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Large Deviations for the Discounted Aggregate Claims in Time-Dependent Risk Model with Constant Interest Force

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Abstract. This paper achieves the weakly asymptotic formulas of the large deviations for the discounted aggregate claims in a time-dependent risk model with widely upper orthant dependent and dominatedly-varying-tailed claims, where the time-dependence structure is defined by a conditional tail probability of the claim size given the inter-arrival time before the claim. Further, if the claims are consistently varying tailed or regularly varying tailed, some asymptotic formulas of the large deviations are established.

1. Introduction and main results

In the paper, we consider a time-dependent risk model, in which the claim sizes $\{X_n, n \geq 1\}$ form a sequence of nonnegative, identically distributed, but not necessarily independent, random variables (r.v.s) with common distribution F, and their inter-arrival times $\{\theta_n, n \geq 1\}$, not necessarily independent of $\{X_n, n \geq 1\}$, form a sequence of independent, identically distributed (i.i.d.) and nonnegative r.v.s with common distribution G. To avoid triviality, we assume that neither F nor G is degenerate at 0. Denote the claim-arrival times by $\tau_0 = 0$, $\tau_n = \sum_{i=1}^{n} \theta_i$, $n \geq 1$, which constitute a renewal counting process

$$N(t) = \sup\{n \geq 1, \tau_n \leq t\}, \quad t \geq 0,$$

with a finite mean function $\lambda(t) = EN(t) = \sum_{n=1}^{\infty} P(\tau_n \leq t)$ for any $t \geq 0$. Note that $\lambda(t) \to \infty$ as $t \to \infty$, and $\lambda(t) \sim \lambda t$ when $E\theta_1 = \lambda^{-1} > 0$, namely, $\lim_{t\to\infty} \lambda(t)/\lambda t = 1$. Let $r \geq 0$ be a constant interest force, then the first n discounted aggregate claims and the discounted aggregate claims up to time t are expressed as, respectively,

$$D_r(n) = \sum_{i=1}^{n} X_i e^{-r\tau_i}, \quad n \geq 1, \tag{1}$$

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and

$$D_r(t) = \sum_{i=1}^{N(t)} X_i e^{-r\tau_i}, \quad t \geq 0. \tag{2}$$

Firstly, we introduce some notions and notations. For two positive functions $a(\cdot)$ and $b(\cdot)$, we write $a(x) \lesssim b(x)$ if $\limsup_{x\to\infty} a(x)/b(x) \leq 1$, write $a(x) \gtrsim b(x)$ if $\liminf_{x\to\infty} a(x)/b(x) \geq 1$, write $a(x) \sim b(x)$ if both, write $a(x) = o(1)b(x)$ if $\lim_{x\to\infty} a(x)/b(x) = 0$, write $a(x) \asymp b(x)$ if $0 < \liminf_{x\to\infty} a(x)/b(x) \leq \limsup_{x\to\infty} a(x)/b(x) < \infty$. For a proper distribution V supported on $(-\infty, \infty)$, denote its tail by $\overline{V}(x) = 1 - V(x)$ and its upper and lower Matuszewska indices by, respectively, for any $y > 0$,

$$J_V^+ = -\lim_{y\to\infty} \frac{\log \overline{V}_*(y)}{\log y}, \quad \text{with} \quad \overline{V}_*(y) =: \liminf_{x\to\infty} \frac{\overline{V}(yx)}{\overline{V}(x)},$$

and

$$J_V^- = -\lim_{y\to\infty} \frac{\log \overline{V}^*(y)}{\log y}, \quad \text{with} \quad \overline{V}^*(y) =: \limsup_{x\to\infty} \frac{\overline{V}(yx)}{\overline{V}(x)}.$$

Next we present some common classes of heavy-tailed distributions. Say that a distribution V supported on $[0, \infty)$ belongs to the dominated variation class, denoted by $V \in \mathscr{D}$, if

$$\overline{V}_*(y) > 0 \quad \text{for all} \quad 0 < y < 1;$$

belongs to the consistent variation class, denoted by $V \in \mathscr{C}$, if

$$L_V =: \lim_{y\downarrow 1} \overline{V}_*(y) = 1;$$

belongs to the regular variation class, denoted by $V \in \mathscr{R}_{-\alpha}$, $0 < \alpha < \infty$, if

$$\lim_{x\to\infty} \overline{V}(xy)/\overline{V}(x) = y^{-\alpha} \quad \text{for all} \quad y > 0.$$

More generally, we say that a distribution V supported on $(-\infty, \infty)$ belongs to a distribution class if $V(x)\mathbf{1}_{\{x\geq 0\}}$ belongs to the same class, where $\mathbf{1}_A$ denotes the indicator function of set A. In conclusion,

$$\mathscr{R}_{-\alpha} \subset \mathscr{C} \subset \mathscr{D}.$$

For more details of heavy-tailed distributions and their applications, we refer the readers to Bingham et al. (1987) and Embrechts et al. (1997).

In what follows, we present some dependence structures, among which the first one was introduced by Wang et al. (2013).

Definition 1.1. *Say that r.v.s $\{\xi_n, n \geq 1\}$ are widely upper orthant dependent (WUOD), if there exists a finite positive real sequence $\{g_U(n), n \geq 1\}$ such that for each $n \geq 1$ and for all $x_i \in (-\infty, \infty)$, $1 \leq i \leq n$,*

$$P\left(\bigcap_{i=1}^{n} \{\xi_i > x_i\}\right) \leq g_U(n) \prod_{i=1}^{n} P(\xi_i > x_i). \tag{3}$$

From the definition of WUOD r.v.s, Wang et al. (2013) gave a proposition below.

Proposition 1.1. *(i) Let $\{\xi_n, n \geq 1\}$ be WUOD r.v.s. If $\{f_n(\cdot), n \geq 1\}$ are a sequence of nondecreasing functions, then $\{f_n(\xi_n), n \geq 1\}$ are still WUOD r.v.s.*
(ii) If $\{\xi_n, n \geq 1\}$ are nonnegative and WUOD r.v.s, then for each $n \geq 1$ and any $s > 0$,

$$E \exp\left\{s \sum_{i=1}^{n} \xi_i\right\} \leq g_U(n) \sum_{i=1}^{n} E(e^{s\xi_i}).$$

Denote by X and θ the generic r.v.s of the claim sizes and their inter-claim times, respectively. Asimit and Badescu (2010) proposed a dependence structure for random pair (X,θ). Precisely, there exists some measurable function $h(\cdot):[0,\infty)\longmapsto(0,\infty)$ such that for any $t\geq 0$,

$$P(X>x|\theta=t)\sim P(X>x)h(t),\quad \text{as } x\to\infty. \tag{4}$$

If t is not a possible value of θ, the conditional probability in (4) is understood as an unconditional one, and then $h(t)=1$. We notice that, adopting the term of Li et al. (2010), the dependence structure defined by (4) is called the time-dependence, which allows both positive and negative dependence and is easily verifiable for some common bivariate copulas, see Li et al. (2010).

