PC^1(J) : u(0) = \sum_{i=1}^{m-2} \alpha_i u^{\triangle}(\eta_i),\ \ u \text{ is nonnegative, nondecreasing on } [0,\infty) \text{ and } u^{\triangle} \text{ is nonincreasing on } [0,\infty)\Big\}.$$

Let the nonnegative continuous concave functionals $\alpha,\ \psi$ and the nonnegative continuous convex functionals $\beta,\ \phi$ be defined on the cone P by

$$\psi(u) = \min_{t\in[0,\xi]} \frac{u(t)}{t}, \qquad \beta(u) = \sup_{t\in[0,\infty)} \frac{u^{\triangle}(t)}{1+t},$$

$$\phi(u) = \sup_{t\in[0,\infty)} \frac{u(t)}{1+t^2},$$

$$\alpha(u) = \frac{\omega^2}{\omega^4+1} \min_{t\in[\frac{1}{\omega^2},\omega^2]} u(t) = \frac{\omega^2}{\omega^4+1} u\left(\frac{1}{\omega^2}\right)$$

for $u \in P$, where $\dfrac{1}{\omega^2} \in \mathbb{T}$, $\dfrac{1}{\omega^2} < \sigma(\eta_1)$ and $\omega^2 > \max\{1, \eta_{m-2}\}$.

Lemma 2.1. *Let* $(H1) - (H2)$ *hold. Then* $u \in PC^1(J) \cap C^2(J')$ *is a solution of the IBVP* (1.1) *if and only if u is a solution of the following integral equation*

$$u(t) = \sum_{i=1}^{m-2} \alpha_i \varphi_q\Big(A_u + \int_{\eta_i}^{\infty} h(\tau) f(\tau, u(\tau), u^{\triangle}(\tau))\nabla\tau + \sum_{\eta_i<t_k} \bar{I}_k(u(t_k))\Big) + \int_0^t \varphi_q\Big(A_u + \int_s^{\infty} h(\tau) f(\tau, u(\tau), u^{\triangle}(\tau))\nabla\tau + \sum_{s<t_k} \bar{I}_k(u(t_k))\Big)\triangle s + \sum_{t_k<t} I_k(u(t_k)),$$

where

$$\varphi_q(A_u) = \sum_{j=1}^{m-2} \beta_j \Big\{ \sum_{i=1}^{m-2} \alpha_i \varphi_q\Big(A_u + \int_{\eta_i}^{\infty} h(\tau) f(\tau, u(\tau), u^{\triangle}(\tau))\nabla\tau + \sum_{\eta_j<t_k} \bar{I}_k(u(t_k))\Big) + \int_0^{\eta_j} \varphi_q\Big(A_u + \int_s^{\infty} h(\tau) f(\tau, u(\tau), u^{\triangle}(\tau))\nabla\tau + \sum_{s<t_k} \bar{I}_k(u(t_k))\Big)\triangle s + \sum_{t_k<\eta_j} I_k(u(t_k))\Big\}. \tag{2.1}$$

Proof. First, suppose that $u \in PC^1(J) \cap C^2(J')$ is a solution of problem (1.1). It is easy to see by integration of (1.1) that

$$\varphi_p(u^\triangle(t)) = \varphi_p(u^\triangle(0)) - \int_0^t h(\tau)f(\tau, u(\tau), u^\triangle(\tau))\nabla\tau - \sum_{t_k<t} \bar{I}_k(u(t_k)).$$

Taking limit for $t \to \infty$, we get

$$\varphi_p(u^\triangle(\infty)) = \varphi_p(u^\triangle(0)) - \int_0^\infty h(\tau)f(\tau, u(\tau), u^\triangle(\tau))\nabla\tau - \sum_{k=1}^{\infty} \bar{I}_k(u(t_k)).$$

If we choose $A_u = \varphi_p(u^\triangle(\infty))$, we have

$$u^\triangle(t) = \varphi_q\Big(A_u + \int_0^\infty h(\tau)f(\tau, u(\tau), u^\triangle(\tau))\nabla\tau - \int_0^t h(\tau)f(\tau, u(\tau), u^\triangle(\tau))\nabla\tau + \sum_{t<t_k} \bar{I}_k(u(t_k))\Big).$$

Integrating above, we get

$$u(t) = \sum_{i=1}^{m-2} \alpha_i\varphi_q\Big(A_u + \int_{\eta_i}^\infty h(\tau)f(\tau, u(\tau), u^\triangle(\tau))\nabla\tau + \sum_{\eta_i<t_k} \bar{I}_k(u(t_k))\Big) + \int_0^t \varphi_q\Big(A_u + \int_s^\infty h(\tau)f(\tau, u(\tau), u^\triangle(\tau))\nabla\tau$$
$$+ \sum_{s<t_k} \bar{I}_k(u(t_k))\Big)\triangle s + \sum_{t_k<t} I_k(u(t_k)).$$

Conversely, if u is a solution of (1.1), direct differentiation of (1.1) implies, for $t = t_k$,

$$u^\triangle(t) = \varphi_q\Big(A_u + \int_0^\infty h(\tau)f(\tau, u(\tau), u^\triangle(\tau))\nabla\tau - \int_0^t h(\tau)f(\tau, u(\tau), u^\triangle(\tau))\nabla\tau + \sum_{t<t_k} \bar{I}_k(u(t_k))\Big).$$

Obviously,

$$\begin{cases} [\varphi_p(u^\triangle(t))]^\nabla + h(t)f(t, u(t), u^\triangle(t)) = 0, \quad t \in (0,\infty), \quad t \neq t_k, \\ u(0) = \sum_{i=1}^{m-2} \alpha_i u^\triangle(\eta_i), \quad u^\triangle(\infty) = \sum_{i=1}^{m-2} \beta_i u(\eta_i), \\ u(t_k^+) - u(t_k^-) = I_k(u(t_k)), \quad \varphi_p(u^\triangle(t_k^+)) - \varphi_p(u^\triangle(t_k^-)) = -\bar{I}_k(u(t_k)), \quad k \in \mathbb{N}. \end{cases}$$

The proof is complete. □

Lemma 2.2. *If* $(H1) - (H3)$ *hold, then for* $u \in C_{ld}^\triangle[0,\infty)$*, there exists a unique* $A_u \in (-\infty,\infty)$ *satisfying* (2.1).

