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Suborbital Graphs for a Non-Transitive Action of the Normalizer

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Abstract. In this paper, we investigate a suborbital graph for the normalizer of $\Gamma_0(n)$ in $PSL(2,\mathbb{R})$, where n will be of the form 3^2p^2, p is a prime and $p > 3$. Then we give edge and circuit conditions on graphs arising from the non-transitive action of the normalizer.

1. Introduction

1.1. Triangle Groups

A triangle group is denoted by (l, m, n) where l, m, n are positive integers or ∞ and has the presentation

$$\{x, y, z \mid x^l = y^m = z^n = xyz = 1\}$$

under the convention that if any of l, m, n is ∞ we ignore the corresponding relation.

For a geometric interpretation of the triangle group (l, m, n) we consider a triangle T with angles $\frac{\pi}{l}, \frac{\pi}{m}, \frac{\pi}{n}$ in a space X, where X is either the sphere, or the Euclidean plane, or the hyperbolic plane. Then, the group generated by the reflections of X in the sides of T has a subgroup of index 2, consisting of orientation preserving transformation, isomorphic to (l, m, n).The integers l, m, n determine completely the space X, namely X is

$$\text{the sphere if and only if } \frac{\pi}{l} + \frac{\pi}{m} + \frac{\pi}{n} > 1$$
$$\text{the Euclidean plane if and only if } \frac{\pi}{l} + \frac{\pi}{m} + \frac{\pi}{n} = 1$$
$$\text{the hyperbolic plane if and only if } \frac{\pi}{l} + \frac{\pi}{m} + \frac{\pi}{n} < 1.$$

One of the most interesting, and definitely the most thoroughly studied, triangle groups in the literature is the modular group Γ defined as

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \ a, b, c, d \in \mathbb{Z}, \ ad - bc = 1 \right\}$$

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(see [9, 17] for more detailed). Seeing Γ as a triangle group we get the isomorphism $\Gamma \simeq (2,3,\infty)$. The cusp set of $\hat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$, and of course Γ acts on it by

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \frac{x}{y} \rightarrow \frac{ax+by}{cx+dy},$$

under the usual convention to write ∞ as a fraction with denominator 0.

1.2. The Normalizer

$\Gamma_0(n) = \{g \in \Gamma : c \equiv 0 \pmod{n}\}$ is a well known congruence subgroup of the classical modular group Γ. The normalizer turns to be a very important group in the study of moonshine and for this reason has been studied by many authors [5,6,13]. It consists exactly of the matrices

$$\begin{pmatrix} ae & b/h \\ cn/h & de \end{pmatrix}, ade^2 - bcn/h^2 = e,$$

where $e \parallel \frac{n}{h^2}$ and h is the largest divisor of 24 for which $h^2|n$ with understandings that the determinant e of the matrix is positive, and that $r \parallel s$ means that $r|s$ and $(r, s/r) = 1$ (r is called an exact divisor of s).

In some ways triangle groups are the simplest Fuchsian groups, in [5] it is shown that maps (tessellations of orientable surfaces) can be parametrized by subgroups of Fuchsian groups containing a period 2 and that the regular maps correspond to normal subgroups. For these reasons, the authors found all values for which $Nor(n)$ is a triangle group as follows.

Lemma 1.1. ([2]) *$Nor(n)$ is a triangle group for precisely 26 values of n.*
If $n = 1, 2^2, 2^4, 2^6, 3^2, 2^2.3^2, 2^4.3^2, 2^6.3^2$, then $Nor(n)$ has signature $(2,3,\infty)$.
If $n = 2, 2^3, 2^5, 2^7, 2.3^2, 2^3.3^2, 2^5.3^2, 2^7.3^2$, then $Nor(n)$ has signature $(2,4,\infty)$.
If $n = 3, 2^2.3, 2^4.3, 2^6.3, 3^3, 2^2.3^3, 2^4.3^3, 2^6.3^3$, then $Nor(n)$ has signature $(2,6,\infty)$.