It is well-known that there are increasing researchers having studied the asymptotics and uniform asymptotics of the ruin-related quantities in the standard or non-standard renewal risk models with constant interest force $r\geq 0$. For example, see Tang (2007), Hao and Tang (2008), Li et al. (2010), Liu et al. (2012), Wang et al. (2013), Gao and Liu (2013), Gao et al. (2015), Jiang et al. (2015), Liu and Gao (2016), among others. Meanwhile, more and more attention has been paid to the large deviations for the aggregate claims with constant interest force $r=0$. For the case that $\{X_n,n\geq 1\}$ are i.i.d. r.v.s with finite mean, some results were obtained, namely,

$$P(S(n)-nEX>x)\sim n\overline{F}(x) \tag{5}$$

holds uniformly for all $x\geq\gamma n$, and

$$P(S(t)-E(S(t))>x)\sim\lambda(t)\overline{F}(x) \tag{6}$$

holds uniformly for all $x\geq\gamma\lambda(t)$. See, for example, Tang et al. (2001), Ng et al. (2003), Ng et al. (2004), and many others. Additionally, the relations (5) and (6) were still extended to the case that $\{X_n,n\geq 1\}$ are dependent r.v.s with finite mean, which can be found in Kaas and Tang (2005), Tang (2006), Liu (2009), Chen et al. (2011), Chen and Yuen (2012), Wang et al. (2012), Yang et al. (2012), Bi and Zhang (2013), He et al. (2013), Tang and Bai (2015), Liu et al. (2017), and references therein.

Motivated by the above references, in this paper we consider a time-dependent risk model with constant interest force $r\geq 0$, the claim sizes following a certain dependence structure and (X,θ) satisfying the dependence structure defined by (4), and study the asymptotic behaviors of the large deviations for discounted aggregate claims. More importantly, in comparison to the corresponding results with (X,θ) mutually independent, our main results successfully capture the impact of the dependence between X and θ.

The main results of this paper are given below, where the claim sizes $\{X_n,n\geq 1\}$ and their inter-arrival times $\{\theta_n,n\geq 1\}$ satisfy a assumption as follows:

Assumption A: *when $m=n\geq 1$, X_n and θ_m are dependent such that (4) holds uniformly for all $t\in(0,\infty)$; when $m\neq n\geq 1$, X_n and θ_m are mutually independent.*

In the first main result, we deal with the large deviations for the first n discounted aggregate claims in the time-dependent risk model.

Theorem 1.1. *Consider the first n discounted aggregate claims described by (1) with WUOD claim sizes $\{X_n,n\geq 1\}$ satisfying $EX<\infty$ and $\sup_{n\geq 1}g_U(n)n^{-\epsilon_0}<\infty$ for some constant $\epsilon_0>0$. If $F\in\mathscr{D}$ with $J_F^->0$, $E\theta=\lambda^{-1}>0$ and Assumption A holds, then for every fixed $\gamma>0$,*

$$\sum_{i=1}^{n}P\left(X_ie^{-r(\theta^*+\tau_{i-1})}>x\right)\lesssim P(D_r(n)>x)\lesssim L_F^{-1}\sum_{i=1}^{n}P\left(X_ie^{-r(\theta^*+\tau_{i-1})}>x\right) \tag{7}$$

holds uniformly for all $x\geq\gamma n$, where θ^* is a r.v., independent of $\{X_n,n\geq 1\}$ and $\{\theta_n,n\geq 1\}$, with a proper distribution given by

$$P(\theta^*\in dt)=h(t)P(\theta\in dt). \tag{8}$$

The uniformity of (7) is understood as

$$\limsup_{n\to\infty}\sup_{x\geq\gamma n}\frac{P(D_r(n)>x)}{\sum_{i=1}^n P\left(X_i e^{-r(\theta^*+\tau_{i-1})}>x\right)}\leq L_F^{-1}$$

and

$$\liminf_{n\to\infty}\inf_{x\geq\gamma n}\frac{P(D_r(n)>x)}{\sum_{i=1}^n P\left(X_i e^{-r(\theta^*+\tau_{i-1})}>x\right)}\geq 1.$$

According to Theorem 1.1, we now propose two special cases when $F\in\mathscr{C}$ and $F\in\mathscr{R}_{-\alpha}$.

Corollary 1.1. *Let the conditions of Theorem 1.1 be true. If $F\in\mathscr{C}$, then for every fixed $\gamma>0$,*

$$P(D_r(n)>x)\sim\sum_{i=1}^n P\left(X_i e^{-r(\theta^*+\tau_{i-1})}>x\right)$$

holds uniformly for all $x\geq\gamma n$, that is

$$\lim_{n\to\infty}\sup_{x\geq\gamma n}\left|\frac{P(D_r(n)>x)}{\sum_{i=1}^n P\left(X_i e^{-r(\theta^*+\tau_{i-1})}>x\right)}-1\right|=0.$$

Furthermore, if $F\in\mathscr{R}_{-\alpha}$, then for every fixed $\gamma>0$,

$$P(D_r(n)>x)\sim\overline{F}(x)\sum_{i=1}^n E\left(e^{-\alpha r(\theta^*+\tau_{i-1})}\right)$$

holds uniformly for all $x\geq\gamma n$.

In the second main results, we turn to consider the large deviations for the discounted aggregate claims up to time t described by relation (2).

Theorem 1.2. *Consider the discounted aggregate claims up to time t described by (2). If the conditions of Theorem 1.1 are true, then for every fixed $\gamma>0$,*

$$\int_{0-}^t\overline{F}(xe^{rs})d\lambda^*(s)\lesssim P(D_r(t)>x)\lesssim L_F^{-1}\int_{0-}^t\overline{F}(xe^{rs})d\lambda^*(s)\tag{9}$$

holds uniformly for all $x\geq\gamma t$, where $\lambda^*(\cdot)$ is the mean function of the delayed renewal process $\{N^*(t),t\geq 0\}$ generated by $\{\theta^*,\theta_n,n\geq 2\}$.

The uniformity of (9) is understood as

$$\limsup_{t\to\infty}\sup_{x\geq\gamma t}\frac{P(D_r(t)>x)}{\int_{0-}^t\overline{F}(xe^{rs})d\lambda^*(s)}\leq L_F^{-1}$$

and

$$\liminf_{t\to\infty}\inf_{x\geq\gamma t}\frac{P(D_r(t)>x)}{\int_{0-}^t\overline{F}(xe^{rs})d\lambda^*(s)}\geq 1.$$

Similarly to Corollary 1.1, Theorem 1.2 can lead to the following corollary.

Corollary 1.2. *Under the conditions of Theorem 1.2, if $F\in\mathscr{C}$, then for every fixed $\gamma>0$,*

$$P(D_r(t)>x)\sim\int_{0-}^t\overline{F}(xe^{rs})d\lambda^*(s)$$

holds uniformly for all $x \geq \gamma t$, that is

$$\lim_{t\to\infty} \sup_{x\geq\gamma t} \left| \frac{P(D_r(t) > x)}{\int_{0-}^{t} \overline{F}(xe^{rs})d\lambda^*(s)} - 1 \right| = 0.$$

Furthermore, if $F \in \mathscr{R}_{-\alpha}$, then for every fixed $\gamma > 0$,

$$P(D_r(t) > x) \sim \overline{F}(x) \int_{0-}^{t} e^{-\alpha rs} d\lambda^*(s)$$

holds uniformly for all $x \geq \gamma t$.

In the rest of this paper, we will prove the main results in Section 3 after giving some lemmas in Section 2.

2. Some Lemmas

In order to prove Theorems 1.1 and 1.2, we now give some lemmas, among which the first one is due to Proposition 2.2.1 of Bingham et al. (1987) and Lemma 3.5 of Tang and Tsitsiashvili (2003).