Proof. For $u \in C_{ld}^\triangle[0,\infty)$, define

$$H_u(c) = \varphi_q(c) - \sum_{j=1}^{m-2} \beta_j\Big\{\sum_{i=1}^{m-2} \alpha_i\varphi_q\Big(c + \int_{\eta_i}^\infty h(\tau)f(\tau, u(\tau), u^\triangle(\tau))\nabla\tau + \sum_{\eta_j<t_k} \bar{I}_k(u(t_k))\Big)$$
$$+ \int_0^{\eta_j} \varphi_q\Big(c + \int_s^\infty h(\tau)f(\tau, u(\tau), u^\triangle(\tau))\nabla\tau + \sum_{s<t_k} \bar{I}_k(u(t_k))\Big)\triangle s + \sum_{t_k<\eta_j} I_k(u(t_k))\Big\}.$$

Then $H_u(c) \in C((-\infty,\infty), \mathbb{R})$. By $(H1)$, $(H3)$ we get $H_u(0) \leq 0$. We will consider two cases to prove

$H_u(c) = 0$ has unique solution on $(-\infty, \infty)$, which means there exists a unique $A_u \in (-\infty, \infty)$ satisfying (2.1).

Case 1: $H_u(0) = 0$.

By $H_u(0) = 0$, we get

$$\begin{aligned} H_u(0) = -\sum_{j=1}^{m-2}\beta_j\Big\{\sum_{i=1}^{m-2}\alpha_i\varphi_q\Big(\int_{\eta_i}^{\infty}h(\tau)f(\tau,u(\tau),u^{\triangle}(\tau))\nabla\tau + \sum_{\eta_j<t_k}\bar{I}_k(u(t_k))\Big) \\ + \int_0^{\eta_j}\varphi_q\Big(\int_s^{\infty}h(\tau)f(\tau,u(\tau),u^{\triangle}(\tau))\nabla\tau + \sum_{s<t_k}\bar{I}_k(u(t_k))\Big)\triangle s + \sum_{t_k<\eta_j}I_k(u(t_k))\Big\} = 0. \end{aligned}$$

Hence

$$\begin{aligned} \beta_j\Big\{\sum_{i=1}^{m-2}\alpha_i\varphi_q\Big(\int_{\eta_i}^{\infty}h(\tau)f(\tau,u(\tau),u^{\triangle}(\tau))\nabla\tau + \sum_{\eta_j<t_k}\bar{I}_k(u(t_k))\Big) + \int_0^{\eta_j}\varphi_q\Big(\int_s^{\infty}h(\tau)f(\tau,u(\tau),u^{\triangle}(\tau))\nabla\tau \\ + \sum_{s<t_k}\bar{I}_k(u(t_k))\Big)\triangle s + \sum_{t_k<\eta_j}I_k(u(t_k))\Big\} = 0 \quad (j = 1,\ 2,\ 3,\cdots, m-2). \end{aligned}$$

So we get

$$\varphi_p(\beta_j)\Big\{\varphi_p(\alpha_i)\Big(\int_{\eta_i}^{\infty}h(\tau)f(\tau,u(\tau),u^{\triangle}(\tau))\nabla\tau + \sum_{\eta_j<t_k}\bar{I}_k(u(t_k))\Big)\Big\} = 0,$$

$$\varphi_p(\beta_j)\int_0^{\eta_i}\varphi_q\Big(\int_s^{\infty}h(\tau)f(\tau,u(\tau),u^{\triangle}(\tau))\nabla\tau + \sum_{s<t_k}\bar{I}_k(u(t_k))\Big)\triangle s = 0,$$

$$\beta_j\sum_{t_k<\eta_j}I_k(u(t_k)) = 0 \quad (i = 1,\ 2,\ 3,\cdots, m-2, \quad j = 1,\ 2,\ 3,\cdots, m-2).$$

$$\begin{aligned} H_u(c) &= \varphi_q(c) - \sum_{j=1}^{m-2}\varphi_q(\varphi_p(\beta_j))\Big\{\sum_{i=1}^{m-2}\varphi_q(\varphi_p(\alpha_i))\varphi_q\Big(c + \int_{\eta_i}^{\infty}h(\tau)f(\tau,u(\tau),u^{\triangle}(\tau))\nabla\tau + \sum_{\eta_j<t_k}\bar{I}_k(u(t_k))\Big) \\ &\quad + \int_0^{\eta_j}\varphi_q\Big(c + \int_s^{\infty}h(\tau)f(\tau,u(\tau),u^{\triangle}(\tau))\nabla\tau + \sum_{s<t_k}\bar{I}_k(u(t_k))\Big)\triangle s + \sum_{t_k<\eta_j}I_k(u(t_k))\Big\} \\ &= \varphi_q(c) - \sum_{j=1}^{m-2}\sum_{i=1}^{m-2}\varphi_q\Big(c\varphi_p(\beta_j\alpha_i) + \varphi_p(\beta_j\alpha_i)\int_{\eta_i}^{\infty}h(\tau)f(\tau,u(\tau),u^{\triangle}(\tau))\nabla\tau + \varphi_p(\beta_j\alpha_i)\sum_{\eta_j<t_k}\bar{I}_k(u(t_k))\Big) \\ &\quad - \sum_{j=1}^{m-2}\int_0^{\eta_j}\varphi_q\Big(c\varphi_p(\beta_j) + \varphi_p(\beta_j)\int_s^{\infty}h(\tau)f(\tau,u(\tau),u^{\triangle}(\tau))\nabla\tau + \varphi_p(\beta_j)\sum_{s<t_k}\bar{I}_k(u(t_k))\Big)\triangle s - \sum_{j=1}^{m-2}\sum_{t_k<\eta_j}\beta_jI_k(u(t_k)) \\ &= \varphi_q(c)\Big\{1 - \sum_{j=1}^{m-2}\beta_j\Big(\sum_{i=1}^{m-2}\alpha_i + \eta_j\Big)\Big\}. \end{aligned}$$

Hence we get there exists a unique $c = 0$ satisfying $H_u(c) = 0$.

Case 2: $H_u(0) \neq 0$, i.e., $H_u(0) < 0$.

(i) When $c \in (-\infty, 0)$

$$H_u(c) \leq \varphi_q(c)\Big\{1 - \sum_{j=1}^{m-2}\beta_j\Big(\sum_{i=1}^{m-2}\alpha_i + \eta_j\Big)\Big\} < 0.$$

So when $c \in (-\infty, 0)$, $H_u(0) \neq 0$.