1.3. Motivation

The modular group acts transitively on $\hat{\mathbb{Q}}$ and in a paper of Jones, Singerman, Wicks, the suborbital graphs were studied and the most basic one turn out to be the well-known Farey graph [9].

Suborbital graphs of the normalizer were studied by same idea. All circuits in the suborbital graph were found when n is a square-free positive integer [11,12] and when n satisfies the condition of transitive action [12]. Then, non-transitive cases have been examined to reach the general statement [7,10]. Our intuitive conclusion obtained from all these studies is that the general case is related the cases which $Nor(n)$ is a triangle group. The transitive action is automatically provided for n values, which $Nor(n)$ is also a triangle group. In this case, circuits in graphs are given in [12]. In non-transitive cases, if the decomposition of n contains n values which provide normalizer to be triangle group, there would be a circuit in the graphs. If not, graphs would be a forest. In the way of verification of this hypothesis, taking one of those values, we examine the combinatorial properties of $Nor(n)$.

2. Main Results

Throughout the paper, n will be of the from 3^2p^2, where p is a prime and $p > 3$. In this case, since $h = 2^{\min\{3,[\alpha/2]\}}3^{\min\{1,[\beta/2]\}}$, h is equal to 3 for $n = 2^\alpha 3^\beta p_3^{\alpha_3} \cdots p_r^{\alpha_r}$. As $e \parallel \frac{n}{h^2}$, e must be $1, p^2$. Hence, $Nor(3^2p^2)$ consists of the following two types of the element:

$$T_1 = \begin{pmatrix} a & b/3 \\ 3p^2c & d \end{pmatrix} : ad - bcp^2 = 1, \; T_2 = \begin{pmatrix} ap^2 & b/3 \\ 3p^2c & dp^2 \end{pmatrix} : adp^2 - bc = 1.$$

2.1. Transitive Action

Lemma 2.1. ([2]) *Let n have the prime power decomposition as* $2^{\alpha_1} \cdot 3^{\alpha_2} \cdot p_3^{\alpha_3} \cdots p_r^{\alpha_r}$. *Then* $Nor(n)$ *acts transitively on* $\hat{\mathbb{Q}}$ *if and only if* $\alpha_1 \leq 7$, $\alpha_2 \leq 3$ *and* $\alpha_i \leq 1$ *for* $i = 3, \ldots, r$.

Hence, the following theorem holds.

Theorem 2.2. $Nor(3^2p^2)$ *is not transitive on* $\hat{\mathbb{Q}}$.

Therefore, we will find a maximal subset of $\hat{\mathbb{Q}}$ on which $Nor(3^2p^2)$ acts transitively. For this,

Lemma 2.3. ([7]) *Let* $d|n$. *Then the orbit* $\begin{pmatrix} a \\ d \end{pmatrix}$ *of* a/d *with* $(a, d) = 1$ *under* $\Gamma_0(n)$ *is the set* $\left\{x/y \in \hat{\mathbb{Q}} : (n, y) = d, a \equiv x\frac{y}{d} \pmod{(d, n/d)}\right\}$. *Furthermore the number of orbits* $\begin{pmatrix} a \\ d \end{pmatrix}$ *with* $d|n$ *under* $\Gamma_0(n)$ *is just* $\varphi(d, N/d)$ *where* $\varphi(n)$ *is Euler's totient function which is the number of positive integers less than or equal to n that are coprime to n.*

In the view of the above theorem, we can give the following

Theorem 2.4. *The orbits of* $\Gamma_0(3^2p^2)$ *on* $\hat{\mathbb{Q}}$ *are as follows;*