Lemma 2.1. *For a distribution V supported on $(-\infty, \infty)$, the following assertions hold:*
(i) $V \in \mathscr{D} \Leftrightarrow J_V^+ < \infty \Leftrightarrow L_V > 0$;
(ii) if $V \in \mathscr{D}$, then for all $p > J_V^+$, $x^{-p} = o(1)\overline{V}(x)$ as $x \to \infty$;
(iii) if $V \in \mathscr{D}$, then for all $0 < p_1 < J_V^-$ and $p_2 > J_V^+$, there exist $C_i > 0$ and $D_i > 0$, $i = 1, 2$ such that

$$\frac{\overline{V}(y)}{\overline{V}(x)} \geq C_1 \left(\frac{x}{y}\right)^{p_1}, \quad x \geq y \geq D_1,$$

and

$$\frac{\overline{V}(y)}{\overline{V}(x)} \leq C_2 \left(\frac{x}{y}\right)^{p_2}, \quad x \geq y \geq D_2.$$

The following second lemma comes from Theorem 3.3 of Cline and Samorodnitsky (1994).

Lemma 2.2. *If ξ is a r.v. distributed by $V \in \mathscr{D}$, and η is a nonnegative r.v. independent of ξ and satisfying $E\eta^p < \infty$ for some $p > J_V^+$, then $\overline{V}(x) \asymp P(\xi\eta > x)$ as $x \to \infty$.*

Lemma 2.3. *Let $\{\xi_n, n \geq 1\}$ be WUOD r.v.s with common distribution V and mean 0. If $E(\xi_1^+)^\beta < \infty$ for some $\beta > 1$ and $\sup_{n\geq 1} g_U(n)n^{-\epsilon_0} < \infty$ for some constant $\epsilon_0 > 0$, then for each fixed $\gamma > 0$ and $p > 0$, there exist $\delta > 0$ and $C_3 = C(\delta, \gamma) > 0$, irrespective to x and n, such that for all $x \geq \gamma n$ and $n \geq 1$,*

$$P\left(\sum_{i=1}^{n} \xi_i > x\right) \leq n\overline{V}(\delta x) + C_3 x^{-p}.$$

Proof. For any fixed $\delta > 0$, we write $\widetilde{\xi_i} = \min\{\xi_i, \delta x\}$, $i \geq 1$, which, by Proposition 1.1(i), are still WUOD. Following the proof of Lemma 2.3 of Tang (2006), it suffices to prove that

$$P\left(\sum_{i=1}^{n} \widetilde{\xi_i} > x\right) \leq C_3 x^{-p} \tag{10}$$

holds uniformly for all $x \geq \gamma n$ and all large $n \geq 1$. In fact, for $1 < q < \min\{\beta, 2\}$, by the proof of Lemma 2.3 of Tang (2006), Proposition 1.1(ii) and the WUOD property, we have

$$\begin{aligned} P\left(\sum_{i=1}^{n} \widetilde{\xi}_i > x\right) &\leq g_U(n) \exp\left\{\frac{1}{\delta} + \frac{\delta^{q-1} x^q \overline{V}(\delta x)}{E(\xi_1^+)^q}\right\} \left(\frac{\delta^{q-1} \gamma x^{q-1}}{E(\xi_1^+)^q}\right)^{-\frac{1}{2\delta}} \\ &\leq \sup_{n\geq 1} g_U(n) n^{-\epsilon_0} C_3 x^{-\frac{q-1-2\epsilon_0\delta}{2\delta}}, \end{aligned} \tag{11}$$

where the coefficient C_3 is given by

$$C_3 = \sup_{x\geq 0} \exp\left\{\frac{1}{\delta} + \frac{\delta^{q-1} x^q \overline{V}(\delta x)}{E(\xi_1^+)^q}\right\} \left(\frac{\delta^{q-1} \gamma}{E(\xi_1^+)^q}\right)^{-\frac{1}{2\delta}} < \infty.$$

Hence, with some $\delta > 0$ such that $\frac{q-1-2\epsilon_0\delta}{2\delta} > p$, it follows from (11) that inequality (10) holds for all $x \geq \gamma n$ and all large $n \geq 1$. □

Lemma 2.4. *Consider the time-dependent risk model introduced in Section 1 with claim sizes $\{X_n, n \geq 1\}$ and inter-arrival times $\{\theta_n, n \geq 1\}$ satisfying Assumption A, then for any fixed $\gamma > 0$, it holds uniformly for all $x \geq \gamma n$ that*

$$\sum_{i=1}^{n} P(X_i e^{-r\tau_i} > x) \sim \sum_{i=1}^{n} P\left(X_i e^{-r(\theta^* + \tau_{i-1})} > x\right), \quad \text{as} \quad n \to \infty,$$

where θ^* is the one appearing in Theorem 1.1. Furthermore, if $F \in \mathscr{R}_{-\alpha}$, then for any fixed $\gamma > 0$, it holds uniformly for all $x \geq \gamma n$ that

$$\sum_{i=1}^{n} P(X_i e^{-r\tau_i} > x) \sim \overline{F}(x) \sum_{i=1}^{n} E\left(e^{-\alpha r(\theta^* + \tau_{i-1})}\right), \quad \text{as} \quad n \to \infty.$$

Proof. By Assumption A and relation (8), it holds uniformly for all $x \geq \gamma n$ that

$$\begin{aligned} \sum_{i=1}^{n} P(X_i e^{-r\tau_i} > x) &= \sum_{i=1}^{n} \int_{0-}^{\infty} \int_{0-}^{\infty} P(X_i e^{-r(u+v)} > x | \theta_i = v) dP(\tau_{i-1} \leq u) dG(v) \\ &\sim \sum_{i=1}^{n} \int_{0-}^{\infty} \int_{0-}^{\infty} P(X_i e^{-r(u+v)} > x) h(v) dP(\tau_{i-1} \leq u) dG(v) \\ &= \sum_{i=1}^{n} \int_{0-}^{\infty} \int_{0-}^{\infty} P(X_i e^{-r(u+v)} > x) dP(\tau_{i-1} \leq u) dP(\theta^* \leq v) \\ &= \sum_{i=1}^{n} P\left(X_i e^{-r(\theta^* + \tau_{i-1})} > x\right), \quad \text{as} \quad n \to \infty. \end{aligned} \tag{12}$$

By the second last step of (12) and $F \in \mathscr{R}_{-\alpha}$, it holds uniformly for all $x \geq \gamma n$ that

$$\begin{aligned} \sum_{i=1}^{n} P(X_i e^{-r\tau_i} > x) &\sim \overline{F}(x) \sum_{i=1}^{n} \int_{0^-}^{\infty} \int_{0^-}^{\infty} e^{-\alpha r(u+v)} dP(\tau_{i-1} \leq u) dP(\theta^* \leq v) \\ &= \overline{F}(x) \sum_{i=1}^{n} E\left(e^{-\alpha r(\theta^* + \tau_{i-1})}\right), \quad \text{as} \quad n \to \infty. \end{aligned}$$

This ends the proof of Lemma 2.4. □

Following the proof of Lemma 2.3 of Chen and Yuen (2012) with slight modifications, we find out that Lemma 2.3 of Chen and Yuen (2012) still holds under some dependence structures introduced in Section 1.

