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Existence and Global Attractivity of Positive Periodic Solutions for a Delayed Predator-Prey Model with Mutual Interference and Functional Response

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Abstract. In this paper, by using comparison principle of differential equations, continuation theorem of coincidence degree theory and Lyapunov function, a delayed predator-prey model with mutual interference and functional response is studied. Some sufficient conditions which guarantee the permanence of positive solutions of the model and the existence and global attractivity of a positive periodic solution of the model are obtained. Some results in the related literature are extended. Furthermore, some numerical simulations have been performed to substantiate our analytical findings.

1. Introduction

Predator-prey model is one of the dominant theme in both ecology and mathematical ecology due to its universal existence and importance with many concerned biological systems [1]. In 1971, During his research of the capturing behavior between two populations, Hassell [2] established a general predator-prey model by considering the factors of density dependence, functional response and mutual interference as follows

$$\begin{cases} \dot{x} = xg(x) - p(x)y^m, \\ \dot{y} = y(-s + cp(x)y^{m-1} - q(y)) \end{cases} \tag{1}$$

where $x(t)$ and $y(t)$ stand for the population densities of the prey and the predator at time t, respectively, m $(0 < m \leq 1)$ is mutual interference constant, $p(x)$ is the predator functional response to prey. In recent years, the dynamic of special types of (1) have been discussed by many researchers [3-10]. In 2008, Wang and Zhu

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[3] discussed the global attractivity of positive periodic solution of the following model

$$\begin{cases} \dot{x}(t) = x(t)\Big(r_1(t) - b_1(t)x(t)\Big) - \dfrac{c_1(t)x(t)}{k + x(t)}y^m(t), \\ \dot{y}(t) = y(t)\Big(-r_2(t) - b_2(t)y(t)\Big) + \dfrac{c_2(t)x(t)}{k + x(t)}y^m(t), \end{cases} \quad (0 < m < 1). \tag{2}$$

Then, in [5], they investigated a predator-prey model with modified Leslie-Gower Holling-type II schemes

$$\begin{cases} \dot{x}(t) = x(t)\Big(r_1(t) - b(t)x(t) - \dfrac{a_1(t)y(t)}{k_1 + x(t)}\Big), \\ \dot{y}(t) = y(t)\Big(r_2(t) - \dfrac{a_2(t)y(t)}{k_2 + x(t)}\Big). \end{cases} \tag{3}$$

In 2010, Wang et al. [8] investigated predator-prey model with mutual interference and Holling III type functional response

$$\begin{cases} \dot{x}(t) = x(t)\Big(r_1(t) - b_1(t)x(t)\Big) - \dfrac{c_1(t)x^2(t)}{k^2 + x^2(t)}y^m(t), \\ \dot{y}(t) = y(t)\Big(-r_2(t) - b_2(t)y(t)\Big) + \dfrac{c_2(t)x^2(t)}{k^2 + x^2(t)}y^m(t), \end{cases} \quad (0 < m < 1). \tag{4}$$

In 2011, Lv and Du [9] also discussed the model (4) and improved the main conditions in [8]. Later, the permanence and existence of a unique globally attractive positive almost periodic solution of the model (4) were considered by Zhang et al. [10].

As a matter of fact, the predation efficiency of predator was effected not only by the density of prey but also by itself. Therefore Rosenzweig and MacArthur [11] expressed the predator functional response by $\Phi(x, y)$ and obtained a more realistic predator-prey model

$$\begin{cases} \dot{x} = xg(x) - y^m\Phi(x, y), \\ \dot{y} = y\Big(-s + cy^{m-1}\Phi(x, y) - q(y)\Big), \end{cases} \quad (0 < m \le 1). \tag{5}$$

The predator-prey model with mutual interference and Beddington-DeAngelis functional response of the following form

$$\begin{cases} \dot{x}(t) = x(t)\Big(r_1(t) - b_1(t)x(t)\Big) - \dfrac{k_1(t)x(t)}{a(t) + b(t)x(t) + c(t)y(t)}y^m(t), \\ \dot{y}(t) = y(t)\Big(-r_2(t) - b_2(t)y(t)\Big) + \dfrac{k_2(t)x(t)}{a(t) + b(t)x(t) + c(t)y(t)}y^m(t), \end{cases} \tag{6}$$

was studied by Lin and Chen [12] for the permanence and existence of a positive almost periodic solution and by Guo and Chen [13] for the existence and global attractivity of positive periodic solution respectively. The other researches on system (6) have been given by Cantrell and Cosner [14], Hwang [15, 16] and Fan and Kuang [17].

In fact, more general predator-prey model is the following Kolmogorov-type (see [18, 19])

$$\begin{cases} \dot{x}(t) = x(t)F_1(x(t), y(t)), \\ \dot{y}(t) = y(t)F_2(x(t), y(t)). \end{cases} \tag{7}$$

For system (7), there are also many authors who considered its dynamic behavior, for instance, we can see the references [20-24] and that cited therein.

On the other hand, any model of species dynamics without delays, as was pointed by Kuang [25], is an approximation at best. The importance and usefulness of time-delays in realistic models were also pointed out in the classical monographs of Macdonald [26] and Gopalsamy [27]. Many scholars, such as Fan [28], Xu, et al. [29], Egami and Hirano [30], Lu [31], Wang [32], Zhu [33], Tripathi, et al. [34], Teng [35] and Wang et al. [36-38], have studied the delay predator-prey model in recent years.

Motivated by the above researches, in this paper, we consider a general class of delayed predator-prey model as the following form

$$\begin{cases} \dot{x}(t) = x(t)\Big(r_1(t) - a_1(t)x(t) - a_2(t)x(t-\tau)\Big) - \dfrac{a_3(t)x^n(t)y^m(t)}{c_1(t) + c_2(t)x^n(t) + c_3(t)y^n(t)}, \\ \dot{y}(t) = y(t)\Big(-r_2(t) - b_1(t)y(t) - b_2(t)y(t-\tau)\Big) + \dfrac{b_3(t)x^n(t)y^m(t)}{d_1(t) + d_2(t)x^n(t) + d_3(t)y^n(t)}, \end{cases} \tag{8}$$

where r_1 is intrinsic growth rate of the prey in the absence of the predator and r_2 is death rate of the predator, a_1 and b_1 are decay rates of the prey and the predator in competition among their own populations, a_2 and b_2 are decay rates of the prey and the predator effected by harmful environmental for a period of past time, a_3 is consumption coefficient for the predator consuming the prey, b_3 is coefficient of transformation from the prey to the predator. $r_i(t)$ $(i = 1, 2)$, $a_i(t)$, $b_i(t)$ $(i = 1, 3)$ and $c_i(t)$, $d_i(t)$ $(i = 1, 2)$ are positive ω-periodic functions, $a_2(t)$, $b_2(t)$, $c_3(t)$ and $d_3(t)$ are nonnegative ω-periodic functions, $t \in \mathbb{R}_+ = [0, +\infty)$, delay $\tau > 0$ and integer $n \geq 2$.

System (8) may be regarded as a delayed Kolmogorov-type system or, more specifically, a delayed Rosenzweig-MacArthur type system. From the viewpoint of Rosenzweig and MacArthur, the functional response $\Phi(x, y)$ should reflect the reality that the predation ability depends not only on the prey numbers but also on the predator density. This predator dependence is demonstrated in Beddington-DeAngelis functional response in system (6), but, is not reflected in systems (2)-(4). In order to describe the complexity of the real predator-prey ecological system, we choose the functional response as the following form

$$\Phi(x, y) = \frac{\alpha(t)x^n(t)}{\beta_1(t) + \beta_2(t)x^n(t) + \beta_3(t)y^n(t)}.$$

The term $\beta_3(t)y^n(t)$ added in the denominator of $\Phi(x, y)$ can reflect the predator dependence. When $n = 1$ there are a lot of valuable literatures, such as [3–5, 12–17, 39–45], in which the behavior of the system has been intensively studied in recent years. When $n = 2$, the functional response is Holling III type and when $\beta_3(t) = 0$, i.e., there is no predator dependence in functional response, its dynamic behavior has been investigated in many articles, however, for the case of $\beta_3(t) \neq 0$, little literature has been found on the research. When n is a general positive integer, the functional response is regarded as Holling $(n + 1)$ type, Wang and Sun [46] studied the following system

$$\begin{cases} \dot{x} = \gamma x(1 - h(x)) - \dfrac{yx^n}{a + x^n}, \\ \dot{y} = y\Big(-e + \mu \dfrac{x^n}{a + x^n}\Big) \end{cases} \tag{9}$$

and gave a necessary and sufficient condition on the uniqueness of limit cycles, which extends the previous relevant results of Sugie et al. [47]. In fact, since $m = 1$, there is no mutual interference in system (9). As far as we know, the research on the Holling $(n + 1)$ type system with predator dependence in functional response is less.

We claim that system (8) is essentially different from systems (6) and (9) because of the influence of the constant n and the mutual interference. In order to show this influence we give the following example.

Example 1.1. In system (6), by selecting

$$r_1(t) = 3.99 + 0.01\sin t,\quad b_1(t) = 2.00 - 0.1\sin t,\quad k_1(t) = 0.011 + 0.001\sin t,$$
$$r_2(t) = 0.41 + 0.01\sin t,\quad b_2(t) = 0.08 - 0.01\sin t,\quad k_2(t) = 0.099 + 0.001\sin t,$$
$$a(t) = 1,\quad b(t) = 2 + \sin t,\quad c(t) = 3 - \sin t,\quad m = 1/2$$

and initial values $(x(0), y(0)) = (1.8, 0.1)$ and $(x(0), y(0)) = (2.3, 1.5)$, we obtain orbits of system (6) as in Fig.1.