(ii) When $c \in (0, \infty)$,

$$\begin{aligned} H_u(c) &= \varphi_q(c)\Big\{1 - \sum_{j=1}^{m-2}\beta_j\Big[\sum_{i=1}^{m-2}\alpha_i\varphi_q\Big(1 + \frac{\int_{\eta_i}^{\infty} h(\tau)f(\tau, u(\tau), u^{\triangle}(\tau))\nabla\tau + \sum_{\eta_j<t_k}\bar{I}_k(u(t_k))}{c}\Big) \\ &\quad + \int_0^{\eta_j}\Big(1 + \frac{\int_s^{\infty} h(\tau)f(\tau, u(\tau), u^{\triangle}(\tau))\nabla\tau + \sum_{s<t_k}\bar{I}_k(u(t_k))}{c}\Big)\triangle s + \frac{\sum_{t_k<\eta_j} I_k(u(t_k))}{\varphi_q(c)}\Big]\Big\} \\ &= \varphi_q(c)\overline{H}_u(c), \end{aligned}$$

where

$$\begin{aligned} \overline{H}_u(c) = 1 - \sum_{j=1}^{m-2}\beta_j\Big[&\sum_{i=1}^{m-2}\alpha_i\varphi_q\Big(1 + \frac{\int_{\eta_i}^{\infty} h(\tau)f(\tau, u(\tau), u^{\triangle}(\tau))\nabla\tau + \sum_{\eta_j<t_k}\bar{I}_k(u(t_k))}{c}\Big) \\ &+ \int_0^{\eta_j}\Big(1 + \frac{\int_s^{\infty} h(\tau)f(\tau, u(\tau), u^{\triangle}(\tau))\nabla\tau + \sum_{s<t_k}\bar{I}_k(u(t_k))}{c}\Big)\triangle s + \frac{\sum_{t_k<\eta_j} I_k(u(t_k))}{\varphi_q(c)}\Big]. \end{aligned}$$

It is clear that $\overline{H}_u(c)$ is strictly increasing on $[0, \infty)$.

Let

$$\bar{c} = \frac{\varphi_p\Big[\max\{1, 2^{q-2}\}\sum_{j=1}^{m-2}\beta_j\Big(\sum_{i=1}^{m-2}\alpha_i + \eta_j\Big)\varphi_q\Big(\int_0^{\infty} h(\tau)f(\tau, u(\tau), u^{\triangle}(\tau))\nabla\tau + \sum_{k=1}^{\infty}\bar{I}_k(u(t_k))\Big) + \sum_{j=1}^{m-2}\beta_j\sum_{k=1}^{\infty} I_k(u(t_k))\Big]}{\varphi_p\Big[1 - \max\{1, 2^{q-2}\}\sum_{j=1}^{m-2}\beta_j\Big(\sum_{i=1}^{m-2}\alpha_i + \eta_j\Big)\Big]}.$$

Since the operator φ_q satisfies the inequalities

$$\varphi_q(x + y) \leq \varphi_q(x) + \varphi_q(y), \quad \text{for } q < 2,$$
$$\varphi_q(x + y) \leq 2^{q-2}[\varphi_q(x) + \varphi_q(y)], \quad \text{for } q \geq 2,$$

we have $\overline{H}_u(\bar{c}) \geq 0$. So $H_u(\bar{c}) = \varphi_q(\bar{c})\overline{H}_u(\bar{c}) \geq 0$. The intermediate value theorem guarantees that there exists a $c_0 \in (0, \bar{c}] \subset (0, \infty)$ such that $H_u(c_0) = 0$. If there exists two constants $c_i \in (-\infty, 0)(i = 1,\ 2)$ satisfying $H_u(c_1) = H_u(c_2) = 0$, then $\overline{H}_u(c_1) = \overline{H}_u(c_2) = 0$. Since $\overline{H}_u(c)$ is strictly increasing on $(0, \infty)$, we get $c_1 = c_2$. Therefore $H_u(c) = 0$ has a unique solution on $(0, \infty)$. Combining (i), (ii) and $H_u(0) \neq 0$, we obtain that $H_u(c) = 0$ has a unique solution on $(-\infty, \infty)$ The proof is complete. □

Remark. From the proof of Lemma 2.1, we know that for $u \in C_{ld}^{\triangle}[0, \infty)$,

$$A_u \in \left[0, \frac{\varphi_p\left[\max\{1,2^{q-2}\}\sum_{j=1}^{m-2}\beta_j\left(\sum_{i=1}^{m-2}\alpha_i+\eta_j\right)\varphi_q\left(\int_0^\infty h(\tau)f(\tau,u(\tau),u^\triangle(\tau))\nabla\tau+\sum_{k=1}^{\infty}\bar{I}_k(u(t_k))\right)+\sum_{j=1}^{m-2}\beta_j\sum_{k=1}^{\infty}I_k(u(t_k))\right]}{\varphi_p\left[1-\max\{1,2^{q-2}\}\sum_{j=1}^{m-2}\beta_j\left(\sum_{i=1}^{m-2}\alpha_i+\eta_j\right)\right]}\right].$$

Moreover, if $H_u(0) = 0$, then $A_u = 0$; if $H_u(0) \neq 0$, then $A_u \neq 0$.

Lemma 2.3. *If* $(H1)$ *holds, then* $\sup_{t\in[0,\infty)} \frac{u(t)}{1+t^2} \leq M \sup_{t\in[0,\infty)} \frac{u^\triangle(t)}{1+t}$ *for* $u \in P$*, where* $M = 1 + \sum_{i=1}^{m-2}\alpha_i$.

Proof. For $u \in P$, one arrives at $u^\triangle(\eta_i) \leq u^\triangle(0)$. Hence

$$\begin{aligned} u(t) &= \int_0^t u^\triangle(s)\triangle s + u(0) \\ &\leq tu^\triangle(0) + u(0) \\ &= tu^\triangle(0) + \sum_{i=1}^{m-2}\alpha_i u^\triangle(\eta_i) \\ &\leq tu^\triangle(0) + u^\triangle(0)\sum_{i=1}^{m-2}\alpha_i \\ &= u^\triangle(0)\left[t + \sum_{i=1}^{m-2}\alpha_i\right]. \end{aligned}$$

So,

$$\begin{aligned} \frac{u(t)}{1+t^2} &\leq \frac{u^\triangle(0)}{1+t^2}\left[t+\sum_{i=1}^{m-2}\alpha_i\right] \\ &< u^\triangle(0)\left[1+\sum_{i=1}^{m-2}\alpha_i\right] \\ &= Mu^\triangle(0). \end{aligned}$$

i.e.,

$$\sup_{t\in[0,\infty)} \frac{u(t)}{1+t^2} \leq Mu^\triangle(0) = M \sup_{t\in[0,\infty)} \frac{u^\triangle(t)}{1+t}.$$