Lemma 2.5. *Under the conditions of Theorem 1.2, for any fixed $p > J_F^+$, there exists some constant $C_4 > 0$ such that, uniformly for all $x \geq 0$, $t \geq 0$ and $n \geq 1$,*

$$P(S_n > x, \tau_n \leq t) \leq C_4 n^{p+1}\overline{F}(x)P(\tau_{n-1} \leq t).$$

The lemma below is from Theorem 1(i) of Kočetova et al. (2009).

Lemma 2.6. *Let the inter-arrival times $\{\theta_n, n \geq 1\}$ form a sequence of i.i.d. and nonnegative r.v.s with mean $\lambda^{-1} > 0$. Then it holds for every $a > \lambda$ and some $b > 1$ that*

$$\lim_{t\to\infty}\sum_{n>at} b^n P(\tau_n \leq t) = 0.$$

The last lemma establishes the law of large numbers of $\{N^*(t), t \geq 0\}$, which is appearing in Theorem 1.2.

Lemma 2.7. *If $E\theta = \lambda^{-1} > 0$, then it holds for any $0 < \varepsilon < 1$ and any function $\gamma(\cdot) : [0,\infty) \mapsto (0,\infty)$ with $\gamma(t) \uparrow \infty$ as $t \to \infty$ that*

$$\lim_{t\to\infty}\sup_{x\geq\gamma(t)} P\left(\left|\frac{N^*(t)}{\lambda t} - 1\right| > \varepsilon\right) = 0.$$

Proof. See the proof of Lemma 2.1 of Chen and Yuen (2012). □

3. Proof of Theorems

In this section, we proceed to prove the main results of this paper.

Proof of Theorem 1.1. All limit relationships in this proof are taken as $n \to \infty$ unless stated otherwise. Note that, for any $0 < \delta < 1$,

$$\begin{aligned} P(D_r(n) > x) &= P\left(\sum_{i=1}^{n} X_i e^{-r\tau_i} > x, \bigcup_{1\leq i<j\leq n}\{X_i > \delta x, X_j > \delta x\}\right) \\ &\quad + P\left(\sum_{i=1}^{n} X_i e^{-r\tau_i} > x, \bigcap_{i=1}^{n}\{X_i \leq \delta x\}\right) \\ &\quad + P\left(\sum_{i=1}^{n} X_i e^{-r\tau_i} > x, \bigcup_{i=1}^{n}\{X_i > \delta x, X_j \leq \delta x, 1 \leq j \neq i \leq n\}\right) \\ &=: \sum_{i=1}^{3} I_i(x,n). \end{aligned} \tag{13}$$

For $I_1(x,n)$, by the WUOD property, we have

$$\begin{aligned} &\limsup_{n\to\infty}\sup_{x\geq\gamma n}\frac{I_1(x,n)}{\sum_{i=1}^{n} P(X_i e^{-r(\theta^*+\tau_{i-1})} > x)} \\ &\leq \limsup_{n\to\infty}\sup_{x\geq\gamma n}\frac{\sum_{1\leq i<j\leq n} P(X_i > \delta x, X_j > \delta x)}{nP(X_1 e^{-r\theta^*} > x)} \end{aligned} \tag{14}$$

$$\leq g_U(2)\limsup_{n\to\infty}\sup_{x\geq\gamma n}\frac{\sum_{1\leq i<j\leq n}P(X_i>\delta x)P(X_j>\delta x)}{nP(X_1e^{-r\theta^*}>x)}$$
$$=g_U(2)\limsup_{n\to\infty}\sup_{x\geq\gamma n}\frac{(n\overline{F}(\delta x))^2}{nP(X_1e^{-r\theta^*}>x)}$$
$$\leq g_U(2)\limsup_{n\to\infty}\sup_{x\geq\gamma n}\frac{\overline{F}(x)}{P(X_1e^{-r\theta^*}>x)}\limsup_{n\to\infty}\sup_{x\geq\gamma n}\frac{\overline{F}(\delta x)}{\overline{F}(x)}\limsup_{n\to\infty}\sup_{x\geq\gamma n}n\overline{F}(\delta x)$$
$$\leq\frac{g_U(2)}{\gamma}\limsup_{x\to\infty}\frac{\overline{F}(x)}{P(X_1e^{-r\theta^*}>x)}\limsup_{x\to\infty}\frac{\overline{F}(\delta x)}{\overline{F}(x)}\limsup_{x\to\infty}x\overline{F}(\delta x). \tag{15}$$

The condition $EX<\infty$ implies that

$$\lim_{x\to\infty}x\overline{F}(x)=0. \tag{16}$$

Then, by $F\in\mathscr{D}$, Lemma 2.2, (15) and (16), it holds uniformly for all $x\geq\gamma n$ that

$$I_1(x,n)=o(1)\sum_{i=1}^{n}P(X_ie^{-r(\theta^*+\tau_{i-1})}>x). \tag{17}$$

For $I_2(x,n)$, we write $\widetilde{X}_i=\min\{X_i,\delta x\}$, $i\geq 1$. By Proposition 1.1(i), $\{\widetilde{X}_i-E\widetilde{X}_i, i\geq 1\}$ are still WUOD. Clearly, it holds that

$$\begin{aligned}I_2(x,n) &= P\left(\sum_{i=1}^{n}\widetilde{X}_ie^{-r\tau_i}>x\right)\\ &= P\left(\sum_{i=1}^{n}(\widetilde{X}_i-E\widetilde{X}_i)e^{-r\tau_i}+E\widetilde{X}_1\sum_{i=1}^{n}e^{-r\tau_i}>x\right)\\ &\leq P\left(\sum_{i=1}^{n}(\widetilde{X}_i-E\widetilde{X}_i)e^{-r\tau_i}>\frac{x}{2}\right)+P\left(E\widetilde{X}_1\sum_{i=1}^{n}e^{-r\tau_i}>\frac{x}{2}\right)\\ &\leq P\left(\sum_{i=1}^{n}(\widetilde{X}_i-E\widetilde{X}_i)>\frac{x}{2}\right)+(2E\widetilde{X}_1)^px^{-p}E\left(\sum_{i=1}^{n}e^{-r\tau_i}\right)^p,\end{aligned}$$

where the second term of the last step follows from Markov's inequlity. For any $p>0$ and $n\geq 1$, we have

$$E\left(\sum_{i=1}^{n}e^{-r\tau_i}\right)^p\leq E\left(\sum_{i=1}^{\infty}\left(e^{-r\theta_1}\right)^i\right)^p=E\left(\frac{e^{-r\theta_1}}{1-e^{-r\theta_1}}\right)^p,$$

which, along with (10), yields that

$$I_2(x,n)\leq\left[C_32^p+(2E\widetilde{X}_1)^pE\left(\frac{e^{-r\theta_1}}{1-e^{-r\theta_1}}\right)^p\right]x^{-p}. \tag{18}$$

For notational convenience, we set $C_5=C_32^p+(2E\widetilde{X}_1)^pE(e^{-r\theta_1}/(1-e^{-r\theta_1}))^p$. By (18), $F\in\mathscr{D}$, Lemmas 2.1(ii) and 2.2, we get that

$$\begin{aligned}\limsup_{n\to\infty}\sup_{x\geq\gamma n}\frac{I_2(x,n)}{\sum_{i=1}^{n}P(X_ie^{-r(\theta^*+\tau_{i-1})}>x)} &\leq C_5\limsup_{x\to\infty}\frac{x^{-p}}{P(X_1e^{-r\theta^*}>x)}\\ &\leq C_5\limsup_{x\to\infty}\frac{x^{-p}}{\overline{F}(x)}\limsup_{x\to\infty}\frac{\overline{F}(x)}{P(X_1e^{-r\theta^*}>x)}\\ &= 0.\end{aligned} \tag{19}$$