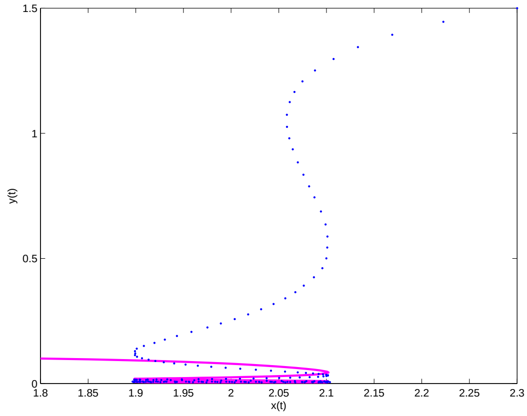

Fig.1. The orbits of system (6).

In system (9), by selecting $h(x) = x$, $\gamma = 3.99 + 0.01\sin t$, $e = 0.41 + 0.01\sin t$, $a = 1$, $\mu = 0.099 + 0.001\sin t$, $n = 4$ and initial values $(x(0), y(0)) = (1.8, 0.1)$ and $(x(0), y(0)) = (2.3, 1.5)$, we obtain orbits of system (9) as in Fig.2.

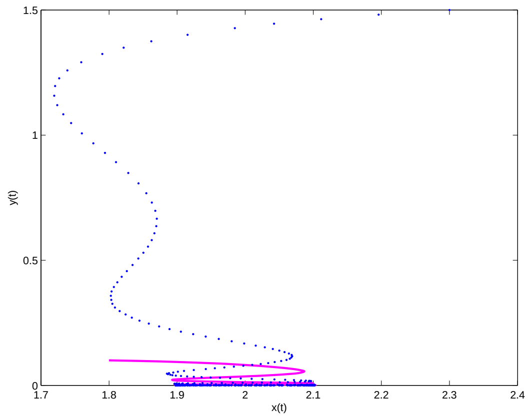

Fig.2. The orbits of system (9).

In system (8), by selecting

$$r_1(t) = 3.99 + 0.01\sin t,\quad a_1(t) = 2.00 - 0.1\sin t,\quad a_2(t) = 0,$$
$$r_2(t) = 0.41 + 0.01\sin t,\quad b_1(t) = 0.08 - 0.01\sin t,\quad b_2(t) = 0,$$
$$a_3(t) = 0.011 + 0.001\sin t,\quad c_1(t) = 1,\quad c_2(t) = 2 + \sin t,\quad c_3(t) = 3 - \sin t,$$
$$b_3(t) = 0.099 + 0.001\sin t,\quad d_1(t) = 1,\quad d_2(t) = 0.26 + 0.01\sin t,\quad d_3(t) = 0.02 - 0.01\sin t,$$
$$n = 4,\quad m = 1/2$$

and initial values $(x(0), y(0)) = (1.8, 0.1)$ and $(x(0), y(0)) = (2.3, 1.5)$, we obtain orbits of system (8) as in Fig.3.

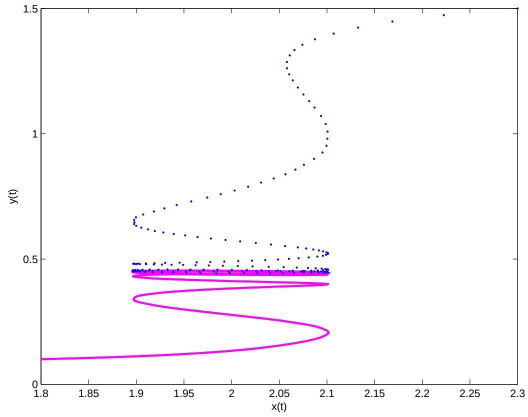

Fig.3. The orbits of system (8).

From Figs. 1-3, we see that, with same assumptions of initial values, systems (6) and (9) are not uniformly persistent because their predators are extinct finally, but systems (8)(with no delay) has positive periodic orbits. This shows that the dynamics behavior of system (8)(with no delay) is different from that of systems (6) and (9). Thus, it is meaningful to study the dynamics behavior of system (8).

Let $C^+ = C((-\infty, 0), \mathbb{R}^2_+)$ where $\mathbb{R}^2_+ = \{(x, y)^T : x, y \in \mathbb{R}_+\}$ and define initial value conditions in view of the biological reasons as follows

$$\begin{cases}(x(s), y(s))^T = (\phi_1(s), \phi_2(s))^T \in C^+, \ s \in [-\tau, 0],\\ x(0) = \phi_1(0) > 0, \ y(0) = \phi_2(0) > 0.\end{cases} \tag{10}$$

The rest of this paper is organized as follows. In the next section, by using the Comparison Principle in ordinary differential equation and some analytical techniques, we study the permanence of positive solutions of delayed predator-prey model (8) with initial value conditions (10). In Section 3, by applying continuation theorem of coincidence degree theory, we prove the existence of positive periodic solution of system (8). Section 4 is devote to the global attractivity. By constructing a suitable Lyapunov functional, we present some sufficient conditions to guarantee the existence, uniqueness and global attractivity of a positive periodic solution for system (8). In the last section, we perform some carefully designed numerical simulations to validate our analytical findings.

2. Permanence of positive solutions

Throughout this paper, for continuous ω-periodic function $f(t)$, we denote

$$f = \inf_{t\in[0,\omega]}\{f(t)\}, \quad F = \sup_{t\in[0,\omega]}\{f(t)\}, \quad \bar{f} = \frac{1}{\omega}\int_0^\omega f(t)\mathrm{d}t. \tag{11}$$

Using similar method as of the proof of Lemma 2.1 in [10], we state the following lemma of which the proof will be omitted.

Lemma 2.1. *All of solutions of initial value problem (8) with (10) are positive.*

In order to obtain the permanence of positive solutions of (8) with (10), we first give the following lemmas.

Lemma 2.2. *If $x(t)$ is a solution of initial value problem*

$$\begin{cases} \dot{x}(t) \le (\ge) x(t)(p - qx(t)),\\ x(0) = x_0 > 0\end{cases} \tag{12}$$

where p and q are positive constants, then $\limsup x(t) \le p/q$ $(\liminf x(t) \ge p/q)$.

Proof. It is easy to see that the solution of the following initial value problem

$$\begin{cases} \dot{u}(t) = u(t)(p - qu(t)), \\ u(0) = x_0 \end{cases}$$

is $u(t) = \frac{p}{q+(px_0^{-1}-q)\exp(-pt)}$. Applying the Comparison Principle in ordinary differential equation, we have

$$x(t) \le (\ge) \frac{p}{q + (px_0^{-1} - q)\exp(-pt)}.$$

Then $\limsup x(t) \le p/q$ $(\liminf x(t) \ge p/q)$. □

Using similar proof as of Lemma 2.2 we easy get the following lemma.

Lemma 2.3. *If $x(t)$ is a solution of initial value problem*

$$\begin{cases} \dot{x}(t) \le (\ge) x(t)(-p + qx^{m-1}(t)), \ (0 < m < 1), \\ x(0) = x_0 > 0 \end{cases} \tag{13}$$

where p and q are positive constants, then $\limsup x(t) \le (p/q)^{\frac{1}{1-m}} \left(\liminf x(t) \ge (p/q)^{\frac{1}{1-m}}\right)$.

For the convenience in next expression we make the following denotations

$$L_1 := \frac{R_1}{a_1}, \ L_2 := \Big(\frac{B_3}{r_2 d_2}\Big)^{\frac{1}{1-m}},$$
$$l_1 := \frac{r_1 c_1}{c_1(A_1 + e^{-\delta\tau}A_2) + A_3 L_1^{n-2} L_2^m}, \ l_2 := \Big(\frac{b_3 l_1^n}{(R_2 + (B_1 + B_2)L_2)(D_1 + D_2 L_1^n + D_3 L_2^n)}\Big)^{\frac{1}{1-m}}$$

where $\delta := r_1 - (A_1 + A_2)L_1 - \frac{A_3}{c_1} L_1^{n-1} L_2^m$.

After a simple calculation, we see that $0 < l_1 < L_1$ and $0 < l_2 < L_2$.

Theorem 2.4. *System (8) with initial value (10) is permanent, that is, all solutions $(x(t), y(t))$ of system (8) satisfy*

$$\begin{aligned} l_1 \le \liminf_{t\to+\infty} x(t) \le \limsup_{t\to+\infty} x(t) \le L_1, \\ l_2 \le \liminf_{t\to+\infty} y(t) \le \limsup_{t\to+\infty} y(t) \le L_2. \end{aligned} \tag{14}$$

Proof. From Theorem 2.4, all solutions of system (8) with initial value (10) are positive. Then from (8) and (10), we get

$$\dot{x}(t) \le x(t)(R_1 - a_1 x(t)), \quad x(0) = \phi_1(0) > 0.$$

By using Lemma 2.2, we have

$$\limsup_{t\to+\infty} x(t) \le \frac{R_1}{a_1} = L_1. \tag{15}$$

Similarly, from (8) and (10), we also get

$$\dot{y}(t) \le y(t)\Big(-r_2 + \frac{B_3}{d_2} y^{m-1}(t)\Big), \quad y(0) = \phi_2(0) > 0.$$

By using Lemma 2.3, we obtain

$$\limsup_{t\to+\infty} y(t) \le \Big(\frac{B_3}{d_2 r_2}\Big)^{\frac{1}{1-m}} = L_2. \tag{16}$$

From (15) and (16), for any small enough positive constant ε, there exists a positive number T_1 such that $x(t) \le L_1 + \varepsilon$ and $y(t) \le L_2 + \varepsilon$ for all $t \ge T_1$. Then we get, from first equation of (8), that

$$\frac{\dot{x}(t)}{x(t)} \ge r_1 - (L_1+\varepsilon)(A_1+A_2) - \frac{A_3}{c_1}(L_1+\varepsilon)^{n-1}(L_2+\varepsilon)^m, \quad t \ge T_1 + \tau.$$

Denoting $\delta(\varepsilon) = r_1 - (L_1+\varepsilon)(A_1+A_2) - \frac{A_3}{c_1}(L_1+\varepsilon)^{n-1}(L_2+\varepsilon)^m$ and integrating above inequality from $t-\tau$ to t, we get $x(t-\tau) \le e^{-\delta(\varepsilon)\tau}x(t)$. Then, from first equation of (8), we obtain that

$$\dot{x}(t) \ge x(t)\Big(r_1 - \Big(A_1 + e^{-\delta(\varepsilon)\tau}A_2 + \frac{A_3}{c_1}(L_1+\varepsilon)^{n-2}(L_2+\varepsilon)^m\Big)x(t)\Big).$$