The proof is complete. □

From Lemma 2.2, we get

$$\|u\| = \max\left\{ \sup_{t\in[0,\infty)} \left|\frac{u(t)}{1+t^2}\right|, \sup_{t\in[0,\infty)} \left|\frac{u^{\triangle}(t)}{1+t}\right| \right\}$$

$$\leq \max\left\{ M \sup_{t\in[0,\infty)} \frac{u^{\triangle}(t)}{1+t}, \sup_{t\in[0,\infty)} \frac{u^{\triangle}(t)}{1+t} \right\}$$

$$= M \sup_{t\in[0,\infty)} \frac{u^{\triangle}(t)}{1+t}.$$

Lemma 2.4. *If $u \in P$, then $\alpha(u) \geq \dfrac{1}{\omega^4+1}\phi(u)$.*

Proof. Since u is concave and nonnegative,

$$\begin{aligned}\alpha(u) &= \frac{\omega^2}{\omega^4+1} u\left(\frac{1}{\omega^2}\right)\\ &= \frac{\omega^2}{\omega^4+1} u\left(\frac{\omega^2+\omega^2t^2-1}{\omega^2+\omega^2t^2} . \frac{1+t^2-t}{\omega^2+\omega^2t^2-1} + \frac{t}{\omega^2+\omega^2t^2}\right)\\ &\geq \frac{1}{\omega^4+1} \frac{u(t)}{1+t^2}\end{aligned}$$

for $u \in P,\ t \in [0,\infty)$. The proof is complete. □

Now define an operator $T : P \to PC^1(J)$ by

$$\begin{aligned}Tu(t) = &\sum_{i=1}^{m-2} \alpha_i \varphi_q\left(A_u + \int_{\eta_i}^{\infty} h(\tau) f(\tau, u(\tau), u^{\triangle}(\tau))\nabla\tau + \sum_{\eta_i<t_k} \bar{I}_k(u(t_k))\right)\\ &+ \int_0^t \varphi_q\left(A_u + \int_s^{\infty} h(\tau) f(\tau, u(\tau), u^{\triangle}(\tau))\nabla\tau + \sum_{s<t_k} \bar{I}_k(u(t_k))\right)\triangle s + \sum_{t_k<t} I_k(u(t_k))\end{aligned}$$

where A_u satisfies (2.1).

Lemma 2.5. *Suppose that $(H1)-(H7)$ hold, then $T : P \to P$ is completely continuous.*

Proof. We divide the proof into four steps.

Step 1: We show that $TP \subset P$.

For $u \in P$, we have

$$\begin{aligned}
\sum_{i=1}^{m-2}\alpha_i (Tu)^{\Delta}(\eta_i) &= \sum_{i=1}^{m-2}\alpha_i\varphi_q\Big(A_u + \int_0^{\infty} h(\tau)f(\tau,u(\tau),u^{\Delta}(\tau))\nabla\tau + \sum_{\eta_i<t_k}\bar{I}_k(u(t_k)) \\
&- \int_0^{\eta_i} h(\tau)f(\tau,u(\tau),u^{\Delta}(\tau))\nabla\tau\Big) \\
&= \sum_{i=1}^{m-2}\alpha_i\varphi_q\Big(A_u + \int_{\eta_i}^{\infty} h(\tau)f(\tau,u(\tau),u^{\Delta}(\tau))\nabla\tau + \sum_{\eta_i<t_k}\bar{I}_k(u(t_k))\Big) \\
&= (Tu)(0),
\end{aligned}$$

$$\begin{aligned}
Tu(t) &= \sum_{i=1}^{m-2}\alpha_i\varphi_q\Big(A_u + \int_{\eta_i}^{\infty} h(\tau)f(\tau,u(\tau),u^{\Delta}(\tau))\nabla\tau + \sum_{\eta_i<t_k}\bar{I}_k(u(t_k))\Big) \\
&+ \int_0^t \varphi_q\Big(A_u + \int_s^{\infty} h(\tau)f(\tau,u(\tau),u^{\Delta}(\tau))\nabla\tau + \sum_{s<t_k}\bar{I}_k(u(t_k))\Big)\Delta s + \sum_{t_k<t} I_k(u(t_k)) > 0,
\end{aligned}$$

$$\begin{aligned}
Tu(t_1) &= \sum_{i=1}^{m-2}\alpha_i\varphi_q\Big(A_u + \int_{\eta_i}^{\infty} h(\tau)f(\tau,u(\tau),u^{\Delta}(\tau))\nabla\tau + \sum_{\eta_i<t_k}\bar{I}_k(u(t_k))\Big) \\
&+ \int_0^{t_1} \varphi_q\Big(A_u + \int_s^{\infty} h(\tau)f(\tau,u(\tau),u^{\Delta}(\tau))\nabla\tau + \sum_{s<t_k}\bar{I}_k(u(t_k))\Big)\Delta s + \sum_{t_k<t_1} I_k(u(t_k)) \\
&\leq \sum_{i=1}^{m-2}\alpha_i\varphi_q\Big(A_u + \int_{\eta_i}^{\infty} h(\tau)f(\tau,u(\tau),u^{\Delta}(\tau))\nabla\tau + \sum_{\eta_i<t_k}\bar{I}_k(u(t_k))\Big) \\
&+ \int_0^{t_2} \varphi_q\Big(A_u + \int_s^{\infty} h(\tau)f(\tau,u(\tau),u^{\Delta}(\tau))\nabla\tau + \sum_{s<t_k}\bar{I}_k(u(t_k))\Big)\Delta s + \sum_{t_k<t_2} I_k(u(t_k)) \\
&= Tu(t_2)
\end{aligned}$$

$$\begin{aligned}
(Tu)^{\Delta}(t_1) &= \varphi_q\Big(A_u + \int_0^{\infty} h(\tau)f(\tau,u(\tau),u^{\Delta}(\tau))\nabla\tau - \int_0^{t_1} h(\tau)f(\tau,u(\tau),u^{\Delta}(\tau))\nabla\tau + \sum_{t_1<t_k}\bar{I}_k(u(t_k))\Big) \\
&> \varphi_q\Big(A_u + \int_0^{\infty} h(\tau)f(\tau,u(\tau),u^{\Delta}(\tau))\nabla\tau - \int_0^{t_2} h(\tau)f(\tau,u(\tau),u^{\Delta}(\tau))\nabla\tau + \sum_{t_2<t_k}\bar{I}_k(u(t_k))\Big) \\
&= (Tu)^{\Delta}(t_2)
\end{aligned}$$

Hence Tu is nonnegative, nondecreasing, $(Tu)^{\Delta}$ is nonincreasing on $[0,\infty)$, i.e., $TP \subset P$.