For $I_3(x,n)$, we see that for any $0<\rho<1$,

$$
\begin{aligned}
I_3(x,n) &= \sum_{i=1}^{n} P\left(X_i e^{-r\tau_i} + \sum_{j=1,j\neq i}^{n} X_j e^{-r\tau_j} > x, X_i > \delta x, \bigcap_{1\le j\neq i\le n}\{X_j \le \delta x\}\right) \\
&\le \sum_{i=1}^{n} P(X_i e^{-r\tau_i} > (1-\rho)x) + \sum_{i=1}^{n} P\left(\sum_{j=1,j\neq i}^{n} \widetilde{X}_j e^{-r\tau_j} > \rho x\right) \\
&\le \sum_{i=1}^{n} P(X_i e^{-r\tau_i} > (1-\rho)x) + nP\left(\sum_{j=1}^{n} \widetilde{X}_j e^{-r\tau_j} > \rho x\right) \\
&=: I_{31}(x,n) + I_{32}(x,n). \qquad (20)
\end{aligned}
$$

For $I_{31}(x,n)$, by $F\in\mathscr{D}$, Assumption A and Lemma 2.4, it holds uniformly for all $x\ge\gamma n$ that

$$
\begin{aligned}
& I_{31}(x,n) \sim \sum_{i=1}^{n} P(X_i e^{-r(\theta^*+\tau_{i-1})} > (1-\rho)x) \\
&= \sum_{i=1}^{n} \int_{0^-}^{\infty}\int_{0^-}^{\infty} P(X_i e^{-r(u+v)} > (1-\rho)x) dP(\tau_{i-1}\le u) dP(\theta^*\le v) \\
&= \sum_{i=1}^{n} \int_{0^-}^{\infty}\int_{0^-}^{\infty} \left(\frac{P(X_i e^{-r(u+v)} > x)}{P(X_i e^{-r(u+v)} > (1-\rho)x)}\right)^{-1} P(X_i e^{-r(u+v)} > x) dP(\tau_{i-1}\le u) dP(\theta^*\le v) \\
&\lesssim \left(\overline{F}_*\left((1-\rho)^{-1}\right)\right)^{-1} \sum_{i=1}^{n} \int_{0^-}^{\infty}\int_{0^-}^{\infty} P(X_i e^{-r(u+v)} > x) dP(\tau_{i-1}\le u) dP(\theta^*\le v) \\
&= \left(\overline{F}_*\left((1-\rho)^{-1}\right)\right)^{-1} \sum_{i=1}^{n} P\left(X_i e^{-r(\theta^*+\tau_{i-1})} > x\right),
\end{aligned}
$$

which, along with the arbitrariness of $0<\rho<1$, proves that, uniformly for all $x\ge\gamma n$,

$$
I_{31}(x,n) \lesssim L_F^{-1} \sum_{i=1}^{n} P\left(X_i e^{-r(\theta^*+\tau_{i-1})} > x\right). \qquad (21)
$$

For $I_{32}(x,n)$, choosing p in (18) such that $p > J_F^+ + 1$ and using Lemma 2.2 and (18), we have

$$
\begin{aligned}
& \limsup_{n\to\infty} \sup_{x\ge\gamma n} \frac{I_{32}(x,n)}{\sum_{i=1}^{n} P(X_i e^{-r(\theta^*+\tau_{i-1})} > x)} \\
&\le \limsup_{n\to\infty} \sup_{x\ge\gamma n} \frac{C_5 \rho^{-p} x^{-(p-1)}}{\gamma \overline{F}(x)} \limsup_{n\to\infty} \sup_{x\ge\gamma n} \frac{\overline{F}(x)}{P(X_1 e^{-r\theta^*} > x)} \\
&= 0. \qquad (22)
\end{aligned}
$$

Hence, we substitute (21) and (22) into (20) to obtain that, uniformly for all $x\ge\gamma n$,

$$
I_3(x,n) \lesssim L_F^{-1} \sum_{i=1}^{n} P\left(X_i e^{-r(\theta^*+\tau_{i-1})} > x\right). \qquad (23)
$$

Consequently, by (13), (17), (19) and (23), it holds uniformly for all $x\ge\gamma n$ that

$$
P(D_r(n) > x) \lesssim L_F^{-1} \sum_{i=1}^{n} P\left(X_i e^{-r(\theta^*+\tau_{i-1})} > x\right). \qquad (24)
$$

On the other hand, we derive by (13) that

$$\begin{aligned}
P(D_r(n) > x) &\geq P\left(\sum_{i=1}^{n} X_i e^{-r\tau_i} > x, \bigcup_{i=1}^{n} \{X_i > \delta x, X_j \leq \delta x, 1 \leq j \neq i \leq n\}\right) \\
&= \sum_{i=1}^{n} P\left(\sum_{i=1}^{n} X_i e^{-r\tau_i} > x, X_i > \delta x, \bigcap_{1\leq j\neq i\leq n} \{X_j \leq \delta x\}\right) \\
&\geq \sum_{i=1}^{n} P\left(X_i e^{-r\tau_i} > x, X_i > \delta x, \bigcap_{1\leq j\neq i\leq n} \{X_j \leq \delta x\}\right) \\
&\geq \sum_{i=1}^{n} P\left(X_i e^{-r\tau_i} > x, \bigcap_{1\leq j\neq i\leq n} \{X_j \leq \delta x\}, e^{-r\tau_i} \leq \delta^{-1}\right) \\
&\quad + \sum_{i=1}^{n} P\left(X_i > \delta x, \bigcap_{1\leq j\neq i\leq n} \{X_j \leq \delta x\}, e^{-r\tau_i} > \delta^{-1}\right) \\
&\geq \sum_{i=1}^{n} P\left(X_i e^{-r\tau_i} > x, e^{-r\tau_i} \leq \delta^{-1}\right) + \sum_{i=1}^{n} P\left(X_i > \delta x, e^{-r\tau_i} > \delta^{-1}\right) \\
&\quad - \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} P\left(X_i e^{-r\tau_i} > x, X_j > \delta x, e^{-r\tau_i} \leq \delta^{-1}\right) \\
&\quad - \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} P\left(X_i > \delta x, X_j > \delta x, e^{-r\tau_i} > \delta^{-1}\right) \\
&\geq \sum_{i=1}^{n} P\left(X_i e^{-r\tau_i} > x\right) - \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} P\left(X_i > \delta x, X_j > \delta x\right) \\
&=: I_4(x,n) - I_5(x,n).
\end{aligned} \tag{25}$$

For $I_4(x,n)$, by Lemma 2.4, it holds uniformly for all $x \geq \gamma n$ that

$$I_4(x,n) \sim \sum_{i=1}^{n} P\left(X_i e^{-r(\theta^* + \tau_{i-1})} > x\right). \tag{26}$$

For $I_5(x,n)$, similarly to the derivation of (17), we obtain that, uniformly for all $x \geq \gamma n$,

$$I_5(x,n) = o(1) \sum_{i=1}^{n} P(X_i e^{-r(\theta^* + \tau_{i-1})} > x). \tag{27}$$

Thus, from (25) to (27), it holds uniformly for all $x \geq \gamma n$ that

$$P(D_r(n) > x) \gtrsim \sum_{i=1}^{n} P\left(X_i e^{-r(\theta^* + \tau_{i-1})} > x\right).$$