By using Lemma 2.2, we see that

$$\liminf_{t\to+\infty} x(t) \ge \frac{c_1 r_1}{c_1(A_1 + e^{-\delta(\varepsilon)\tau}A_2) + A_3(L_1+\varepsilon)^{n-2}(L_2+\varepsilon)^m}. \tag{17}$$

When $\varepsilon \to 0$, inequality (17) leads to

$$\liminf_{t\to+\infty} x(t) \ge \frac{c_1 r_1}{c_1(A_1 + e^{-\delta\tau}A_2) + A_3 L_1^{n-2} L_2^m} = l_1. \tag{18}$$

From second equation of (8) with (15), (16) and (18), there exists a $T_2 > T_1$ such that, for all $t \ge T_2$,

$$\dot{y}(t) \ge y(t)\Big(-(R_2 + (B_1+B_2)(L_2+\varepsilon)) + \frac{b_3(l_1+\varepsilon)^n}{D_1 + D_2(L_1+\varepsilon)^n + D_3(L_2+\varepsilon)^n} y^{m-1}(t)\Big).$$

Therefore, from Lemma 2.3 and letting $\varepsilon \to 0$, we get

$$\liminf_{t\to+\infty} y(t) \ge \Big(\frac{b_3 l_1^n}{(R_2 + (B_1+B_2)L_2)(D_1 + D_2 L_1^n + D_3 L_2^n)}\Big)^{\frac{1}{m-1}} = l_2$$

The proof is completed. □

By method of the proof of Theorem 2.4, we easy to obtain the following permanence result for system (4).

Corollary 2.5. *System (4) with initial value*

$$x(0) = \phi_1(0) > 0, \; y(0) = \phi_2(0) > 0 \tag{19}$$

is permanent, that is, all solutions $(x(t), y(t))$ of system (4) with initial value (19) satisfy

$$\begin{aligned} &l_1^* \le \liminf_{t\to+\infty} x(t) \le \limsup_{t\to+\infty} x(t) \le L_1^*, \\ &l_2^* \le \liminf_{t\to+\infty} y(t) \le \limsup_{t\to+\infty} y(t) \le L_2^* \end{aligned} \tag{20}$$

where

$$L_1^* := \frac{R_1}{b_1}, \; L_2^* := \Big(\frac{C_2}{r_2}\Big)^{\frac{1}{1-m}},$$

$$l_1^* := \frac{k^2 r_1}{k^2 B_1 + C_1 L_2^{*m}}, \; l_2^* := \Big(\frac{c_2 l_1^{*2}}{(R_2 + B_2 L_2^*)(k^2 + L_1^{*2})}\Big)^{\frac{1}{1-m}}.$$

Remark 2.6. *It is easy to see that* $0 < l_1^* \le L_1^*$ *and* $0 < l_2^* \le L_2^*$ *hold without any other extra conditions except for the basic assumptions for coefficients of system (4). Therefore, Corollary 2.5 improves Theorem 3.1 in [10].*

3. Existences of periodic solutions

Suppose $(x(t), y(t))^T$ is an arbitrary positive solution of system (8) and let $u(t) = \ln x(t)$ and $v(t) = \ln y(t)$, then system (8) can be changed into

$$\begin{cases} \dot{u}(t) = r_1(t) - a_1(t)e^{u(t)} - a_2(t)e^{u(t-\tau)} - \dfrac{a_3(t)e^{(n-1)u(t)}e^{mv(t)}}{c_1(t) + c_2(t)e^{nu(t)} + c_3(t)e^{nv(t)}}, \\ \dot{v}(t) = -r_2(t) - b_1(t)e^{v(t)} - b_2(t)e^{v(t-\tau)} + \dfrac{b_3(t)e^{nu(t)}e^{(m-1)v(t)}}{d_1(t) + d_2(t)e^{nu(t)} + d_3(t)e^{nv(t)}}. \end{cases} \tag{21}$$

Denoting the right terms of first equation and second equation in (21) by $F_1(t, u(t), v(t))$ and $F_2(t, u(t), v(t))$ respectively and considering system

$$\begin{cases} \dot{u}(t) = \lambda F_1(t, u(t), v(t)), \\ \dot{v}(t) = \lambda F_2(t, u(t), v(t)) \end{cases} \tag{22}$$

where $\lambda \in (0, 1]$, we have the following lemma.

Lemma 3.1. *Suppose* $(u(t), v(t))^T$ *is a* ω*-periodic solution of (22), then there exists a positive number* S_1 *such that* $|u(t)| + |v(t)| \le S_1$ *where* S_1 *will be calculated as in the following proof.*

Proof. Since $(u(t), v(t))^T$ is periodic, the following discussion will be restricted to $t \in [0, \omega]$. Integrating the first equation of (22) from 0 to ω and in view of $\int_0^\omega \dot{u}(t)\mathrm{d}t = 0$, we get

$$\int_0^\omega r_1(t)\mathrm{d}t = \int_0^\omega \left(a_1(t)e^{u(t)} + a_2(t)e^{u(t-\tau)} + \frac{a_3(t)e^{(n-1)u(t)}e^{mv(t)}}{c_1(t) + c_2(t)e^{nu(t)} + c_3(t)e^{nv(t)}} \right) \mathrm{d}t. \tag{23}$$

Therefore,

$$\int_0^\omega |\dot{u}(t)|\mathrm{d}t = \lambda \int_0^\omega |F_1(t, u(t), v(t))|\mathrm{d}t \le \int_0^\omega 2r_1(t)\mathrm{d}t = 2\bar{r}_1\omega. \tag{24}$$

Suppose $\eta_1, \xi_1, \eta_2, \xi_2 \in [0, \omega]$ such that

$$u(\eta_1) = \min_{t\in[0,\omega]} u(t), \;\; u(\xi_1) = \max_{t\in[0,\omega]} u(t), \;\; v(\eta_2) = \min_{t\in[0,\omega]} v(t), \;\; v(\xi_2) = \max_{t\in[0,\omega]} v(t). \tag{25}$$

Then we see that $\dot{u}(\eta_1) = \dot{u}(\xi_1) = \dot{v}(\eta_2) = \dot{v}(\xi_2) = 0$.

From (23) and (25), we have

$$\int_0^\omega r_1(t)\mathrm{d}t \ge \int_0^\omega \left(a_1(t)e^{u(t)} + a_2(t)e^{u(t-\tau)} \right) \mathrm{d}t \ge \int_0^\omega \left(a_1(t) + a_2(t) \right) e^{u(\eta_1)}\mathrm{d}t = \omega(\bar{a}_1 + \bar{a}_2)e^{u(\eta_1)}.$$

Therefore

$$e^{u(\eta_1)} \le \frac{1}{\omega(\bar{a}_1 + \bar{a}_2)} \int_0^\omega r_1(t)\mathrm{d}t = \frac{\bar{r}_1}{(\bar{a}_1 + \bar{a}_2)}. \tag{26}$$

From (24), we have

$$u(t) - u(\eta_1) = \int_{\eta_1}^t \dot{u}(t)\mathrm{d}t \le \left| \int_{\eta_1}^t \dot{u}(t)\mathrm{d}t \right| \le \int_0^\omega |\dot{u}(t)|\mathrm{d}t \le 2\bar{r}_1\omega.$$

Therefore, for all $t \in [0, \omega]$,

$$u(t) \leq u(\eta_1) + 2\bar{r}_1\omega \leq \ln \frac{\bar{r}_1}{(\bar{a}_1 + \bar{a}_2)} + 2\bar{r}_1\omega := U_1. \tag{27}$$

Let $t = \xi_2$ in the second equation of (22) and in view of $\dot{v}(\xi_2) = 0$, we obtain

$$r_2(\xi_2) = -b_1(\xi_2)e^{v(\xi_2)} - b_2(\xi_2)e^{v(\xi_2-\tau)} - \frac{b_3(\xi_2)e^{nu(\xi_2)}e^{(m-1)v(\xi_2)}}{d_1(\xi_2) + d_2(\xi_2)e^{nu(\xi_2)} + d_3(\xi_2)e^{nv(\xi_2)}}. \tag{28}$$

Then

$$\begin{aligned} b_1(\xi_2)e^{v(\xi_2)} &= -r_2(\xi_2) - b_2(\xi_2)e^{v(\xi_2-\tau)} + \frac{b_3(\xi_2)e^{nu(\xi_2)}e^{(m-1)v(\xi_2)}}{d_1(\xi_2) + d_2(\xi_2)e^{nu(\xi_2)} + d_3(\xi_2)e^{nv(\xi_2)}} \\ &\leq \frac{b_3(\xi_2)}{d_2(\xi_2)}e^{(m-1)v(\xi_2)}. \end{aligned}$$

Further

$$v(\xi_2) \leq \frac{1}{2-m} \ln \frac{b_3(\xi_2)}{b_1(\xi_2)d_2(\xi_2)} \leq \frac{1}{2-m} \ln \frac{B_3}{b_1 d_2}.$$

Therefore, for all $t \in [0, \omega]$, we have

$$v(t) \leq \frac{1}{2-m} \ln \frac{B_3}{b_1 d_2} := V_1. \tag{29}$$

Letting $t = \eta_1$ in first equation and $t = \eta_2$ in second equation of (22) and noticing $\dot{u}(\eta_1) = \dot{v}(\eta_2) = 0$, we have

$$\begin{cases} r_1(\eta_1) = a_1(\eta_1)e^{u(\eta_1)} + a_2(\eta_1)e^{u(\eta_1-\tau)} + \dfrac{a_3(\eta_1)e^{(n-1)u(\eta_1)}e^{mv(\eta_1)}}{c_1(\eta_1) + c_2(\eta_1)e^{nu(\eta_1)} + c_3(\eta_1)e^{nv(\eta_1)}}, \\ r_2(\eta_2) = -b_1(\eta_2)e^{v(\eta_2)} - b_2(\eta_2)e^{v(\eta_2-\tau)} + \dfrac{b_3(\eta_2)e^{nu(\eta_2)}e^{(m-1)v(\eta_2)}}{d_1(\eta_2) + d_2(\eta_2)e^{nu(\eta_2)} + d_3(\eta_2)e^{nv(\eta_2)}}. \end{cases} \tag{30}$$

Now we estimate lower bound of $u(t)$.