Step 2: We show that $T : P \to P$ is continuous.
Let $u_n \to u$ as $n \to \infty$ in P, then there exists r_0 such that

$$\max\left\{\|u\|, \sup_{n\in\mathbb{N}\setminus\{0\}} \|u_n\|\right\} < r_0.$$

By using (H6), we get $F(t,u,v)$ is bounded on $[0,\infty)_\pi \times [0,r_0]^2$. Set

$$B_0 = sup\Big\{F(t,u,v) \mid (t,u,v) \in [0,\infty)_\pi \times [0,r_0]^2\Big\},$$

$$D_0 = \sup_{t\in[0,\infty)} \left\{\sum_{k=1}^{\infty} \bar{I}_k(u(t_k))\right\}.$$

We get

$$|\varphi_p((Tu_n)^\Delta(t)) - \varphi_p((Tu)^\Delta(t))|$$

$$= \Bigg| A_n + \int_0^\infty h(\tau)f(\tau,u_n(\tau),u_n^\Delta(\tau))\nabla\tau - \int_0^t h(\tau)f(\tau,u_n(\tau),u_n^\Delta(\tau))\nabla\tau + \sum_{t<t_k} \bar{I}_k(u_n(t_k))$$

$$-A_u - \int_0^\infty h(\tau)f(\tau,u(\tau),u^\Delta(\tau))\nabla\tau + \int_0^t h(r)f(\tau,u(\tau),u^\Delta(\tau))\nabla\tau - \sum_{t<t_k} \bar{I}_k(u(t_k))\Bigg|$$

$$\le |A_n - A_u| + \int_t^\infty h(\tau)|f(\tau,u_n(\tau),u_n^\Delta(\tau)) - f(\tau,u(\tau),u^\Delta(\tau))|\nabla\tau + \sum_{t<t_k} |\bar{I}_k(u_n(t_k))$$

$$- \bar{I}_k(u(t_k))|$$

$$\le 2\Lambda + \int_t^\infty h(\tau)\left|F\left(\tau, \frac{u_n(\tau)}{1+\tau^2}, \frac{u_n^\Delta(\tau)}{1+\tau}\right) - F\left(\tau, \frac{u(\tau)}{1+\tau^2}, \frac{u^\Delta(\tau)}{1+\tau}\right)\right|\nabla\tau + \sum_{t<t_k} |\bar{I}_k(u_n(t_k))$$

$$- \bar{I}_k(u(t_k))|$$

$$\le 2\Lambda + \int_0^\infty h(\tau)\left|F\left(\tau, \frac{u_n(\tau)}{1+\tau^2}, \frac{u_n^\Delta(\tau)}{1+\tau}\right) - F\left(\tau, \frac{u(\tau)}{1+\tau^2}, \frac{u^\Delta(\tau)}{1+\tau}\right)\right|\nabla\tau + \sum_{k=1}^{\infty} |\bar{I}_k(u_n(t_k))$$

$$- \bar{I}_k(u(t_k))|$$

$$\le 2\Lambda + 2B_0\varphi_p(\Omega) + 2D_0.$$

Hence

$$\sup_{t\in[0,\infty)} \left|\frac{(Tu_n)^\Delta(t) - (Tu)^\Delta(t)}{1+t}\right| \to 0, \ \ n\to\infty.$$

From Lemma 2.3, we get

$$\sup \left|\frac{(Tu_n)(t) - (Tu)(t)}{1+t^2}\right| \to 0, \ \ n\to\infty.$$

Hence $T : P \to P$ is continuous.

Step 3: We show that $T : P \to P$ is relatively compact.
Let W be any bounded subset of P, then there exists $L > 0$ such that $\|u\| < L$. Set

$$B_L = \sup\Big\{F(t,u,v) \mid (t,u,v) \in [0,\infty)_\pi \times [0,L]^2\Big\}.$$

For $u \in W$, we have,

$$\begin{aligned}
\|Tu\| &= \max\left\{\sup_{t\in[0,\infty)}\left|\frac{(Tu)(t)}{1+t^2}\right|, \sup_{t\in[0,\infty)}\left|\frac{(Tu)^{\Delta}(t)}{1+t}\right|\right\}\\
&\le \max\left\{M\sup_{t\in[0,\infty)}\left|\frac{(Tu)^{\Delta}(t)}{1+t}\right|, \sup_{t\in[0,\infty)}\left|\frac{(Tu)^{\Delta}(t)}{1+t}\right|\right\}\\
&= M\sup_{t\in[0,\infty)}\left|\frac{(Tu)^{\Delta}(t)}{1+t}\right|\\
&= M(Tu)^{\Delta}(0)\\
&= M\varphi_q\Big(A_u + \int_0^{\infty} h(\tau)f(\tau,u(\tau),u^{\Delta}(\tau))\nabla\tau + \sum_{k=1}^{\infty}\bar{I}_k(u(t_k))\Big)\\
&= M\varphi_q\Big(A_u + \int_0^{\infty} h(\tau)F\Big(\tau,\frac{u(\tau)}{1+\tau^2},\frac{u^{\Delta}(\tau)}{1+\tau}\Big)\nabla\tau + \sum_{k=1}^{\infty}\bar{I}_k(u(t_k))\Big)\\
&\le M\varphi_q(\Lambda + B_L\varphi_p(\Omega) + D_0)
\end{aligned}$$

Hence TW is uniformly bounded.

Now we show that $(TW)^{\Delta}$ is locally equicontinuous on $[0,\infty)$. For any $v > 0$, $t_1, t_2 \in [0,v]$ and $u \in W$, without loss of generality, we may assume that $t_1 > t_2$.

$$\begin{aligned}
&|\varphi_p((Tu)^{\Delta}(t_1)) - \varphi_p((Tu)^{\Delta}(t_2))|\\
&= \Big|A_u + \int_0^{\infty} h(\tau)f(\tau,u(\tau),u^{\Delta}(\tau))\nabla\tau - \int_0^{t_1} h(\tau)f(\tau,u(\tau),u^{\Delta}(\tau))\nabla\tau\\
&+ \sum_{t_1<t}\bar{I}_k(u(t_k)) - A_u - \int_0^{\infty} h(\tau)f(\tau,u(\tau),u^{\Delta}(\tau))\nabla\tau\\
&+ \int_0^{t_2} h(\tau)f(\tau,u(\tau),u^{\Delta}(\tau))\nabla\tau - \sum_{t_2<t}\bar{I}_k(u(t_k))\Big|\\
&= \Big|\int_{t_1}^{t_2} h(\tau)f(\tau,u(\tau),u^{\Delta}(\tau))\nabla\tau + \sum_{t_1<t_k<t_2}\bar{I}_k(u(t_k))\Big|\\
&= \Big|\int_{t_1}^{t_2} h(\tau)F\Big(\tau,\frac{u(\tau)}{1+\tau^2},\frac{u^{\Delta}(\tau)}{1+\tau}\Big)\nabla\tau + \sum_{t_1<t_k<t_2}\bar{I}_k(u(t_k))\Big|\\
&\le \int_0^{\infty} h(\tau)F\Big(\tau,\frac{u(\tau)}{1+\tau^2},\frac{u^{\Delta}(\tau)}{1+\tau}\Big)\nabla\tau + \sum_{k=1}^{\infty}\bar{I}_k(u(t_k))\\
&\le \varphi_p(\Omega)B_0 + D_0.
\end{aligned}$$

Hence $\left|\dfrac{(Tu)^{\Delta}(t_1)-(Tu)^{\Delta}(t_2)}{1+t}\right| \to 0,$ as $t_1 \to t_2$. This implies that $(TW)^{\Delta}$ is equicontinuous on $[0,v]$. Since v is arbitrary, $(TW)^{\Delta}$ is locally continuous on $[0,v]$.