This, along with (24), implies that relation (7) holds uniformly for all $x \geq \gamma n$. □

Proof of Theorem 1.2. In this proof, all limit relationships are taken as $t \to \infty$ without special statement. On the one hand, for any $t \geq 0$ and $i \geq 1$,

$$\sum_{i=1}^{\infty} P\left(X_i e^{-r(\theta^*+\tau_{i-1})} > x\right) = \sum_{i=1}^{\infty} P\left(X_i e^{-r(\theta^*+\tau_{i-1})} \mathbf{1}_{\{\theta^*+\tau_{i-1}\leq t\}} > x\right) + \sum_{i=1}^{\infty} P\left(X_i e^{-r(\theta^*+\tau_{i-1})} \mathbf{1}_{\{\theta^*+\tau_{i-1}> t\}} > x\right). \tag{28}$$

By $F \in \mathscr{D}$ and Lemma 2.1(iii), we have

$$\begin{aligned}
&\limsup_{t\to\infty} \sup_{x\geq\gamma t} \frac{\sum_{i=1}^{\infty} P\left(X_i e^{-r(\theta^*+\tau_{i-1})} \mathbf{1}_{\{\theta^*+\tau_{i-1}> t\}} > x\right)}{\sum_{i=1}^{\infty} P\left(X_i e^{-r(\theta^*+\tau_{i-1})} \mathbf{1}_{\{\theta^*+\tau_{i-1}\leq t\}} > x\right)} \\
&= \limsup_{t\to\infty} \sup_{x\geq\gamma t} \frac{\sum_{i=1}^{\infty} \int_t^{\infty} \overline{F}(xe^{rs}) dP(\theta^*+\tau_{i-1}\leq s)}{\sum_{i=1}^{\infty} \int_{0-}^{t} \overline{F}(xe^{rs}) dP(\theta^*+\tau_{i-1}\leq s)} \\
&= \limsup_{t\to\infty} \sup_{x\geq\gamma t} \frac{\sum_{i=1}^{\infty} \int_t^{\infty} \left(\frac{\overline{F}(x)}{\overline{F}(xe^{rs})}\right)^{-1} dP(\theta^*+\tau_{i-1}\leq s)}{\sum_{i=1}^{\infty} \int_{0-}^{t} \left(\frac{\overline{F}(x)}{\overline{F}(xe^{rs})}\right)^{-1} dP(\theta^*+\tau_{i-1}\leq s)} \\
&\leq \limsup_{t\to\infty} \frac{C_2 \sum_{i=1}^{\infty} \int_t^{\infty} e^{-rp_1 s} dP(\theta^*+\tau_{i-1}\leq s)}{C_1 \sum_{i=1}^{\infty} \int_{0-}^{t} e^{-rp_2 s} dP(\theta^*+\tau_{i-1}\leq s)} \\
&= \limsup_{t\to\infty} \frac{C_2 \sum_{i=1}^{\infty} E\left(e^{-rp_1(\theta^*+\tau_{i-1})} \mathbf{1}_{\{\theta^*+\tau_{i-1}> t\}}\right)}{C_1 \sum_{i=1}^{\infty} E\left(e^{-rp_2(\theta^*+\tau_{i-1})} \mathbf{1}_{\{\theta^*+\tau_{i-1}\leq t\}}\right)} \\
&= 0.
\end{aligned} \tag{29}$$

By (28) and (29), we obtain that, uniformly for all $x \geq \gamma t$,

$$\begin{aligned}
\sum_{i=1}^{\lfloor\lambda t\rfloor} P\left(X_i e^{-r(\theta^*+\tau_{i-1})} > x\right) &\leq \sum_{i=1}^{\infty} P\left(X_i e^{-r(\theta^*+\tau_{i-1})} > x\right) \\
&\sim \sum_{i=1}^{\infty} P\left(X_i e^{-r(\theta^*+\tau_{i-1})} \mathbf{1}_{\{\theta^*+\tau_{i-1}\leq t\}} > x\right),
\end{aligned}$$

where $\lfloor\lambda t\rfloor$ denotes the integer part of λt. On the other hand, for any $t \geq 0$,

$$\sum_{i=1}^{\infty} P\left(X_i e^{-r(\theta^*+\tau_{i-1})} > x\right) = \sum_{i=1}^{\lfloor\lambda t\rfloor} P\left(X_i e^{-r(\theta^*+\tau_{i-1})} > x\right) + \sum_{i=\lfloor\lambda t\rfloor+1}^{\infty} P\left(X_i e^{-r(\theta^*+\tau_{i-1})} > x\right). \tag{30}$$

By $F \in \mathscr{D}$ and Lemma 2.1(iii), we get

$$\begin{aligned}
&\limsup_{t\to\infty} \sup_{x\geq\gamma t} \frac{\sum_{i=\lfloor\lambda t\rfloor+1}^{\infty} P\left(X_i e^{-r(\theta^*+\tau_{i-1})} > x\right)}{\sum_{i=1}^{\lfloor\lambda t\rfloor} P\left(X_i e^{-r(\theta^*+\tau_{i-1})} > x\right)} \\
&= \limsup_{t\to\infty} \sup_{x\geq\gamma t} \frac{\sum_{i=\lfloor\lambda t\rfloor+1}^{\infty} \int_{0^-}^{\infty} \overline{F}(xe^{rs}) dP(\theta^*+\tau_{i-1}\leq s)}{\sum_{i=1}^{\lfloor\lambda t\rfloor} \int_{0^-}^{\infty} \overline{F}(xe^{rs}) dP(\theta^*+\tau_{i-1}\leq s)}
\end{aligned} \tag{31}$$

$$
\begin{aligned}
&= \limsup_{t\to\infty}\sup_{x\geq\gamma t}\frac{\sum_{i=\lfloor\lambda t\rfloor+1}^{\infty}\int_{0^-}^{\infty}\left(\frac{\overline{F}(x)}{\overline{F}(xe^{rs})}\right)^{-1}dP(\theta^*+\tau_{i-1}\leq s)}{\sum_{i=1}^{\lfloor\lambda t\rfloor}\int_{0^-}^{\infty}\left(\frac{\overline{F}(x)}{\overline{F}(xe^{rs})}\right)^{-1}dP(\theta^*+\tau_{i-1}\leq s)}\\
&\leq \limsup_{t\to\infty}\frac{C_2\sum_{i=\lfloor\lambda t\rfloor+1}^{\infty}\int_{0^-}^{\infty}e^{-rp_1s}dP(\theta^*+\tau_{i-1}\leq s)}{C_1\sum_{i=1}^{\lfloor\lambda t\rfloor}\int_{0^-}^{\infty}e^{-rp_2s}dP(\theta^*+\tau_{i-1}\leq s)}\\
&= \limsup_{t\to\infty}\frac{C_2\sum_{i=\lfloor\lambda t\rfloor+1}^{\infty}E\left(e^{-rp_1(\theta^*+\tau_{i-1})}\right)}{C_1\sum_{i=1}^{\lfloor\lambda t\rfloor}E\left(e^{-rp_2(\theta^*+\tau_{i-1})}\right)}\\
&= 0. \qquad (32)
\end{aligned}
$$