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}; \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix}; \begin{pmatrix} 1 \\ 3^2 \end{pmatrix}; \begin{pmatrix} 1 \\ p^2 \end{pmatrix}; \begin{pmatrix} 1 \\ 3p^2 \end{pmatrix}, \begin{pmatrix} 2 \\ 3p^2 \end{pmatrix}; \begin{pmatrix} 1 \\ 3^2p^2 \end{pmatrix}; \begin{pmatrix} 1 \\ p \end{pmatrix}, \begin{pmatrix} 2 \\ p \end{pmatrix} \cdots \begin{pmatrix} p-1 \\ p \end{pmatrix};$$
$$\begin{pmatrix} 1 \\ 3p \end{pmatrix}, \begin{pmatrix} p+2 \\ 3p \end{pmatrix}, \cdots \begin{pmatrix} 2p-1 \\ 3p \end{pmatrix}; \begin{pmatrix} 1 \\ 3^2p \end{pmatrix}, \begin{pmatrix} p+2 \\ 3^2p \end{pmatrix}, \cdots \begin{pmatrix} 2p-1 \\ 3^2p \end{pmatrix}.$$

Proof. Let us denote the representatives of the orbits by $\begin{pmatrix} a \\ d \end{pmatrix}$. The possible values of d are $1, 3, 3^2, p, 3p, 3^2p, p^2, 3p^2, 3^2p^2$ by Lemma 2.3. Hence, the number of non-conjugate classes of these orbits with Euler formula are $1; 2; p-1$ and $2(p-1)$ for $1, 3^2, p^2, 3^2p^2$; $3, 3p^2$; $p, 3^2p$ and $3p$ respectively. Consequently, the number of orbits of $\Gamma_0(3^2p^2)$ on $\hat{\mathbb{Q}}$ is $4p + 4$. □

Theorem 2.5. *The set* $\hat{\mathbb{Q}}(3^2p^2) := \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cup \begin{pmatrix} 1 \\ 3 \end{pmatrix} \cup \begin{pmatrix} 2 \\ 3 \end{pmatrix} \cup \begin{pmatrix} 1 \\ 3^2 \end{pmatrix} \cup \begin{pmatrix} 1 \\ p^2 \end{pmatrix} \cup \begin{pmatrix} 1 \\ 3p^2 \end{pmatrix} \cup \begin{pmatrix} 2 \\ 3p^2 \end{pmatrix} \cup \begin{pmatrix} 1 \\ 3^2p^2 \end{pmatrix}$, *is a maximal orbit of* $Nor(3^2p^2)$ *on* $\hat{\mathbb{Q}}$.

Proof. Let us consider the orbit $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ under the action of the elements of $Nor(3^2p^2)$. For the element T_1, taking into account $det(T_1)$, we suppose that a, d-odd, one of b and c is even . Hence,

(i) If $3 \nmid d$; then $T_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3a+b \\ 3(3p^2c+d) \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

(ii) If $3 \nmid d$ and $3a + b$-even; then $T_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2a_0 \\ 3(3p^2c+d) \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

(iii) If $3 \parallel d$; then $T_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3a+b \\ 3^2(p^2c+d_0) \end{pmatrix} = \begin{pmatrix} 1 \\ 3^2 \end{pmatrix}$.

For the element T_2, taking into account $det(T_2)$, we suppose that a, d-odd, one of b and c is even . Hence,

(iv) If $3 \nmid d$; then $T_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3ap^2+b \\ 3p^2(3c+d) \end{pmatrix} = \begin{pmatrix} 1 \\ 3p^2 \end{pmatrix}$.

(v) If $3 \nmid d$ and $3ap^2 + b$-even; then $T_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2a_0 \\ 3p^2(3c+d) \end{pmatrix} = \begin{pmatrix} 2 \\ 3p^2 \end{pmatrix}$.

(vi) If $3 \parallel d$; then $T_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3ap^2 + b \\ 3^2p^2(c + d_0) \end{pmatrix} = \begin{pmatrix} 1 \\ 3^2p^2 \end{pmatrix}$.

Lastly, we suppose that a, d-even and b, c-odd for the element T_2.

(vii) If $3 \parallel b$; then $T_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} ap^2 + b_0 \\ p^2(3c + d) \end{pmatrix} = \begin{pmatrix} 1 \\ p^2 \end{pmatrix}$.