Step 4: We show that $T : P \to P$ is equiconvergent at $+\infty$.
For $u \in W$, we have

$$\begin{aligned}
\left|(Tu)^{\Delta}(t)\right| &= \left|\varphi_q\left(A_u + \int_t^{\infty} h(\tau)f(\tau,u(\tau),u^{\Delta}(\tau))\nabla\tau + \sum_{t<t_k}\bar{I}_k(u(t_k))\right)\right| \\
&\le \left|\varphi_q\left(A_u + \int_0^{\infty} h(\tau)f(\tau,u(\tau),u^{\Delta}(\tau))\nabla\tau + \sum_{k=1}^{\infty}\bar{I}_k(u(t_k))\right)\right| \\
&= \left|\varphi_q\left(A_u + \int_0^{\infty} h(\tau)F\left(\tau,\frac{u(\tau)}{1+\tau^2},\frac{u^{\Delta}(\tau)}{1+\tau}\right)\nabla\tau + \sum_{k=1}^{\infty}\bar{I}_k(u(t_k))\right)\right| \\
&= \varphi_q(\Lambda + B_0\varphi(\Omega) + D_0).
\end{aligned}$$

Hence

$$\lim_{t\to\infty}\left|\frac{(Tu)^{\Delta}(t)}{1+t}\right| \to 0$$

From Lemma 2.3, we get

$$\lim_{t\to\infty}\left|\frac{(Tu)(t)}{1+t^2}\right| \to 0.$$

Therefore $T : P \to P$ is equiconvergent at $+\infty$. The proof is complete. □

3. Main Result

Let α, ψ be nonnegative continuous concave functionals on P, and let β and ϕ be nonnegative continuous convex functionals on P then for positive numbers $r,\ j,\ l$ and R, we define the sets:

$$Q(\alpha,\beta,r,R) = \{u \in P \mid r \le \alpha(u),\ \beta(u) \le R\},$$
$$U(\psi,j) = \{u \in Q(\alpha,\beta,r,R) \mid j \le \psi(u)\},$$
$$V(\phi,l) = \{u \in Q(\alpha,\beta,r,R) \mid \phi(u) \le l\}.$$

In the next theorem we also assume the following condition:

$(H8)$ There exist nonnegative constants $c_k,\ d_k,\ e_k,\ f_k$ such that

$$I_k(u) \le c_k u + d_k, \quad \forall u \in \mathbb{R}^+,\ k \in \mathbb{N},$$

$$\bar{I}_k(u) \le \varphi_p(e_k)\varphi_p(u) + f_k, \quad \forall u \in \mathbb{R}^+,\ k \in \mathbb{N},$$

$$c^* = \sum_{k=1}^{\infty}(t^2+1)c_k < \infty, \quad d^* = \sum_{k=1}^{\infty} d_k < \infty,$$

$$e^* = \max\{1, 2^{q-2}\}\sum_{k=1}^{\infty}(t^2+1)e_k < \infty, \quad f^* = \max\{1, 2^{q-2}\}\varphi_q\left(\sum_{k=1}^{\infty} f_k\right) < \infty.$$

For the convenience, we take the notations:

$$\Omega = \varphi_q\left(\int_0^{\infty} h(\tau)\nabla\tau\right),$$

$$A = (\omega^4+1)\left[\varphi_q\left(\int_{\frac{1}{\omega^2}}^{\omega^2} h(\tau)\nabla\tau\right)\right]^{-1},$$

$$B = \frac{\left[1-\max\{1,2^{q-2}\}\sum_{j=1}^{m-2}\beta_j\left(\sum_{i=1}^{m-2}\alpha_i+\eta_j\right)\right]R - \max\{1,2^{q-2}\}\left[\sum_{j=1}^{m-2}\beta_j[c^*MR+d^*] + \max\{1,2^{q-2}\}[e^*MR+f^*]\right]}{\Omega[\max\{1,2^{q-2}\}]^2}.$$

Theorem 3.1. *Assume* $(H1)-(H7)$ *hold. If there exist constants* r, j, l, R *with* $\max\left\{r(\omega^4+1), \frac{j+R}{2}\left[\frac{1}{2}+\sum_{i=1}^{m-2}\alpha_i\right]\right\} \leq$
l, $\max\left\{j, \frac{2r(\omega^4+1)}{1+\omega^2\sum_{i=1}^{m-2}\alpha_i} - j, \frac{\max\{1,2^{q-2}\}\left[d^*\sum_{j=1}^{m-2}\beta_j + f^*\max\{1,2^{q-2}\}\right]}{1-\max\{1,2^{q-2}\}\left[\sum_{j=1}^{m-2}\beta_j\left(\sum_{i=1}^{m-2}\alpha_i+\eta_j\right)+c^*M\sum_{j=1}^{m-2}\beta_j+e^*M\max\{1,2^{q-2}\}\right]}\right\} < R$ *and suppose that* f *satisfies the following conditions:*

(C1) $F(t,u,u^{\triangle}) \geq \varphi_p(rA)$, *for* $(t,u,u^{\triangle}) \in \left[\frac{1}{\omega^2}, \omega^2\right]\times\left[\frac{r}{\omega^2}, l\right]\times[0,R]$;

(C2) $F(t,u,u^{\triangle}) \leq \varphi_p(B)$, *for* $(t,u,u^{\triangle}) \in [0,\infty)\times[0,MR]\times[0,R]$.