If $u(\eta_1) \geq 0$, then $u(t) \geq 0$ for all t and the lower bound of $u(t)$ is 0.

If $u(\eta_1) < 0$, then $e^{(n-1)u(\eta_1)} \leq e^{u(\eta_1)}$ for $n \geq 2$. It is easy to obtain from (30) that

$$r_1(\eta_1) \leq a_1(\eta_1)e^{u(\eta_1)} + a_2(\eta_1)e^{u(\eta_1-\tau)} + \frac{a_3(\eta_1)e^{u(\eta_1)}e^{mv(\eta_1)}}{c_1(\eta_1)} \leq \left(a_1(\eta_1) + a_2(\eta_1) + \frac{a_3(\eta_1)}{c_1(\eta_1)}e^{mv(\eta_1)}\right)e^{u(\xi_1)}.$$

Therefore,

$$e^{u(\xi_1)} \geq r_1(\eta_1)\left(a_1(\eta_1) + a_2(\eta_1) + \frac{a_3(\eta_1)}{c_1(\eta_1)}e^{mv(\eta_1)}\right)^{-1} \geq \frac{c_1 r_1}{c_1 A_1 + c_1 A_2 + A_3 e^{mV_1}},$$

i.e.

$$u(\xi_1) \geq \ln \frac{c_1 r_1}{c_1 A_1 + c_1 A_2 + A_3 e^{mV_1}}.$$

On the other hand

$$u(\xi_1) - u(t) = \int_t^{\xi_1} \dot{u}(s)\mathrm{d}s \leq \left|\int_t^{\xi_1} \dot{u}(s)\mathrm{d}s\right| \leq \int_0^{\omega} |\dot{u}(s)|\mathrm{d}s \leq 2\bar{r}_1\omega,$$

then

$$u(t) \geq u(\xi_1) - 2\bar{r}_1\omega \geq \ln \frac{c_1 r_1}{c_1 A_1 + c_1 A_2 + A_3 e^{mV_1}} - 2\bar{r}_1\omega := E_1.$$

Let $U_0 = \min\{0, E_1\}$, then, for all $t \in [0, \omega]$,

$$u(t) \geq U_0. \tag{31}$$

Next we estimate lower bound of $v(t)$.

If $v(\eta_2) \geq 0$, then 0 is lower bound of $v(t)$.

If $v(\eta_2) < 0$, then $e^{(1-m)v(\eta_2)} \geq e^{(2-m)v(\eta_2)}$ and $e^{nv(\eta_2)} < 1$. Therefore, from the second equation of (30), we have

$$\Big(r_2(\eta_2) + b_1(\eta_2) + b_2(\eta_2)e^{v(\eta_2-\tau)}\Big)e^{(1-m)v(\eta_2)} \geq \frac{b_3(\eta_2)e^{nu(\eta_2)}}{d_1(\eta_2) + d_2(\eta_2)e^{nu(\eta_2)} + d_3(\eta_2)}.$$

Hence,

$$\begin{aligned} v(\eta_2) &\geq \frac{1}{1-m} \ln \frac{b_3(\eta_2)e^{nu(\eta_2)}}{\Big(r_2(\eta_2) + b_1(\eta_2) + b_2(\eta_2)e^{v(\eta_2-\tau)}\Big)\Big(d_1(\eta_2) + d_2(\eta_2)e^{nu(\eta_2)} + d_3(\eta_2)\Big)} \\ &\geq \frac{1}{1-m} \ln \frac{b_3 e^{nU_0}}{(R_2 + B_1 + B_2 e^{V_1})(D_1 + D_2 e^{nU_1} + D_3)} := E_2. \end{aligned}$$

Therefore, for $t \in [0, \omega]$, $v(t) \geq E_2$. Letting $V_0 = \min\{0, E_2\}$, we have,

$$v(t) \geq V_0, \quad \text{for } t \in [0, \omega]. \tag{32}$$

From (27), (29), (31) and (32), we know, for $t \in [0, \omega]$, that

$$U_0 \leq u(t) \leq U_1, \quad V_0 \leq v(t) \leq V_1. \tag{33}$$

Denoting $E_3 = \max\{|U_0|, |U_1|\}$, $E_4 = \max\{|V_0|, |V_1|\}$ and $S_1 = E_3 + E_4$, we have

$$|u(t)| + |v(t)| \leq S_1. \tag{34}$$

□

Suppose $(u, v)^T$ is a constant solution of system (21), then

$$\begin{cases} r_1(t) - a_1(t)e^u - a_2(t)e^u - \dfrac{a_3(t)e^{(n-1)u}e^{mv}}{c_1(t) + c_2(t)e^{nu} + c_3(t)e^{nv}} = 0, \\ -r_2(t) - b_1(t)e^v - b_2(t)e^v + \dfrac{b_3(t)e^{nu}e^{(m-1)v}}{d_1(t) + d_2(t)e^{nu} + d_3(t)e^{nv}} = 0. \end{cases}$$

Integrating two sides of above equations on $[0, \omega]$ and applying integral mean theorem, we get

$$\begin{cases} \bar{r}_1 - (\bar{a}_1 + \bar{a}_2)e^u - \dfrac{\bar{a}_3 e^{(n-1)u}e^{mv}}{c_1(t_1) + c_2(t_1)e^{nu} + c_3(t_1)e^{nv}} = 0, \\ -\bar{r}_2 - (\bar{b}_1 + \bar{b}_2)e^v + \dfrac{\bar{b}_3 e^{nu}e^{(m-1)v}}{d_1(t_2) + d_2(t_2)e^{nu} + d_3(t_2)e^{nv}} = 0 \end{cases} \tag{35}$$

where $t_1, t_2 \in [0, \omega]$.

Consider the following equations

$$\begin{cases} \bar{r}_1 - (\bar{a}_1 + \bar{a}_2)e^u - \mu \dfrac{\bar{a}_3 e^{(n-1)u} e^{mv}}{c_1(t_1) + c_2(t_1)e^{nu} + c_3(t_1)e^{nv}} = 0, \\ -\mu \bar{r}_2 - (\bar{b}_1 + \bar{b}_2)e^v + \dfrac{\bar{b}_3 e^{nu} e^{(m-1)v}}{d_1(t_2) + d_2(t_2)e^{nu} + d_3(t_2)e^{nv}} = 0 \end{cases} \tag{36}$$

where $\mu \in [0, 1]$ is a parameter.

Lemma 3.2. *Suppose $(u, v)^T$ is a solution of (36), then there exists a positive number S_2 such that $|u| + |v| \leq S_2$ where S_2 will be defined in the following proof.*

Proof. From the first equation of (36) we get $\bar{r}_1 \geq (\bar{a}_1 + \bar{a}_2)e^u$, then

$$u \leq \frac{\bar{r}_1}{\bar{a}_1 + \bar{a}_2} := U_3. \tag{37}$$

From the second equation of (36), we get

$$(\bar{b}_1 + \bar{b}_2)e^v \leq \frac{\bar{b}_3 e^{nu} e^{(m-1)v}}{d_1(t_2) + d_2(t_2)e^{nu} + d_3(t_2)e^{nv}} \leq \frac{\bar{b}_3 e^{(m-1)v}}{d_2(t_2)}.$$

Then

$$v \leq \frac{1}{2-m} \ln \frac{\bar{b}_3}{(\bar{b}_1 + \bar{b}_2)d_2} := V_3. \tag{38}$$

If $u \geq 0$, then 0 is the low bound of u. If $u < 0$, from the first equation of (36), we get

$$\bar{r}_1 \leq (\bar{a}_1 + \bar{a}_2)e^u + \frac{\bar{a}_3 e^u e^{mv}}{c_1(t_1)} \leq \left(\bar{a}_1 + \bar{a}_2 + \frac{\bar{a}_3 e^{mV_3}}{c_1}\right)e^u.$$

Therefore

$$u \geq \ln \frac{c_1 \bar{r}_1}{c_1(\bar{a}_1 + \bar{a}_2) + \bar{a}_3 e^{mV_3}} := H_1.$$

Let $U_2 = \min\{0, H_1\}$, we have,

$$u \geq U_2. \tag{39}$$

If $v \geq 0$, then 0 is the low bound of v. If $u < 0$, from the second equation of (36), we get

$$\bar{r}_2 e^{(1-m)v} + (\bar{b}_1 + \bar{b}_2)e^{(2-m)v} \geq \frac{\bar{b}_3 e^{nu}}{d_1(t_2) + d_2(t_2)e^{nu} + d_3(t_2)e^{nv}}.$$

Then, in view of $(1-m)v \geq (2-m)v$ and $e^{nv} < 1$, we obtain

$$\left(\bar{r}_2 + \bar{b}_1 + \bar{b}_2\right)e^{(1-m)v} \geq \frac{\bar{b}_3 e^{nu}}{d_1(t_2) + d_2(t_2)e^{nu} + d_3(t_2)e^{nv}} \geq \frac{\bar{b}_3 e^{nU_2}}{D_1 + D_2 e^{nU_3} + D_3}.$$

Therefore

$$v \geq \frac{1}{1-m} \ln \frac{\bar{b}_3 e^{nU_2}}{\left(\bar{r}_2 + \bar{b}_1 + \bar{b}_2\right)\left(D_1 + D_2 e^{nU_3} + D_3\right)} := H_2.$$

Letting $V_2 = \min\{0, H_2\}$, we have,

$$v \geq V_2. \tag{40}$$

From (37), (38), (39) and (40), we know that

$$U_2 \leq u(t) \leq U_3, \;\; V_2 \leq v(t) \leq V_3. \tag{41}$$

Denoting $E_5 = \max\{|U_2|, |U_3|\}$, $E_6 = \max\{|V_2|, |V_3|\}$ and $S_2 = E_5 + E_6$, we have

$$|u(t)| + |v(t)| \leq S_2. \tag{42}$$

□

In order to discuss the existence of periodic solutions of system (8), we introduce some definitions and Mawhin's coincidence theorem.