Consequently, $(Nor(3^2p^2), \hat{\mathbb{Q}}(3^2p^2))$ is a transitive permutation group. We now consider the imprimitivity of the action of $Nor(3^2p^2)$ on $\hat{\mathbb{Q}}(3^2p^2)$. □

2.2. Imprimitive Action

Lemma 2.6. ([4]) *Let (G, Δ) be a transitive permutation group. (G, Δ) is primitive if and only if G_α,the stabilizer of $\alpha \in \Delta$, is a maximal subgroup of G for each $\alpha \in \Delta$.*

From the above lemma we see that whenever, for some α, $G_\alpha \lneq H \lneq G$, then Ω admits some G-invariant equivalence relation other than the trivial cases. Because of the transitivity, every element of Ω has the form $g(\alpha)$ for some $g \in G$. Thus one of the non-trivial G-invariant equivalence relation on Ω is given as follows:

$$g(\alpha) \approx g'(\alpha) \text{ if and only if } g' \in gH.$$

The number of blocks (equivalence classes) is the index $|G : H|$ and the block containing α is just the orbit $H(\alpha)$.

We can apply these ideas to the case where G is the $Nor(3^2p^2)$ and Δ is $\hat{\mathbb{Q}}(3^2p^2)$ which is the orbit in Theorem 2.5, G_α is the stabilizer of ∞ in $\hat{\mathbb{Q}}(3^2p^2)$; that is, $Nor(3^2p^2)_\infty = \left\langle \begin{pmatrix} 1 & 1/3 \\ 0 & 1 \end{pmatrix} \right\rangle$, and H is $H_0 := \left\langle \Gamma_0(3^2p^2), A, B \right\rangle$ where

$$A = \begin{pmatrix} ap & b/3p \\ 3pc & dp \end{pmatrix} \text{ and } B := \begin{pmatrix} ap & b/p \\ 3^2pc & dp \end{pmatrix}.$$

Clearly, the relation $Nor(3^2p^2)_\infty < H_0 < Nor(3^2p^2)$ produce an imprimitive action as desired.

2.3. Block Design

Lemma 2.7. ([1]) *The index $|Nor(n) : \Gamma_0(n)| = 2^\rho h^2 \tau$, where ρ is the number of prime factors of n/h^2, $\tau = (\frac{3}{2})^{\varepsilon_1}(\frac{4}{3})^{\varepsilon_2}$,*

$$\varepsilon_1 = \begin{cases} 1 & \text{if } 2^2, 2^4, 2^6 \parallel n \\ 0 & \text{otherwise} \end{cases}, \quad \varepsilon_2 = \begin{cases} 1 & \text{if } 9 \parallel n \\ 0 & \text{otherwise} \end{cases}$$

Using Lemma 2.7, we get following easily:

Theorem 2.8. *There are only two blocks which are* $[\infty]$ *and* $[0]$*. These are as following:*

$$[0] := \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cup \begin{pmatrix} 1 \\ 3 \end{pmatrix} \cup \begin{pmatrix} 2 \\ 3 \end{pmatrix} \cup \begin{pmatrix} 1 \\ 3^2 \end{pmatrix} \text{ and } [\infty] := \begin{pmatrix} 1 \\ p^2 \end{pmatrix} \cup \begin{pmatrix} 1 \\ 3p^2 \end{pmatrix} \cup \begin{pmatrix} 2 \\ 3p^2 \end{pmatrix} \cup \begin{pmatrix} 1 \\ 3^2p^2 \end{pmatrix}.$$

Proof. First, let us calculate the index $|Nor(3^2p^2) : \Gamma_0(3^2p^2)|$ using Lemma 2.7. Since $h = 3$, we have $\rho = 1$. As $3^2 \parallel 3^2p^2$, then $\varepsilon_1 = 0$, $\varepsilon_2 = 1$. Hence, it can be concluded that $|Nor(3^2p^2) : \Gamma_0(3^2p^2)| = 2.3^2.\frac{4}{3} = 24$.