Then the IBVP (1.1) *has at least one positive solution* $u \in P$ *such that*

$$\frac{\omega^2}{\omega^2+1}\min_{t\in[\frac{1}{\omega^2},\omega^2]} u(t) \geq r, \quad \sup_{t\in[0,\infty)}\frac{u^{\triangle}(t)}{1+t} \leq R.$$

Proof. The impulsive boundary value problem (1.1) has a solution $u = u(t)$ if and only if u solves the operator equation $u = Tu$. Thus we set out to verify that the operator T satisfies four functionals fixed point theorem which will prove the existence of a fixed point of T.
We first show that $Q(\alpha, \beta, r, R)$ is bounded and $T : Q(\alpha, \beta, r, R) \to P$ is continuous. For all $u \in Q(\alpha, \beta, r, R)$ with Lemma 2.3, we have

$$\|u\| \leq M \sup_{t\in[0,\infty)} \frac{u^{\triangle}(t)}{1+t} = M\beta(u) \leq MR,$$

which means that $Q(\alpha, \beta, r, R)$ is a bounded set. According to Lemma 2.5, it is clear that $T : Q(\alpha, \beta, r, R) \to P$ is completely continuous.

Let

$$u_0(t) = \frac{j+R}{2}\left[t + \sum_{i=1}^{m-2} \alpha_i\right].$$

Clearly, $u_0 \in P$. By direct calculation,

$$\alpha(u_0) = \frac{\omega^2}{\omega^4+1}\frac{j+R}{2}\left[\frac{1}{\omega^2} + \sum_{i=1}^{m-2} \alpha_i\right] > r,$$

$$\beta(u_0) = \frac{j+R}{2} < R,$$

$$\psi(u_0) > \frac{j+R}{2} \geq j,$$

$$\phi(u_0) \leq \frac{j+R}{2}\left[\frac{1}{2} + \sum_{i=1}^{m-2} \alpha_i\right] \leq l.$$

So,

$$u_0 \in \{u \in U(\psi, j) : \beta(u) < R\} \cap \{u \in V(\phi, l) : r < \alpha(u)\},$$

which means that (i) in Theorem 1.1 is satisfied. For all $u \in Q(\alpha, \beta, r, R)$, with $\alpha(u) = r$ and $l < \phi(Tu)$, from Lemma 2.4, we have

$$\alpha(Tu) \geq \frac{1}{\omega^4+1}\phi(Tu) > \frac{l}{\omega^4+1} \geq r.$$

Thus, $\alpha(Tu) > r$. Hence (*ii*) in Theorem 1.1 is fulfilled.

For $u \in V(\phi, l)$, with $\alpha(u) = r$,

$$\begin{aligned}
\alpha(Tu) &= \frac{\omega^2}{\omega^4+1} \min_{t\in[\frac{1}{\omega^2},\omega^2]} Tu(t) \\
&= \frac{\omega^2}{\omega^4+1} Tu\Big(\frac{1}{\omega^2}\Big) \\
&= \frac{\omega^2}{\omega^4+1}\Bigg\{\sum_{i=1}^{m-2}\alpha_i\varphi_q\Bigg(A_u + \int_{\eta_i}^{\infty} h(\tau)f(\tau,u(\tau),\tau^{\triangle}(\tau))\nabla\tau + \sum_{\eta_i<t_k}\bar{I}_k(u(t_k))\Bigg) \\
&\quad + \int_0^{\frac{1}{\omega^2}} \varphi_q\Bigg(A_u + \int_s^{\infty} h(\tau)f(\tau,u(\tau),u^{\triangle}(\tau))\nabla\tau + \sum_{s<t_k}\bar{I}_k(u(t_k))\Bigg)\triangle s + \sum_{t_k<\frac{1}{\omega^2}} I_k(u(t_k))\Bigg\} \\
&\geq \frac{\omega^2}{\omega^4+1}\int_0^{\frac{1}{\omega^2}} \varphi_q\Bigg(\int_s^{\infty} h(\tau)f(\tau,u(\tau),u^{\triangle}(\tau))\nabla\tau\Bigg)\triangle s \\
&\geq \frac{1}{\omega^4+1}\varphi_q\Bigg(\int_{\frac{1}{\omega^2}}^{\infty} h(\tau)f(\tau,u(\tau),u^{\triangle}(\tau))\nabla\tau\Bigg) \\
&\geq \frac{1}{\omega^4+1}\varphi_q\Bigg(\int_{\frac{1}{\omega^2}}^{\omega^2} h(\tau)f(\tau,u(\tau),u^{\triangle}(\tau))\nabla\tau\Bigg) \\
&= \frac{1}{\omega^4+1}\varphi_q\Bigg(\int_{\frac{1}{\omega^2}}^{\omega^2} h(\tau)F\Big(\tau,\frac{u(\tau)}{1+\tau^2},\frac{u^{\triangle}(\tau)}{1+\tau}\Big)\nabla\tau\Bigg) \\
&\geq \frac{1}{\omega^4+1}\varphi_q\Bigg(\int_{\frac{1}{\omega^2}}^{\omega^2} h(\tau)\varphi_p(rA)\nabla\tau\Bigg) \\
&= \frac{1}{\omega^4+1} rA\varphi_q\Bigg(\int_{\frac{1}{\omega^2}}^{\omega^2} h(\tau)\nabla\tau\Bigg) \\
&= r.
\end{aligned}$$

By using condition ($H8$), we have

$$\sum_{k=1}^{\infty} I_k(u(t_k)) \leq c^*\|u\|_1 + d^*.$$

$$\varphi_q\Bigg(\sum_{k=1}^{\infty}\bar{I}_k(u(t_k))\Bigg) \leq e^*\|u\|_1 + f^*.$$

Thus, for all $u \in U(\psi, j)$, with $\beta(u) = R$, we get

$$\begin{aligned}
\beta(Tu) &= \sup_{t\in[0,\infty)} \frac{(Tu)^{\triangle}(t)}{1+t} \\
&= (Tu)^{\triangle}(0)
\end{aligned}$$

$$= \varphi_q\Big(A_u + \int_0^\infty h(\tau)f(\tau,u(\tau),u^\triangle(\tau))\nabla\tau + \sum_{k=1}^{\infty}\bar{I}_k(u(t_k))\Big)$$

$$\leq \max\{1,2^{q-2}\}\frac{\max\{1,2^{q-2}\}\sum_{j=1}^{m-2}\beta_j\Big(\sum_{i=1}^{m-2}\alpha_i+\eta_j\Big)\varphi_q\Big(\int_0^\infty h(\tau)f(\tau,u(\tau),u^\triangle(\tau))\nabla\tau + \sum_{k=1}^{\infty}\bar{I}_k(u(t_k))\Big) + \sum_{j=1}^{m-2}\beta_j\sum_{k=1}^{\infty}I_k(u(t_k))}{1-\max\{1,2^{q-2}\}\sum_{j=1}^{m-2}\beta_j\Big(\sum_{i=1}^{m-2}\alpha_i+\eta_j\Big)}$$