Definition 3.3. *(see [48]) Let X and Y be both Banach spaces and* $L : DomL(\subset X) \to Y$ *be a linear map. If the following conditions are satisfied*

(a) *ImL is a closed subspace of Y;*

(b) $\dim KerL = codimImL < +\infty$,

then L is called a Fredholm operator.

If L is a Fredholm operator with index zero and there exist continuous projects

$$P : \; X \to \mathrm{Dom}L \;\; \text{and} \;\; Q : \; Y \to Y$$

such that $\mathrm{Im}P = \mathrm{Ker}L$, $\mathrm{Ker}Q = \mathrm{Im}L = \mathrm{Im}(I - Q)$ and

$$X = \mathrm{Ker}L \oplus \mathrm{Ker}P, \quad Y = \mathrm{Im}L \oplus \mathrm{Im}Q.$$

Then map $L_P = L|_{\mathrm{Dom}L \cap \mathrm{Ker}P} : \mathrm{Dom}L \cap \mathrm{Ker}P \to \mathrm{Im}L$ is invertible. Denote inverse of L_P by K_P, then $K_P : \mathrm{Im}L \to \mathrm{Dom}L \cap \mathrm{Ker}P$.

Definition 3.4. *(see [48]) Let* $N : X \to Y$ *be a continuous map and* $\Omega \subset X$ *be any open set. If* $QN(\overline{\Omega})$ *is bounded and* $K_P(I - Q)N(\overline{\Omega})$ *is relative compact in X, then we say N is L-compact on* $\overline{\Omega}$.

Lemma 3.5. *(Mawhin's coincidence theorem, [48]) Let X and Y be both Banach spaces and* $L : \mathrm{Dom}L(\subset X) \to Y$ *be a Fredholm operator with index zero,* $\Omega \subset X$ *be an open bounded set and* $N : \overline{\Omega} \to Y$ *be L-compact on* $\overline{\Omega}$. *If the following conditions hold:*

(i) $Lx \neq \lambda Nx$, $x \in \partial\Omega \cap \mathrm{Dom}L$, $\lambda \in (0, 1)$;

(ii) $Nx \notin \mathrm{Im}L$, $x \in \partial\Omega \cap \mathrm{Ker}L$;

(iii) $\deg\{JQN, \Omega \cap \mathrm{Ker}L, 0\} \neq 0$, *where* $J : \mathrm{Im}Q \to \mathrm{Ker}L$ *is an isomorphism.*

Then the equation $Lx = Nx$ *has at least one solution on* $\overline{\Omega} \cap \mathrm{Dom}L$.

Theorem 3.6. *System* (8) *with initial value* (10) *has at least one positive ω-periodic solution.*

Proof. Suppose that $(x(t), y(t))^T$ is an arbitrary positive solution of system (8) and let $u(t) = \ln x(t)$ and $v(t) = \ln y(t)$, then system (8) is changed into system (21).

Let $X = Y = \{z(t)|\, z(t) = (u(t), v(t))^T \in C(\mathbb{R}, \mathbb{R}^2), z(t+\omega) = z(t)\}$ equipped with the norm

$$\|z(t)\| = \|(u(t), v(t))^T\| = \max_{t\in[0,\omega]} |u(t)| + \max_{t\in[0,\omega]} |v(t)|,$$

then X and Y are both Banach spaces.

We define operators L, P and Q on X as follows

$$\begin{aligned} &L: \ L(z) = \frac{dz}{\mathrm{d}t};\\ &P: \ P(z) = z(0);\\ &Q: \ Q(z) = \frac{1}{\omega}\int_0^\omega z(t)\mathrm{d}t, \end{aligned}$$

and define operator $N : X \to Y$ as the following form

$$N(z) = \begin{pmatrix} r_1(t) - a_1(t)e^{u(t)} - a_2(t)e^{u(t-\tau)} - \dfrac{a_3(t)e^{(n-1)u(t)}e^{mv(t)}}{c_1(t) + c_2(t)e^{nu(t)} + c_3(t)e^{nv(t)}} \\ -r_2(t) - b_1(t)e^{v(t)} - b_2(t)e^{v(t-\tau)} + \dfrac{b_3(t)e^{nu(t)}e^{(m-1)v(t)}}{d_1(t) + d_2(t)e^{nu(t)} + d_3(t)e^{nv(t)}} \end{pmatrix}.$$

Then $\mathrm{Dom}L = \{z(t) \in X : \ z(t) \in C^1(\mathbb{R}, \mathbb{R}^2)\}$, $\mathrm{Ker}L = \mathbb{R}^2$, $\mathrm{Im}Q = \mathbb{R}^2$, $\dim \mathrm{Ker}L = \mathrm{codim}\, \mathrm{Im}L = 2$ and $\mathrm{Im}L = \{z|\, z \in Y, \int_0^\omega z(t)\mathrm{d}t = 0\}$. By Lebesgue dominated convergence theorem, we know $\mathrm{Im}L$ is closed in Y and L is a Fredholm operator with index zero.

Obviously, P and Q are both continuous projections satisfying $\mathrm{Im}P = \mathrm{Ker}L$, $\mathrm{Im}L = \mathrm{Ker}Q = \mathrm{Im}(I - Q)$. Thus operator L on $\mathrm{Dom}L \cap \mathrm{Ker}P$ has a inverse defined by $K_P : \mathrm{Im}L \to \mathrm{Dom}L \cap \mathrm{Ker}P$. By simple calculation we see

$$K_P(z) = \int_0^t z(s)\mathrm{d}s - \frac{1}{\omega}\int_0^\omega \mathrm{d}t \int_0^t z(s)\mathrm{d}s.$$

For any $z(t) \in X$, we obtain

$$\begin{aligned} QN(z) &= Q\Big(F_1(t, u(t), v(t)), F_2(t, u(t), v(t))\Big)^T\\ &= \left(\frac{1}{\omega}\int_0^\omega F_1(t, u(t), v(t))\mathrm{d}t, \frac{1}{\omega}\int_0^\omega F_2(t, u(t), v(t))\mathrm{d}t\right)^T\\ &= (\bar{F}_1, \bar{F}_2)^T \end{aligned}$$

and

$$K_P(I - Q)N(z) = (W_1, W_2)^T$$

where $W_i(t) = \int_0^t F_i(s, u(s), v(s))\mathrm{d}s - \bar{F}_i t - \frac{1}{\omega}\int_0^\omega \int_0^t F_i(s, u(s), v(s))\mathrm{d}s\mathrm{d}t + \frac{\omega}{2}$, $i = 1, 2$. Therefore, by Lebesgue dominated convergence theorem, we know that QN and $K_P(I - Q)N$ are both continuous. For any bounded open set $\Omega \subset X$, $F_i(s, u(s), v(s))$ $(i = 1, 2)$ are bounded on $\overline{\Omega}$, then $QN(\overline{\Omega})$ and $K_P(I - Q)N(\overline{\Omega})$ are both uniformly bounded and equicontinuous. By using Arzela-Ascoli theorem, we know that $QN(\overline{\Omega})$ and $K_P(I - Q)N(\overline{\Omega})$ are both compact. Therefore N is L-compact on $\overline{\Omega}$. Particularly, we take $\Omega = \{z(t)|\, z(t) = (u(t), v(t))^T \in X, \|z(t)\| \le S\}$ where $S = S_1 + S_2 + \varepsilon$ $(\varepsilon > 0)$ and S_1, S_2 are defined as in Lemmas 3.1 and 3.2.

Next, we check the three conditions in Lemma 3.5.

(i) For each $\lambda \in (0,1)$, $z(t) \in \partial\Omega \cap \mathrm{Dom}L$, we have $Lz \neq \lambda Nz$. Otherwise, $z(t)$ is a ω-periodic solution of (22) and then $\|z(t)\| \leq S_1$ will be derived by Lemma 3.1. It is impossible because $\|z(t)\| = S > S_1$ for $z(t) \in \partial\Omega \cap \mathrm{Dom}L$.