Second, we calculate the index $|H_0 : \Gamma_0(3^2p^2)|$ using [1]. It is known that $A^6 \in \Gamma_0(3^2p^2) \Leftrightarrow a+d \neq 3k$ $(k \in \mathbb{Z})$ and that $B^2 \in \Gamma_0(3^2p^2)$. Hence, we have that

$$\{I, A, A^2, A^3, A^4, A^5\} \times \{I, B\} = \{I, B, A, \ldots, AB, \ldots, A^5B\}$$

as cosets. So, we obtain that $|H_0 : \Gamma_0(3^2p^2)| = 12$. Using the equation $|Nor(3^2p^2) : \Gamma_0(3^2p^2)| = |Nor(3^2p^2) : H_0|.|H_0 : \Gamma_0(3^2p^2)|$, we have that$|Nor(3^2p^2) : H_0| = 2$ and that

$$Nor(3^2p^2) = H_0 \cup \begin{pmatrix} a & b/3 \\ 3p^2c & d \end{pmatrix} H_0.$$

As we observed in Theorem 2.8, the orbit $\hat{\mathbb{Q}}(3^2p^2)$ is divided into two blocks as the statement of the theorem taking into account orbit $\begin{pmatrix}1\\1\end{pmatrix}$ under the action of elements of H_0. □

2.4. Suborbital Graphs

In [16], Sims introduced the idea of the suborbital graphs of a permutation group G acting on a set Δ, these are graphs with vertex-set Δ, on which G induces automorphisms. We summarize Sims'theory as follows: Let (G, Δ) be transitive permutation group. Then G acts on $\Delta \times \Delta$ by $g(\alpha, \beta) = (g(\alpha), g(\beta))$ for $g \in G$ and $\alpha, \beta \in \Delta$. The orbits of this action are called *suborbitals* of G. The orbit containing (α, β) is denoted by $O(\alpha, \beta)$. From $O(\alpha, \beta)$ we can form a *suborbital graph* $G(\alpha, \beta)$: its vertices are the elements of Δ, and there is a directed edge from γ to δ if $(\gamma, \delta) \in O(\alpha, \beta)$. A directed edge from γ to δ is denoted by $(\gamma \to \delta)$. If $(\gamma, \delta) \in O(\alpha, \beta)$, then we will say that there exists an edge $(\gamma \to \delta)$ in $G(\alpha, \beta)$ and represent them as hyperbolic geodesics in the upper half plane $\mathbb{H} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$.

If $\alpha = \beta$, the corresponding suborbital graph $G(\alpha, \alpha)$, called the trivial suborbital graph, is *self-paired*: it consists of a loop based at each vertex $\alpha \in \Delta$. By a *circuit* of length m (or a closed edge path), we mean a sequence $v_1 \to v_2 \to \cdots \to v_m \to v_1$ such that $v_i \neq v_j$ for $i \neq j$, where $m \geq 3$. If $m = 3$ or 4 then the circuit is called a triangle or rectangle.

In this study, G and Δ will be the normalizer of $\Gamma_0(N)$ in $PSL(2, \mathbb{R})$ and the extended rational $\hat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$, respectively. Since rational numbers are well ordered, we also used the notations $\gamma \xrightarrow{>} \delta$ or $\gamma \xrightarrow{<} \delta$ according to the order of vertices.

$Nor(3^2p^2)$ acts transitively on $\hat{\mathbb{Q}}(3^2p^2)$, every suborbital $O(\alpha, \beta)$ contains a pair $(\infty, u/p^2)$ for $u/p^2 \in \hat{\mathbb{Q}}(3^2p^2)$. As $Nor(3^2p^2)$ permutes the blocks transitively, all subgraphs corresponding to blocks are isomorphic. Therefore we will only consider the subgraph $F(\infty, u/p^2)$ of $G(\infty, u/p^2)$ whose vertices form the block $[\infty]$.