By (30) and (32), we show that, uniformly for all $x \geq \gamma t$,

$$
\begin{aligned}
\sum_{i=1}^{\lfloor\lambda t\rfloor}P\left(X_ie^{-r(\theta^*+\tau_{i-1})}>x\right) &\sim \sum_{i=1}^{\infty}P\left(X_ie^{-r(\theta^*+\tau_{i-1})}>x\right)\\
&\geq \sum_{i=1}^{\infty}P\left(X_ie^{-r(\theta^*+\tau_{i-1})}\mathbf{1}_{\{\theta^*+\tau_{i-1}\leq t\}}>x\right).
\end{aligned}
$$

Hence, it follows that, uniformly for all $x \geq \gamma t$,

$$
\begin{aligned}
\sum_{i=1}^{\lfloor\lambda t\rfloor}P\left(X_ie^{-r(\theta^*+\tau_{i-1})}>x\right) &\sim \sum_{i=1}^{\infty}P\left(X_ie^{-r(\theta^*+\tau_{i-1})}\mathbf{1}_{\{\theta^*+\tau_{i-1}\leq t\}}>x\right)\\
&= \int_{0-}^{t}\overline{F}(xe^{rs})d\lambda^*(t).
\end{aligned}
$$

Therefore, we will achieve the proof if we prove that, for every fixed $\gamma > 0$,

$$
\sum_{i=1}^{\lfloor\lambda t\rfloor}P\left(X_ie^{-r(\theta^*+\tau_{i-1})}>x\right)\lesssim P(D_r(t)>x)\lesssim L_F^{-1}\sum_{i=1}^{\lfloor\lambda t\rfloor}P\left(X_ie^{-r(\theta^*+\tau_{i-1})}>x\right) \qquad (33)
$$

holds uniformly for all $x \geq \gamma t$.

Note that, for any small $0 < \varepsilon < 1$,

$$
\begin{aligned}
P(D_r(t)>x) &= P\left(\sum_{i=1}^{N(t)}X_ie^{-r\tau_i}>x, N(t)\leq(1+\varepsilon)\lambda t\right)\\
&\quad +P\left(\sum_{i=1}^{N(t)}X_ie^{-r\tau_i}>x, N(t)>(1+\varepsilon)\lambda t\right)\\
&=: J_1(x,t)+J_2(x,t). \qquad (34)
\end{aligned}
$$

For $J_1(x,t)$, by Theorem 1.1, it holds uniformly for all $x \geq \gamma t$ that

$$
\begin{aligned}
J_1(x,t) &\leq P\left(\sum_{i=1}^{(1+\varepsilon)\lambda t}X_ie^{-r\tau_i}>x\right)\\
&= P\left(\sum_{i=1}^{\lfloor(1+\varepsilon)\lambda t\rfloor}X_ie^{-r\tau_i}>x\right)\\
&\lesssim L_F^{-1}\sum_{i=1}^{\lfloor(1+\varepsilon)\lambda t\rfloor}P\left(X_ie^{-r(\theta^*+\tau_{i-1})}>x\right). \qquad (35)
\end{aligned}
$$

For $J_2(x,t)$, by Lemmas 2.2, 2.5 and 2.6, it holds uniformly for all $x \geq \gamma t$ that

$$\begin{aligned} J_2(x,t) &\leq \sum_{n>(1+\varepsilon)\lambda t} P\left(\sum_{i=1}^{n} X_i > x, N(t)=n\right) \\ &\leq \sum_{n>(1+\varepsilon)\lambda t} P\left(\sum_{i=1}^{n} X_i > x, \tau_n \leq t\right) \\ &\leq C_4\overline{F}(x) \sum_{n>(1+\varepsilon)\lambda t} n^{p+1} P(\tau_{n-1} \leq t) \\ &= o(1)\overline{F}(x) \\ &= o(1)\sum_{i=1}^{\lfloor \lambda t \rfloor} P\left(X_i e^{-r(\theta^*+\tau_{i-1})} > x\right). \end{aligned} \tag{36}$$

Hence, combining (34)-(36) and the arbitrariness of ε, it holds uniformly for all $x \geq \gamma t$ that

$$P(D_r(t) > x) \lesssim L_F^{-1} \sum_{i=1}^{\lfloor \lambda t \rfloor} P\left(X_i e^{-r(\theta^*+\tau_{i-1})} > x\right). \tag{37}$$

On the other hand, for any, but small, $0 < \varepsilon < 1$,

$$\begin{aligned} P(D_r(t) > x) &\geq P\left(\sum_{i=1}^{N(t)} X_i e^{-r\tau_i} > x, (1-\varepsilon)\lambda t \leq N(t) \leq (1+\varepsilon)\lambda t\right) \\ &= \sum_{(1-\varepsilon)\lambda t \leq n \leq (1+\varepsilon)\lambda t} P\left(\sum_{i=1}^{n} X_i e^{-r\tau_i} > x, N(t)=n\right) \\ &\geq \sum_{(1-\varepsilon)\lambda t \leq n \leq (1+\varepsilon)\lambda t} \sum_{i=1}^{n} P\left(X_i e^{-r\tau_i} > x, N(t)=n\right) \\ &\quad - \sum_{(1-\varepsilon)\lambda t \leq n \leq (1+\varepsilon)\lambda t} \sum_{1 \leq i < j \leq n} P\left(X_i e^{-r\tau_i} > x, X_j e^{-r\tau_j} > x, N(t)=n\right) \\ &=: J_3(x,t) - J_4(x,t). \end{aligned} \tag{38}$$

For $J_3(x,t)$, by Assumption A and Lemma 2.7, it holds uniformly for $x \geq \gamma t$ that

$$\begin{aligned} J_3(x,t) &= \sum_{(1-\varepsilon)\lambda t \leq n \leq (1+\varepsilon)\lambda t} \sum_{i=1}^{n} \int_{0^-}^{t} \int_{0^-}^{t-v} P\left(X_i e^{-r(u+v)} > x, N(t-u-v)=n-i \middle| \theta_i = v\right) \\ &\qquad dP(\tau_{i-1} \leq u)dG(v) \\ &= \sum_{(1-\varepsilon)\lambda t \leq n \leq (1+\varepsilon)\lambda t} \sum_{i=1}^{n} \int_{0^-}^{t} \int_{0^-}^{t-v} P\left(X_i e^{-r(u+v)} > x \middle| \theta_i = v\right) \\ &\qquad P(N(t-u-v)=n-i)dP(\tau_{i-1} \leq u)dG(v) \end{aligned} \tag{39}$$

$$
\begin{aligned}
&\sim \sum_{(1-\varepsilon)\lambda t\le n\le(1+\varepsilon)\lambda t}\sum_{i=1}^{n}\int_{0^-}^{t}\int_{0^-}^{t-v} P\left(X_i e^{-r(u+v)}>x\right)\\
&\qquad\qquad P(N(t-u-v)=n-i)h(v)dP(\tau_{i-1}\le u)dG(v)\\
&\ge \sum_{i=1}^{(1-\varepsilon)\lambda t} P\left(X_i e^{-r(\theta^*+\tau_{i-1})}>x\right)P((1-\varepsilon)\lambda t\le N^*(t)\le(1+\varepsilon)\lambda t)\\
&= \sum_{i=1}^{\lfloor(1-\varepsilon)\lambda t\rfloor} P\left(X_i e^{-r(\theta^*+\tau_{i-1})}>x\right)P\left(\left|\frac{N^*(t)}{\lambda t}-1\right|\le\varepsilon\right)\\
&\sim \sum_{i=1}^{\lfloor(1-\varepsilon)\lambda t\rfloor} P\left(X_i e^{-r(\theta^*+\tau_{i-1})}>x\right). \qquad (40)
\end{aligned}
$$