(ii) When $z(t) \in \partial\Omega \cap \mathrm{Ker}L$, $\frac{\mathrm{d}z(t)}{\mathrm{d}t} = 0$, i.e., $z(t)$ is a constant vector $(u,v)^T$ with $\|(u,v)^T\| = S_1 + S_2 + \varepsilon$. If $QN(u,v)^T = 0$, then $(u,v)^T$ is a solution of (36) for $\mu = 1$. By Lemma 3.2, we have $\|(u,v)^T\| \leq S_2$ which contradicts to $\|(u,v)^T\| = S_1 + S_2 + \varepsilon$. Thus, for each $z \in \partial\Omega \cap \mathrm{Ker}L$, $QNz \neq 0$

(iii) Choose $J : \mathrm{Im}Q \to \mathrm{Ker}L$ such that $J(z) = z$ for each $z \in \mathrm{Im}Q$. When $z \in \Omega \cap \mathrm{Ker}L$, $z(t) = (u,v)^T$ is a constant vector and satisfies

$$
\begin{aligned}
JQN(u,v)^T &= JQ\Big(F_1(t,u,v), F_2(t,u,v)\Big)^T \\
&= \left(\frac{1}{\omega}\int_0^\omega F_1(t,u,v)\mathrm{d}t, \frac{1}{\omega}\int_0^\omega F_2(t,u,v)\mathrm{d}t\right)^T \\
&= \begin{pmatrix} \bar{r}_1 - (\bar{a}_1 + \bar{a}_2)e^u - \dfrac{\bar{a}_3 e^{(n-1)u} e^{mv}}{c_1(t_1) + c_2(t_1)e^{nu} + c_3(t_1)e^{nv}} \\ -\bar{r}_2 - (\bar{b}_1 + \bar{b}_2)e^v + \dfrac{\bar{b}_3 e^{nu} e^{(m-1)v}}{d_1(t_2) + d_2(t_2)e^{nu} + d_3(t_2)e^{nv}} \end{pmatrix}
\end{aligned}
$$

where t_1, t_2 were defined as in (35). We define $\varphi : \Omega \cap \mathrm{Ker}L \times [0,1] \to X$ as follows

$$
\varphi(u,v,\mu) = \begin{pmatrix} \bar{r}_1 - (\bar{a}_1 + \bar{a}_2)e^u \\ \dfrac{\bar{b}_3 e^{nu} e^{(m-1)v}}{d_1(t_2) + d_2(t_2)e^{nu} + d_3(t_2)e^{nv}} - (\bar{b}_1 + \bar{b}_2)e^v \end{pmatrix} + \mu \begin{pmatrix} -\dfrac{\bar{a}_3 e^{(n-1)u} e^{mv}}{c_1(t_1) + c_2(t_1)e^{nu} + c_3(t_1)e^{nv}} \\ -\bar{r}_2 \end{pmatrix}.
$$

Then $JQN(u,v)^T = \varphi(u,v,1)$. By Lemma 3.2, we see $\varphi(u,v,1) \neq (0,0)^T$. Therefore, using the homotopy invariance theorem of topological degree, we obtain

$$
\begin{aligned}
&\deg\{JQN(u,v)^T, \Omega \cap \mathrm{Ker}L, (0,0)^T\} \\
=&\deg\{\varphi(u,v,1), \Omega \cap \mathrm{Ker}L, (0,0)^T\} \\
=&\deg\{\varphi(u,v,0), \Omega \cap \mathrm{Ker}L, (0,0)^T\} \\
=&\deg\Big\{\Big(\bar{r}_1 - (\bar{a}_1 + \bar{a}_2)e^u, \frac{\bar{b}_3 e^{nu} e^{(m-1)v}}{d_1(t_2) + d_2(t_2)e^{nu} + d_3(t_2)e^{nv}} - (\bar{b}_1 + \bar{b}_2)e^v\Big)^T, \ \Omega \cap \mathrm{Ker}L, \ (0,0)^T\Big\}.
\end{aligned}
$$

Denote

$$
\begin{aligned}
\psi_1(u,v) &:= \bar{r}_1 - (\bar{a}_1 + \bar{a}_2)e^u, \\
\psi_2(u,v) &:= \frac{\bar{b}_3 e^{nu} e^{(m-1)v}}{d_1(t_2) + d_2(t_2)e^{nu} + d_3(t_2)e^{nv}} - (\bar{b}_1 + \bar{b}_2)e^v,
\end{aligned}
$$

and consider the following algebraic equations

$$
\begin{cases} \psi_1(u,v) = 0, \\ \psi_2(u,v) = 0. \end{cases} \tag{43}
$$

From first equation of (43) we get its unique $u^* = \ln \frac{\bar{r}_1}{\bar{a}_1 + \bar{a}_2}$. Substituting it into second equation of (43), we get

$$
\frac{\bar{b}_3 e^{nu^*}}{d_1(t_2) + d_2(t_2)e^{nu^*} + d_3(t_2)e^{nv}} - (\bar{b}_1 + \bar{b}_2)e^{(2-m)v} = 0
$$

which is easily checked to have a unique solution v^* on $\mathbb{R}$. So, equations (43) has unique solution $(u^*, v^*)^T$ on $\Omega \cap \mathrm{Ker}L$.

For convenience, we denote $p(u,v) = d_1(t_2) + d_2(t_2)e^{nu} + d_3(t_2)e^{nv}$. Then

$$\begin{aligned}
&\frac{\partial \psi_1}{\partial u} = -(\bar{a}_1 + \bar{a}_2)e^u,\\
&\frac{\partial \psi_1}{\partial v} = 0,\\
&\frac{\partial \psi_2}{\partial u} = \frac{n\bar{b}_3 e^{nu} e^{(1-m)v}\Big(p(u,v) - n\bar{b}_3 d_2(t_2)e^{nu}\Big)}{p^2(u,v)},\\
&\frac{\partial \psi_2}{\partial v} = \frac{-\bar{b}_3 e^{nu} e^{(m-1)v}\Big((1-m)p(u,v) + nd_3(t_2)e^{nv}\Big)}{p^2(u,v)} - (\bar{b}_1 + \bar{b}_2)e^v.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\deg\{JQN(u,v)^T, \Omega \cap \mathrm{Ker}L, (0,0)^T\} &= \mathrm{sgn}\begin{vmatrix} \frac{\partial \psi_1}{\partial u} & \frac{\partial \psi_1}{\partial v} \\ \frac{\partial \psi_2}{\partial u} & \frac{\partial \psi_2}{\partial v} \end{vmatrix}_{(u^*,v^*)^T} = \mathrm{sgn}\begin{vmatrix} \frac{\partial \psi_1}{\partial u} & 0 \\ \frac{\partial \psi_2}{\partial u} & \frac{\partial \psi_2}{\partial v} \end{vmatrix}_{(u^*,v^*)^T}\\
=&\mathrm{sgn}\left[(\bar{a}_1 + \bar{a}_2)e^{u^*}\Big(\frac{\bar{b}_3 e^{nu^*} e^{(m-1)v^*}\Big((1-m)p(u^*,v^*) + nd_3(t_2)e^{nv^*}\Big)}{p^2(u^*,v^*)} + (\bar{b}_1 + \bar{b}_2)e^{v^*}\Big)\right]\\
=&1 \neq 0.
\end{aligned}$$

So far, all of the conditions in Lemma 3.5 have been checked. This implies that system (21) has at least one ω-periodic solution. Further system (8) has at least one positive ω-periodic solution. The proof is completed. □

Remark 3.7. *When $a_2(t) = b_2(t) = c_3(t) = d_3(t) = 0$, $c_1(t) = d_1(t) = k^2$, $c_2(t) = d_2(t) = 1$ and $n = 2$, system (8) is degenerated into system (4). Therefore Theorem 3.6 extends Theorem 3.1 in [8] and Theorem 3.1 in [9].*

4. Global attractivity

Definition 4.1. *Suppose $(\tilde{x}(t), \tilde{y}(t))^T$ is a positive ω-periodic solution of system (8), $(x(t), y(t))^T$ is arbitrary positive solution of system (8) and*

$$\lim_{t\to+\infty} |x(t) - \tilde{x}(t)| = 0, \quad \lim_{t\to+\infty} |y(t) - \tilde{y}(t)| = 0.$$

Then $(\tilde{x}(t), \tilde{y}(t))^T$ is called globally attractive.

Lemma 4.2. *(see [49]) If function f is nonnegative, integrable and uniformly continuous on $[0, +\infty)$, then $\lim_{t\to+\infty} f(t) = 0$.*

From Theorem 2.4, we know that for any enough small positive ε $(< \min\{l_1, l_2\})$ there exists $T(> 0)$ such that, when $t \geq T$, arbitrary positive solution $(x(t), y(t))^T$ of system (8) satisfies that

$$l_1 - \varepsilon \leq x(t) \leq L_1 + \varepsilon, \quad l_2 - \varepsilon \leq y(t) \leq L_2 + \varepsilon. \tag{44}$$

For arbitrary positive ω-periodic solution $(\tilde{x}(t), \tilde{y}(t))^T$ of system (8), if let $u(t) = \ln \tilde{x}(t)$ and $v(t) = \ln \tilde{y}(t)$, then $(u(t), v(t))^T$ satisfies (21). From (33) in proof of Lemma 3.1, we have

$$e^{U_0} \leq \tilde{x}(t) \leq e^{U_1}, \quad e^{V_0} \leq \tilde{y}(t) \leq e^{V_1}. \tag{45}$$

For convenience, we denote

$$\gamma = \min\{l_1, l_2, e^{U_0}, e^{V_0}\},\ \Gamma = \max\{L_1, L_2, e^{U_1}, e^{V_1}\}$$

and

$$g_1(t) = g_1(t,\gamma,\gamma),\ \ g_2(t) = g_2(t,\gamma,\gamma),\ \ G_1(t) = g_1(t,\Gamma,\Gamma),\ \ G_2(t) = g_2(t,\Gamma,\Gamma), \tag{46}$$

where

$$g_1(t,x,y) = c_1(t) + c_2(t)x^n + c_3(t)y^n,\ \ g_2(t,x,y) = d_1(t) + d_2(t)x^n + d_3(t)y^n.$$

Theorem 4.3. *Suppose system (8) with initial condition (10) satisfy*

$$\begin{aligned}(A)\ \sigma_1 = \min_{t\in[0,\omega]} \Big\{ & a_1(t) + (n-1)\gamma^{n+m-2}\frac{a_3(t)(c_1(t)+\gamma^n c_3(t))}{G_1^2(t)} - \Gamma^{2n+m-2}\frac{a_3(t)c_2(t)}{g_1^2(t)} \\ & - n\gamma^{m-1}\Gamma^{n-1}\frac{b_3(t)(d_1(t)+\Gamma^n d_3(t))}{g_2^2(t)} - A_2\Big\} > 0;\end{aligned}$$

$$\begin{aligned}(B)\ \sigma_2 = \min_{t\in[0,\omega]} \Big\{ & b_1(t) + (1-m)\gamma^n\Gamma^{n-2}\frac{b_3(t)}{G_2(t)} + n\gamma^{2n-1}\Gamma^{m-1}\frac{b_3(t)d_3(t)}{G_2^2(t)} \\ & - n\Gamma^{2n+m-1}\frac{a_3(t)c_3(t)}{g_1^2(t)} - m\gamma^{m-1}\Gamma^{n-1}\frac{a_3(t)}{g_1(t)} - B_2\Big\} > 0.\end{aligned}$$

Then system (8) has only one positive ω-periodic solution which is globally attractive.