Theorem 2.9. (Edge condition) *Let r/s and x/y be in the block* $[\infty]$. *Then there is an edge $r/s \to x/y$ in $F(\infty, u/p^2)$ if and only if*

(i) *If* $3^2p^2 \parallel s$, *then* $x \equiv \pm ur \pmod{p^2}, y \equiv \pm us \pmod{p^2}, ry - sx = \pm p^2$

(ii) *If* $3p^2 \parallel s$, *then* $x \equiv \pm 3ur \pmod{p^2}, y \equiv \pm 3us \pmod{p^2}, ry - sx = \pm 3p^2$

(iii) *If* $p^2 \parallel s$, *then* $x \equiv \pm 9ur \pmod{p^2}, y \equiv \pm 9us \pmod{p^2}, ry - sx = \pm p^2$,

(Plus and minus sign correspond to $r/s > x/y$ and $r/s < x/y$, respectively).

Proof. Assume first that $r/s \xrightarrow{>} x/y$ is an edge in $F(\infty, u/p^2)$. It means that there exists some T in the normalizer $Nor(3^2p^2)$ such that T sends the pair $(\infty, u/p^2)$ to the pair $(r/s, x/y)$, that is $T(\infty) = r/s$ and $T(u/p^2) = x/y$.

Case 1. If $3^2p^2 \parallel s$, taking into account that $T = \begin{pmatrix} a & b/3 \\ 3p^2c & d \end{pmatrix}$, suppose that $3 \nmid a$ and $3 \parallel b, c$ by the equation $ad - bcp^2 = 1$. $T(\infty) = \dfrac{a}{3^2p^2c} = \dfrac{r}{s}$ gives that $r = a$ and $s = 3^2p^2c_0$. $T(u/p^2) = \dfrac{au + b_0p^2}{3^2p^2c_0u + dp^2} = \dfrac{x}{y}$ gives that $x \equiv ur \pmod{p^2}, y \equiv us \pmod{p^2}$. Furthermore, we get $ry - sx = p^2$ from the equation

$$\begin{pmatrix} a & b/3 \\ 3p^2c & d \end{pmatrix}\begin{pmatrix} 1 & u \\ 0 & p^2 \end{pmatrix} = \begin{pmatrix} a & au + bp^2/3 \\ 3p^2c & 3p^2cu + dp^2 \end{pmatrix} = \begin{pmatrix} r & s \\ x & y \end{pmatrix}.$$

Case 2. If $3p^2 \parallel s$, taking into account that $T = \begin{pmatrix} a & b/3 \\ 3p^2c & d \end{pmatrix}$, suppose that $3 \nmid a, b, c$ by the equation $ad - bcp^2 = 1$. $T(\infty) = \dfrac{a}{3^2p^2c} = \dfrac{r}{s}$ gives that $r = a$ and $s = 3p^2c$. $T(u/p^2) = \dfrac{au + bp^2/3}{3p^2cu + dp^2} = \dfrac{3au + bp^2}{3^2p^2cu + 3dp^2} = \dfrac{x}{y}$ gives that $x \equiv 3ur \pmod{p^2}, y \equiv 3us \pmod{p^2}$. Furthermore, we get $ry - sx = 3p^2$ from the equation

$$\begin{pmatrix} a & b/3 \\ 3p^2c & d \end{pmatrix}\begin{pmatrix} 1 & u \\ 0 & p^2 \end{pmatrix} = \begin{pmatrix} a & au+bp^2/3 \\ 3p^2c & 3p^2cu+dp^2 \end{pmatrix} = \begin{pmatrix} r & x \\ s & y \end{pmatrix}.$$