For $J_4(x,t)$, by the WUOD property, (16) and Lemma 2.2, it holds uniformly for all $x\ge\gamma t$ that

$$
\begin{aligned}
J_4(x,t) &\le \sum_{(1-\varepsilon)\lambda t\le n\le(1+\varepsilon)\lambda t}\sum_{1\le i<j\le n} P(X_i>x,X_j>x,N(t)=n)\\
&\le \sum_{1\le i<j\le(1+\varepsilon)\lambda t}\sum_{(1-\varepsilon)\lambda t\le n\le(1+\varepsilon)\lambda t} P(N(t)=n|X_i>x,X_j>x)P(X_i>x,X_j>x)\\
&\le \sum_{1\le i<j\le(1+\varepsilon)\lambda t} P(X_i>x,X_j>x)\\
&\le g_U(2)((1+\varepsilon)\lambda t\overline{F}(x))^2\\
&\le \lambda t\overline{F}(x)\frac{g_U(2)(1+\varepsilon)^2\lambda}{\gamma}x\overline{F}(x)\\
&= o(1)\lambda t\overline{F}(x)\\
&= o(1)\sum_{i=1}^{\lfloor\lambda t\rfloor} P\left(X_i e^{-r(\theta^*+\tau_{i-1})}>x\right). \qquad (41)
\end{aligned}
$$

Therefore, combining (38)-(41) and the arbitrariness of ε, we prove that, uniformly for all $x\ge\gamma t$,

$$
P(D_r(t)>x)\gtrsim\sum_{i=1}^{\lfloor\lambda t\rfloor} P\left(X_i e^{-r(\theta^*+\tau_{i-1})}>x\right),
$$

which, along with (37), implies that (33) holds uniformly for all $x\ge\gamma t$, and then the proof is completed. □

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References

[1] A. Asimit, A. Badescu, Extremes on the discounted aggregate claims in a time dependent risk model, Scand. Actuar. J. 2 (2010) 93–104.

[2] X. Bi, S. Zhang, Precise large deviations of aggregate claims in a risk model with regression-type size-dependence, Stat. Probab. Lett. 83(10) (2013) 2248–2255.

[3] N. H. Bingham, C. M. Goldie, J. L. Teugels, Regular Variation, Cambridge University Press, Cambridge, 1987.

[4] Y. Chen, K. C. Yuen, K. W. Ng, Precise large deviations of random sums in presence of negative dependence and consistent variation, Methodol. Comput. Appl. Probab. 13 (2011) 821–833.

[5] Y. Chen, K. C. Yuen, Precise large deviations of aggregate claims in a size-dependent renewal risk model, Insur. Math. Econ. 51(2) (2012) 457–461.

[6] D. B. H. Cline, G. Samorodnitsky, Subexponentiality of the product of independent random raviables, Stoch. Proc. Appl. 49(1) (1994) 75–98.

[7] P. Embrechts, C. Klüppelberg, T. Mikosch, Modelling extremal events for insurance and finance, Springer, Berlin, 1997.
[8] Q. Gao, X. Liu, Uniform asymptotics for the finite-time ruin probability with upper tail asymptotically independent claims and constant force of interest, Stat. Probab. Lett. 83 (2013) 1527–1538.
[9] Q. Gao, E. Zhang, N. Jin, The ultimate ruin probability of a dependent delayed-claim risk model perturbed by diffusion with constant force of interest, Bull. Korean Math. Soc. 52(3) (2015) 895–906.
[10] X. Hao, Q. Tang, A uniform asymptotic estimate for discounted aggregate claims with subexponential tails, Insur. Math. Econ. 43 (2008) 116–120.
[11] W. He, D. Cheng, Y. Wang, Asymptotic lower bounds of precise large deviations with nonnegative and dependent random variables, Statist. Probab. Lett. 83 (2013) 331–338.
[12] T. Jiang, Y. Wang, Y. Chen, H. Xu, Uniform asymptotic estimate for finite-time ruin probabilities of a time-dependent bidimensional renewal model, Insur. Math. Econ. 64 (2015) 45–53.
[13] R. Kaas, Q. Tang, A large deviation result for aggregate claims with dependent claim occurrences, Insur. Math. Econ. 36(3) (2005) 251–259.
[14] J. Kočetova, R. Leipus, J. Šiaulys, A property of the renewal counting process with application to the finite-time ruin probability, Lith. Math. J. 49 (2009) 55–61.
[15] J. Li, Q. Tang, R. Wu, Subexponential tails of discounted aggregate claims in a time-dependent renewal risk model, Adv. Appl. Probab. 42 (2010) 1126–1146.
[16] L. Liu, Precise large deviations for dependent variables with heavy tails, Stat. Probab. Lett. 79(9) (2009) 1290–1298.
[17] X. Liu, Q. Gao, Y. Wang, A note on a dependent risk model with constant interest rate, Statist. Probab. Lett. 82 (2012) 707–712.
[18] X. Liu, Q. Gao, Uniformly asymptotic behavior for the tail probability of discounted aggregate claims in the time-dependent risk model with upper tail asymptotically independent claims, Commun. Statist. Theory Method. 45(18) (2016) 5341–5354.
[19] X. Liu, C. Yu, Q. Gao, Precise large deviations of aggregate claim amount in a dependent renewal risk model, Commun. Statist. Theory Method. 46(5) (2017) 2354–2363.
[20] K. W. Ng, Q. Tang, J. Yan, H. Yang, Precise large deviations for the prospective-loss process, J. Appl. Probab. 40(2) (2003) 391–400.
[21] K. W. Ng, Q. Tang, J. Yan, H. Yang, Precise large deviations for sums of random variables with consistently varying tails, J. Appl. Probab. 41(1) (2004) 93–107.
[22] F. Tang, J. Bai, Precise large deviations for aggregate loss process in a multi-risk model (ENG), J. Korean Math. Soc. 52(3) (2015) 447–467.
[23] Q. Tang, C. Su, T. Jiang, J. Zhang, Large deviations for heavy-tailed random sums in compound renewal model, Statist. Probab. Lett. 52(1) (2001) 91–100.
[24] Q. Tang, G. Tsitsiashvili, Precise estimates for the ruin probability in finite horizon in a discrete-time model with heavy-tailed insurance and financial risks, Stoch. Proc. Appl. 108 (2003) 299–325.
[25] Q. Tang, Insensitivity to negative dependence of the asymptotic behavior of precise large deviations, Electron. J. Probab. 11 (2006) 107–120.
[26] Q. Tang, Heavy tails of discounted aggregrate claims in the continuous-time renewal model, J. Appl. Probab. 44 (2007) 285–294.
[27] K. Wang, Y. Yang, J. Lin, Precise large deviations for widely orthant dependent random variables with dominatedly varying tails, Front. Math. China 7(5) (2012) 919–932.
[28] K. Wang, Y. Wang, Q. Gao, Uniform asymptotics for the finite-time ruin probability of a dependent risk model with a constant interest rate, Methodol. Comput. Appl. Probab. 15(1) (2013) 109–124.
[29] Y. Yang, L. Remigijus, J. Šiaulys, Precise large deviations for compound random sums in the presence of dependence structures, Comput. Math. Appl. 64 (2012) 2074–2083.