Proof. **Proof**. Suppose $(x(t), y(t))^T$ is arbitrary positive solution of system (8), then we know it satisfies (44). Moreover, Theorem 3.6 indicates that system (8) has at least one positive ω-periodic solution $(\tilde{x}(t), \tilde{y}(t))^T$ satisfying condition (45). We choose Lyapunov function as follows

$$V(t) = V_1(t) + V_2(t)$$

where

$$\begin{aligned}V_1(t) &= \Big|\ln x(t) - \ln\tilde{x}(t)\Big| + A_2\int_{t-\tau}^{t}\Big|\ln x(s) - \ln\tilde{x}(s)\Big|ds,\\ V_2(t) &= \Big|\ln y(t) - \ln\tilde{y}(t)\Big| + B_2\int_{t-\tau}^{t}\Big|\ln y(s) - \ln\tilde{y}(s)\Big|ds.\end{aligned}$$

Then

$$\begin{aligned}D^+V_1(t)|_{(1.7)} =& \mathrm{sgn}\Big(x(t)-\tilde{x}(t)\Big)\left(\frac{\dot{x}(t)}{x(t)} - \frac{\dot{\tilde{x}}(t)}{\tilde{x}(t)}\right) + A_2\Big|\ln x(s) - \ln\tilde{x}(s)\Big| - A_2\Big|\ln x(t-\tau) - \ln\tilde{x}(t-\tau)\Big|\\ =& \mathrm{sgn}\Big(x(t)-\tilde{x}(t)\Big)\Bigg[-a_1(t)\Big(x(t)-\tilde{x}(t)\Big) - a_2(t)\Big(x(t-\tau)-\tilde{x}(t-\tau)\Big) - \Bigg(\frac{a_3(t)x^{n-1}(t)y^m(t)}{g_1(t,x,y)}\\ & - \frac{a_3(t)\tilde{x}^{n-1}(t)\tilde{y}^m(t)}{g_1(t,\tilde{x},\tilde{y})}\Bigg)\Bigg] + A_2\Big|x(t)-\tilde{x}(t)\Big| - A_2\Big|x(t-\tau)-\tilde{x}(t-\tau)\Big|.\end{aligned} \tag{47}$$

Since

$$\begin{aligned}
&\frac{a_3(t)x^{n-1}(t)y^m(t)}{g_1(t,x,y)}-\frac{a_3(t)\tilde{x}^{n-1}(t)\tilde{y}^m(t)}{g_1(t,\tilde{x},\tilde{y})}\\
=&\frac{a_3(t)x^{n-1}(t)y^m(t)}{g_1(t,x,y)}-\frac{a_3(t)\tilde{x}^{n-1}(t)y^m(t)}{g_1(t,\tilde{x},\tilde{y})}+\frac{a_3(t)\tilde{x}^{n-1}(t)y^m(t)}{g_1(t,\tilde{x},\tilde{y})}-\frac{a_3(t)\tilde{x}^{n-1}(t)\tilde{y}^m(t)}{g_1(t,\tilde{x},\tilde{y})}\\
=&\frac{a_3(t)y^m(t)}{g_1(t,x,y)g_1(t,\tilde{x},\tilde{y})}\Big[\Big(c_1(t)+c_3(t)\tilde{y}^n(t)\Big)\Big(x^{n-1}(t)-\tilde{x}^{n-1}(t)\Big)-c_2(t)x^{n-1}(t)\tilde{x}^{n-1}(t)\Big(x(t)-\tilde{x}(t)\Big)\\
&+c_3(t)\tilde{x}^{n-1}(t)\Big(\tilde{y}^n(t)-y^n(t)\Big)\Big]+\frac{a_3(t)\tilde{x}^{n-1}(t)}{g_1(t,\tilde{x},\tilde{y})}\Big(y^m(t)-\tilde{y}^m(t)\Big).
\end{aligned}\tag{48}$$

Substituting (48) in to (47) and in view of $A_2 \geq a_2(t)$, we get

$$\begin{aligned}
D^+V_1(t)|_{(1.7)}\leq&-a_1(t)\Big|x(t)-\tilde{x}(t)\Big|+\frac{a_3(t)y^m(t)}{g_1(t,x,y)g_1(t,\tilde{x},\tilde{y})}\Big[-\Big(c_1(t)+c_3(t)\tilde{y}^n(t)\Big)\Big|x^{n-1}(t)-\tilde{x}^{n-1}(t)\Big|\\
&+c_2(t)x^{n-1}(t)\tilde{x}^{n-1}(t)\Big|x(t)-\tilde{x}(t)\Big|+c_3(t)\tilde{x}^{n-1}(t)\Big|\tilde{y}^n(t)-y^n(t)\Big|\Big]+A_2\Big|x(t)-\tilde{x}(t)\Big|\\
&+\frac{a_3(t)\tilde{x}^{n-1}(t)}{g_1(t,\tilde{x},\tilde{y})}\Big|y^m(t)-\tilde{y}^m(t)\Big|.
\end{aligned}\tag{49}$$

Meanwhile,

$$\begin{aligned}
D^+V_2(t)|_{(1.7)}=&\operatorname{sgn}\Big(y(t)-\tilde{y}(t)\Big)\left(\frac{\dot{y}(t)}{y(t)}-\frac{\dot{\tilde{y}}(t)}{\tilde{y}(t)}\right)+B_2\Big|y(s)-\tilde{y}(s)\Big|-B_2\Big|y(t-\tau)-\tilde{y}(t-\tau)\Big|\\
=&\operatorname{sgn}\Big(y(t)-\tilde{y}(t)\Big)\Bigg[-b_1(t)\Big(y(t)-\tilde{y}(t)\Big)-b_2(t)\Big(y(t-\tau)-\tilde{y}(t-\tau)\Big)+\Bigg(\frac{b_3(t)x^n(t)y^{m-1}(t)}{g_2(t,x,y)}\\
&-\frac{b_3(t)\tilde{x}^n(t)\tilde{y}^{m-1}(t)}{g_2(t,\tilde{x},\tilde{y})}\Bigg)\Bigg]+B_2\Big|y(t)-\tilde{y}(t)\Big|-B_2\Big|y(t-\tau)-\tilde{y}(t-\tau)\Big|.
\end{aligned}\tag{50}$$

Since

$$\begin{aligned}
&\frac{b_3(t)x^n(t)y^{m-1}(t)}{g_2(t,x,y)}-\frac{b_3(t)\tilde{x}^n(t)\tilde{y}^{m-1}(t)}{g_2(t,\tilde{x},\tilde{y})}\\
=&\frac{b_3(t)x^n(t)y^{m-1}(t)}{g_2(t,x,y)}-\frac{b_3(t)\tilde{x}^n(t)y^{m-1}(t)}{g_2(t,\tilde{x},\tilde{y})}+\frac{b_3(t)\tilde{x}^n(t)y^{m-1}(t)}{g_2(t,\tilde{x},\tilde{y})}-\frac{b_3(t)\tilde{x}^n(t)\tilde{y}^{m-1}(t)}{g_2(t,\tilde{x},\tilde{y})}\\
=&\frac{b_3(t)y^{m-1}(t)}{g_2(t,x,y)g_2(t,\tilde{x},\tilde{y})}\Big[\Big(d_1(t)+d_3(t)\tilde{y}^n(t)\Big)\Big(x^n(t)-\tilde{x}^n(t)\Big)-d_3(t)\tilde{x}^n(t)\Big(y^n(t)-\tilde{y}^n(t)\Big)\Big]\\
&+\frac{b_3(t)\tilde{x}^n(t)}{g_2(t,\tilde{x},\tilde{y})}\Big(y^{m-1}(t)-\tilde{y}^{m-1}(t)\Big).
\end{aligned}\tag{51}$$

Substituting (51) in to (50) and in view of $B_2 \geq b_2(t)$, we get

$$\begin{aligned}
D^+V_2(t)|_{(1.7)}=&\operatorname{sgn}\Big(y(t)-\tilde{y}(t)\Big)\Big[-b_1(t)\Big(y(t)-\tilde{y}(t)\Big)-b_2(t)\Big(y(t-\tau)-\tilde{y}(t-\tau)\Big)\\
&\frac{b_3(t)y^{m-1}(t)}{g_2(t,x,y)g_2(t,\tilde{x},\tilde{y})}\Big(\Big(d_1(t)+d_3(t)\tilde{y}^n(t)\Big)\Big(x^n(t)-\tilde{x}^n(t)\Big)-d_3(t)\tilde{x}^n(t)\Big(y^n(t)-\tilde{y}^n(t)\Big)\Big)\\
&+\frac{b_3(t)\tilde{x}^n(t)}{g_2(t,\tilde{x},\tilde{y})}\Big(y^{m-1}(t)-\tilde{y}^{m-1}(t)\Big)\Big]+B_2\Big|y(t)-\tilde{y}(t)\Big|-B_2\Big|y(t-\tau)-\tilde{y}(t-\tau)\Big|
\end{aligned}\tag{52}$$

$$\leq -b_1(t)\big|y(t)-\tilde{y}(t)\big| + \frac{b_3(t)y^{m-1}(t)}{g_2(t,x,y)g_2(t,\tilde{x},\tilde{y})}\Big(\big(d_1(t)+d_3(t)\tilde{y}^n(t)\big)\big|x^n(t)-\tilde{x}^n(t)\big| - d_3(t)\tilde{x}^n(t)\big|y^n(t)-\tilde{y}^n(t)\big|\Big) - \frac{b_3(t)\tilde{x}^n(t)}{g_2(t,\tilde{x},\tilde{y})}\big|y^{m-1}(t)-\tilde{y}^{m-1}(t)\big| + B_2\big|y(t)-\tilde{y}(t)\big|.$$