Case 3. If $p^2 \parallel s$, taking into account that $T = \begin{pmatrix} a & b/3 \\ 3p^2c & d \end{pmatrix}$, suppose that $3 \parallel a$ and $3 \nmid b, c$ by the equation $ad - bcp^2 = 1$. $T(\infty) = \frac{a}{3p^2c} = \frac{r}{s}$ gives that $r = a_0$ and $s = p^2c$. $T(u/p^2) = \frac{au + bp^2/3}{3p^2cu + dp^2} = \frac{9a_0u + bp^2}{9p^2cu + 3dp^2} = \frac{x}{y}$ gives that $x \equiv 9ur \pmod{p^2}$, $y \equiv 9us \pmod{p^2}$. Furthermore, we get $ry - sx = p^2$ from the equation

$$\begin{pmatrix} a & b/3 \\ 3p^2c & d \end{pmatrix}\begin{pmatrix} 1 & u \\ 0 & p^2 \end{pmatrix} = \begin{pmatrix} a & au+bp^2/3 \\ 3p^2c & 3p^2cu+dp^2 \end{pmatrix} = \begin{pmatrix} r & x \\ s & y \end{pmatrix}.$$

For the opposite direction, we assume that $3^2p^2 \parallel s$ and $x \equiv ur \pmod{p^2}$, $y \equiv us \pmod{p^2}$, $ry - sx = p^2$. In this case, there exist $b, d \in \mathbb{Z}$ such that $x = ur + bp^2$ and $y = us + dp^2$. If we put these equivalences in $ry - sx = p^2$, we obtain $rd - bs = 1$. So the element $T_0 = \begin{pmatrix} r & b \\ s & d \end{pmatrix}$ is clearly in H_0. For minus sign and another conditions, similar calculations are done. □

2.5. Circuit Condition

It is known that a graph which contains no circuit is called a forest. In introduction part, we also mentioned that the trivial suborbital graphs are self-paired ones. In this section, we will be mainly interested in the remaining non-trivial suborbital graphs.

Theorem 2.10. *Let $F(\infty, u/p^2)$ contains a triangle if and only if $9u^2 \pm 3u + 1 \equiv 0 \pmod{p^2}$.*

Proof. We suppose that there is a triangle such as $\frac{k}{l} \to \frac{m}{n} \to \frac{x}{y} \to \frac{k}{l}$ in $F(\infty, u/p^2)$. Since H_0 permutes the vertices transitively, we may suppose that the triangle has the form $\frac{1}{0} \to \frac{r_0}{s_0p^2} \to \frac{x_0}{y_0p^2} \to \frac{1}{0}$. Furthermore, without loss of generality, suppose $\frac{r_0}{s_0p^2} < \frac{x_0}{y_0p^2}$. From Theorem 2.9.(i), we have that $r_0 \equiv u \pmod{p^2}$ and $s_0 = 1$ from the first edge. Hence, we get the second vertex as $\frac{u}{p^2}$. Applying to Theorem 2.9.(iv) to second edge, we obtain that $x_0 \equiv -9u^2 \pmod{p^2}$ and $uy_0 - x_0 = -1$. Taking into account $x_0 = uy_0 + 1$,

Case 1. If $y_0 = 1$, then second edge will be of the form $\frac{u}{p^2} \to \frac{u+1}{p^2}$. By Theorem 2.9, we have that $u + 1 \equiv -9u^2 \pmod{p^2}$ and $9u^2 + 9u + 1 \equiv 0 \pmod{p^2}$ from second and third edge, respectively. These equivalences gives a contradiction as $u \equiv 0 \pmod{p^2}$.

Case 2. If $y_0 = 2$, then second edge will be of the form $\frac{u}{p^2} \to \frac{2u+1}{2p^2}$. In this case, the third edge $\frac{2u+1}{2p^2} \to \frac{1}{0}$ contradict to Theorem 2.9.

Case 3. If $y_0 = 3$, then second edge will be of the form $\frac{u}{p^2} \to \frac{3u+1}{3p^2}$. In this case, we have that $x_0 \equiv -9u^2 \pmod{p^2}$ and $9u^2 + 3u + 1 \equiv 0 \pmod{p^2}$ by Theorem 2.9.