$$+\max\{1,2^{q-2}\}\varphi_q\Big(\int_0^\infty h(\tau)f(\tau,u(\tau),u^\triangle(\tau))\nabla\tau + \sum_{k=1}^{\infty}\bar{I}_k(u(t_k))\Big)$$

$$= \max\{1,2^{q-2}\}\frac{\sum_{j=1}^{m-2}\beta_j\sum_{k=1}^{\infty}I_k(u(t_k)) + \varphi_q\Big(\int_0^\infty h(\tau)f(\tau,u(\tau),u^\triangle(\tau))\nabla\tau + \sum_{k=1}^{\infty}\bar{I}_k(u(t_k))\Big)}{1-\max\{1,2^{q-2}\}\sum_{j=1}^{m-2}\beta_j\Big(\sum_{i=1}^{m-2}\alpha_i+\eta_j\Big)}$$

$$\leq \frac{\max\{1,2^{q-2}\}\Big\{\sum_{j=1}^{m-2}\beta_j[c^*\|u\|_1+d^*] + \max\{1,2^{q-2}\}\Big[\varphi_q\Big(\int_0^\infty h(\tau)f(\tau,u(\tau),u^\triangle(\tau))\nabla\tau\Big) + \varphi_q\Big(\sum_{k=1}^{\infty}\bar{I}_k\Big)\Big]\Big\}}{1-\max\{1,2^{q-2}\}\sum_{j=1}^{m-2}\beta_j\Big(\sum_{i=1}^{m-2}\alpha_i+\eta_j\Big)}$$

$$\leq \frac{\max\{1,2^{q-2}\}\Big\{\sum_{j=1}^{m-2}\beta_j[c^*M\|u\|_2+d^*] + \max\{1,2^{q-2}\}[\Omega B + e^*M\|u\|_2 + f^*]\Big\}}{1-\max\{1,2^{q-2}\}\sum_{j=1}^{m-2}\beta_j\Big(\sum_{i=1}^{m-2}\alpha_i+\eta_j\Big)}$$

$$= R.$$

Thus (*iii*) and (*v*) in Theorem 1.1 hold. We finally prove that (*iv*) in Theorem 1.1 holds.

For all $u \in Q(\alpha,\beta,r,R)$, with $\beta(u) = R$ and $\psi(Tu) < j$, we have $\beta(Tu) \leq \psi(Tu) < j < R$.

Thus, all conditions of Theorem 1.1 are satisfied. T has a fixed point u in $Q(\alpha,\beta,r,R)$. Therefore, the IBVP (1.1) has at least one positive solution $u \in P$ such that

$$\frac{\omega^2}{\omega^4+1}\min_{t\in[\frac{1}{\omega^2},\omega^2]} u(t) \geq r, \quad \sup_{t\in[0,\infty)}\frac{u^\triangle(t)}{1+t} \leq R.$$

□

4. Example

Example 4.1. To illustrate how our main result can be used in practice we present an example.

Let $\mathbb{T} = \mathbb{R}$, $p = 2$, $m = 4$, $\alpha_1 = \frac{1}{2}$, $\alpha_2 = \frac{1}{5}$, $\beta_1 = \frac{1}{4}$, $\beta_2 = \frac{1}{6}$, $\eta_1 = \frac{1}{3}$, $\eta_2 = \frac{1}{2}$ in the IBVP (1.1). Now we consider the following problem

$$\begin{cases} u^{\triangle\nabla}(t) + h(t)f(t, u(t), u^{\triangle}(t)) = 0, \quad t \neq 1, \\ u(0) = \frac{1}{2}u^{\triangle}\left(\frac{1}{12}\right) + \frac{1}{5}u^{\triangle}\left(\frac{1}{9}\right), \quad u^{\triangle}(\infty) = \frac{1}{4}u\left(\frac{1}{12}\right) + \frac{1}{6}u\left(\frac{1}{9}\right), \\ u(t_k^+) - u(t_k^-) = I_k(u(t_k)), \quad \varphi_p(u^{\triangle}(t_k^+)) - \varphi_p(u^{\triangle}(t_k^-)) = -\bar{I}_k(u(t_k)), \quad k \in \mathbb{N}, \end{cases} \tag{4.1}$$

where

$$F(t, u, v) = \frac{1}{100}\left[\frac{22}{17}u + \frac{v}{10} + 3\right],$$

$$I_k(u) = \frac{2}{3^{k+1}}, \quad \bar{I}_k(u) = \frac{3}{10}\left(\frac{2}{5}\right)^k,$$

$$h(t) = \frac{1}{(1+t)^2}, \quad t \neq 1.$$

Let $c_k = \frac{9}{34}\frac{\left(\frac{1}{2}\right)^k}{1+t^2}, \quad d_k = \frac{2}{3^{k+1}}, \quad e_k = \frac{5}{408}\frac{\left(\frac{1}{2}\right)^k}{1+t^2}, \quad f_k = \frac{3}{10}\left(\frac{2}{5}\right)^k$. Then we easily obtain that $c^* = \frac{9}{17}, \quad d^* = 1, \quad e^* = \frac{5}{204}, \quad f^* = \frac{1}{2}$.

Set $\omega = 2$, by simple calculation we get $A = \frac{85}{3}, \quad B = \frac{1}{3}, \quad M = \frac{17}{10}$.

Choose $j = 9, \ R = 10, \ r = \frac{1}{1000}, \ l = 15$, it is easy to check that $\max\left\{\frac{17}{1000}, \frac{57}{5}\right\} \leq 15, \ \max\left\{9, -\frac{17083}{1900}, \frac{22}{3}\right\} \leq 10$.

$$F(t, u, u^{\triangle}) \geq \frac{204022}{6800000} > \varphi_p(rA) = \frac{85}{3000}, \quad for \ (t, u, u^{\triangle}) \in \left[\frac{1}{4}, 4\right] \times \left[410^{-4}, 15\right] \times [0, 10];$$
$$F(t, u, u^{\triangle}) \leq \frac{26}{100} < \varphi_p(B) = \frac{1}{3}, \quad for \ (t, u, u^{\triangle}) \in [0, \infty) \times [0, 17] \times [0, 10].$$

So, all conditions of the Theorem 3.1 hold. Thus by Theorem 3.1 the IBVP (4.1) has at least one positive solution u such that

$$\frac{4}{5}\min_{t\in[\frac{1}{4},4]} u(t) \geq \frac{1}{1000}, \quad \sup_{t\in[0,\infty)} \frac{u^{\triangle}(t)}{1+t} \leq 10.$$

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