If the inequalities $\frac{r_0}{s_0p^2} > \frac{x_0}{y_0p^2}$ hold then we conclude that $9u^2 - 3u + 1 \equiv 0 \pmod{p^2}$.

For the opposite direction, we assume that $9u^2 \pm 3u + 1 \equiv 0 \pmod{p^2}$. Using Theorem 2.9, it is clear that $\frac{1}{0} \to \frac{u}{p^2} \to \frac{3u \pm 1}{3p^2} \to \frac{1}{0}$ is a triangle in $F(\infty, u/p^2)$. □

Example 2.11. We can use easy number-theoretical techniques to calculate which suborbital graphs contains a triangle. Suppose that p is equal to 13. Since $9u^2 + 3u + 1 \equiv 0 \pmod{13^2}$, then $9u^2 + 3u + 1 \equiv 0 \pmod{13}$, giving $u = 1 + 13k$ such that $k \in \mathbb{Z}$. Hence, we have $9(1 + 13k)^2 + 3(1 + 13k) + 1 \equiv 0 \pmod{169}$, then $1521k^2 + 273k + 13 \equiv 0 \pmod{169}$. As $117k^2 + 21k + 1 \equiv 0 \pmod{13}$, we obtain $k = 8$ and $u = 105$. Since $9(105)^2 + 3(105) + 1 \equiv 0 \pmod{169}$, $F(\infty, 105/169)$ contains a triangle like as in $F(\infty, 6/49)$ for $p = 7$.

3. Conclusion

We know that every prime $p \neq 3$ has the form $3q + 1$ or $3q + 2$ for some integer q. We suppose that $p \equiv 2 \pmod{3}$. In this case $9u^2 + 3u + 1 = 2 + 3t$ for some $t \in \mathbb{Z}$. This equation gives a contradiction that $3 \mid 1$. As a consequence,

Corollary 3.1. *The prime divisors p of $9u^2 \pm 3u + 1$, for any $u \in \mathbb{Z}$, are of the form $p \equiv 1 \pmod{3}$.*

Corollary 3.2. *Let $k/l \to m/n \to x/y \to k/l$ be a triangle in $F(\infty, u/p^2)$. There exists an unique elliptic element Ψ in H_0 of order 3 such that $\Psi(k/l) = m/n$, $\Psi(m/n) = x/y$, $\Psi(x/y) = k/l$.*

Proof. Because of the transitive action, there exists an element Ψ in H_0 maps the triangle $\frac{k}{l} \to \frac{m}{n} \to \frac{x}{y} \to \frac{k}{l}$ to the ideal triangle $\frac{1}{0} \to \frac{u}{p^2} \to \frac{3u \pm 1}{3p^2} \to \frac{1}{0}$ as follows

$$\Psi\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}u\\p^2\end{pmatrix}, \Psi\begin{pmatrix}u\\p^2\end{pmatrix} = \begin{pmatrix}3u \pm 1\\3p^2\end{pmatrix}, \Psi\begin{pmatrix}3u \pm 1\\3p^2\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}.$$

By Theorem 2.10, this means that $9u^2 \pm 3u + 1 \equiv 0 \pmod{p^2}$. Hence,

$$\Psi := \begin{pmatrix} -3^2u & (9u^2 \pm 3u + 1)/p^2 \\ -3^2p^2 & 3^2u + 3 \end{pmatrix}$$

is an elliptic element in H_0 of order 3 and satisfies the desired conditions. Uniqueness is obvious. □

Proof of Corollary 3.1. Let u be any integer and p a prime divisor of $9u^2 \pm 3u + 1$. Then, without any difficulty, it can be easily seen that the normalizer $Nor(3^2p)$, like $Nor(3^2p^2)$, has the elliptic element $\Psi := \begin{pmatrix} -3^2u & (9u^2 \pm 3u + 1)/p \\ -3^2p & 3^2u + 3 \end{pmatrix}$ of order 3. From [2], we get that $p \equiv 1 \pmod{3}$.

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