For any $x_1, x_2 \in [a,b] \subset (0,+\infty)$, we have

$$|\alpha|b^{\alpha-1}|x_1-x_2| \leq |x_1^\alpha - x_2^\alpha| \leq |\alpha|a^{\alpha-1}|x_1-x_2|, \text{ for } \alpha<1,$$
$$\alpha a^{\alpha-1}|x_1-x_2| \leq |x_1^\alpha - x_2^\alpha| \leq \alpha b^{\alpha-1}|x_1-x_2|, \text{ for } \alpha>1.$$

Then from (49) and (52) and in view of (44), (45) and (46) and letting $\varepsilon \to 0$, we get

$$\begin{aligned} D^+V_1(t)|_{(1.7)} \leq& \Big(-a_1(t)-(n-1)\gamma^{n+m-2}\frac{a_3(t)(c_1(t)+\gamma^n c_3(t))}{G_1^2(t)} + \Gamma^{2n+m-2}\frac{a_3(t)c_2(t)}{g_1^2(t)} + A_2\Big)\big|x(t)-\tilde{x}(t)\big| \\ &+ \Big(n\Gamma^{2n+m-1}\frac{a_3(t)c_3(t)}{g_1^2(t)} + m\gamma^{m-1}\Gamma^{n-1}\frac{a_3(t)}{g_1(t)}\Big)\big|y(t)-\tilde{y}(t)\big|, \end{aligned} \tag{53}$$

and

$$\begin{aligned} D^+V_2(t)|_{(1.7)} \leq& \Big(-b_1(t)-n\gamma^{2n-1}\Gamma^{m-1}\frac{b_3(t)d_3(t)}{G_2^2(t)} - (1-m)\gamma^n\Gamma^{m-2}\frac{b_3(t)}{G_2(t)} + B_2\Big)\big|y(t)-\tilde{y}(t)\big| \\ &+ n\gamma^{m-1}\Gamma^{n-1}\frac{b_3(t)(d_1(t)+\Gamma^n d_3(t))}{g_2^2(t)}\big|x(t)-\tilde{x}(t)\big| \end{aligned} \tag{54}$$

Summing (53) and (54), we get, for $t > T > 0$, that

$$\begin{aligned} &D^+V(t)|_{(1.7)} = D^+V_1(t)|_{(1.7)} + D^+V_2(t)|_{(1.7)} \\ \leq& -\Big(a_1(t)+(n-1)\gamma^{n+m-2}\frac{a_3(t)(c_1(t)+\gamma^n c_3(t))}{G_1^2(t)} - \Gamma^{2n+m-2}\frac{a_3(t)c_2(t)}{g_1^2(t)} \\ &- n\gamma^{m-1}\Gamma^{n-1}\frac{b_3(t)(d_1(t)+\Gamma^n d_3(t))}{g_2^2(t)} - A_2\Big)\big|x(t)-\tilde{x}(t)\big| \\ &-\Big(b_1(t)+(1-m)\gamma^n\Gamma^{n-2}\frac{b_3(t)}{G_2(t)} + n\gamma^{2n-1}\Gamma^{m-1}\frac{b_3(t)d_3(t)}{G_2^2(t)} - n\Gamma^{2n+m-1}\frac{a_3(t)c_3(t)}{g_1^2(t)} \\ &- m\gamma^{m-1}\Gamma^{n-1}\frac{a_3(t)}{g_1(t)} - B_2\Big)\big|y(t)-\tilde{y}(t)\big| \\ \leq& -\sigma_1\big|x(t)-\tilde{x}(t)\big| - \sigma_2\big|y(t)-\tilde{y}(t)\big|. \end{aligned}$$

Integrating two sides of above inequality, we have

$$V(t) + \sigma_1\int_T^t |x(s)-\tilde{x}(s)|ds + \sigma_2\int_T^t |y(s)-\tilde{y}(t)|ds \leq V(T) < +\infty.$$

This indicates $|x(s)-\tilde{x}(s)|$ and $|y(t)-\tilde{y}(t)|$ are integrable on $[0,+\infty)$. Moreover, since all of solutions of system (8) are bounded, their derivatives are also bounded, then $|x(t)-\tilde{x}(t)|$ and $|y(t)-\tilde{y}(t)|$ are uniformly continuous. From Lemma 4.2, we obtain

$$\lim_{t\to+\infty}|x(t)-\tilde{x}(t)| = 0, \quad \lim_{t\to+\infty}|y(t)-\tilde{y}(t)| = 0. \tag{55}$$

This proves any positive ω-periodic solution of system (8) is globally attractive.

Next we apply (55) to prove the uniqueness of the positive ω-periodic solution $(\tilde{x}(s), \tilde{y}(t))$. Suppose there is another positive ω-periodic solution $(\tilde{x}^*(s), \tilde{y}^*(t))$, we claim that $(\tilde{x}(s), \tilde{y}(t)) = (\tilde{x}^*(s), \tilde{y}^*(t))$. Otherwise, there exists a $\xi \in [0, \omega]$ such that $\tilde{x}(\xi) \neq \tilde{x}^*(\xi)$ or $\tilde{y}(\xi) \neq \tilde{y}^*(\xi)$. Without lose of generality, we suppose $\tilde{x}(\xi) \neq \tilde{x}^*(\xi)$. Let $\varepsilon_0 = |\tilde{x}(\xi) - \tilde{x}^*(\xi)|$, then $\varepsilon_0 > 0$. However

$$\varepsilon_0 = \lim_{n\to+\infty} |\tilde{x}(\xi + n\omega) - \tilde{x}^*(\xi + n\omega)| = \lim_{t\to+\infty} |\tilde{x}(t) - \tilde{x}^*(t)| = 0.$$

This is a contradiction. Therefore the positive ω-periodic solution $(\tilde{x}(s), \tilde{y}(t))$ is unique.

The proof of the theorem is completed. □

Remark 4.4. *Theorem 4.1 extends the results for global attractivity in [8] and [9].*

5. Simulation

Now we consider the following two examples. In the first example, under well selected parameters, conditions (A) and (B) in Theorem 4.1 are satisfied and then the conclusion in Theorem 4.1 holds. But in the second example, we select other parameters such that the conditions (A) and (B) are not satisfied and then the conclusion of Theorem 4.1 does not hold.

Example 5.1. *In system (8), we select*

$$\begin{aligned}
&r_1(t) = 3.99 + 0.01\sin t, \quad a_1(t) = 2.00 - 0.1\sin t, \quad a_2(t) = 0.03 + 0.01\sin t,\\
&a_3(t) = 0.011 + 0.001\sin t, \quad c_1(t) = 1, \quad c_2(t) = 2 + \sin t, \quad c_3(t) = 3 - \sin t,\\
&r_2(t) = 0.41 + 0.01\sin t, \quad b_1(t) = 0.08 - 0.01\sin t, \quad b_2(t) = 0.009 + 0.001\sin t,\\
&b_3(t) = 0.099 + 0.001\sin t, \quad d_1(t) = 1, \quad d_2(t) = 0.26 + 0.01\sin t, \quad d_3(t) = 0.02 - 0.01\sin t,\\
&n = 2, \quad , m = 1/2, \quad \tau = 0.1.
\end{aligned}$$

By simple calculation, we obtain $\sigma_1 \approx \min_{t\in[0,2\pi]}\{0.192 - 0.1\sin t\} = 0.092 > 0$ *and* $\sigma_2 \approx \min_{t\in[0,2\pi]}\{0.03 - 0.01\sin t\} = 0.02 > 0$. *So by Theorem 4.3 we claim that system (8) has only one globally attractive positive* 2π*-periodic solution. In order to demonstrate the conclusion, we take two sets of initial value as* $(x(s), y(s)) = (1.2, 5)$ *and* $(x(s), y(s)) = (3, 27)$ *for* $s \in [-0.1, 0]$. *Its integral curves and orbits are shown in Figs. 4-6, respectively.*

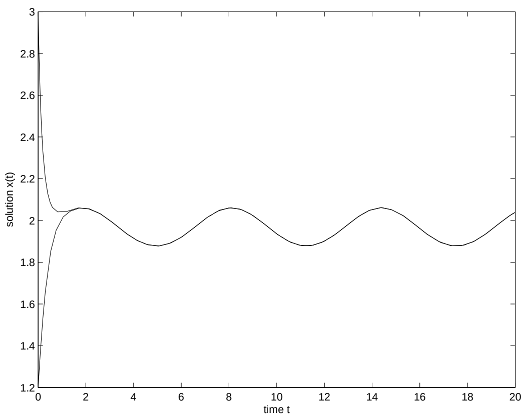

Fig.4. The integral curves of prey.

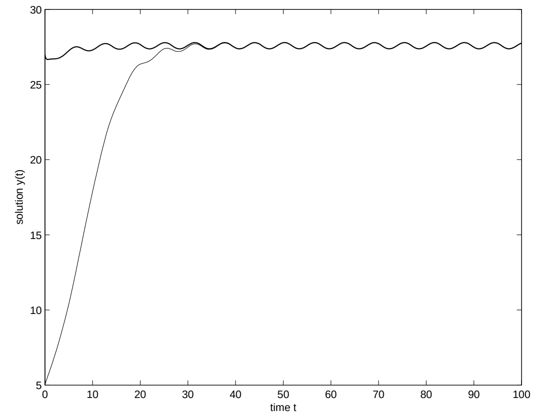

Fig.5. The integral curves of predator.

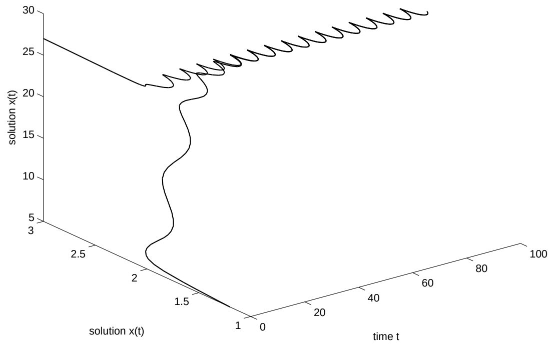

Fig.6. The orbits of predator-prey-time.

From Figs. 1-3, we see that there is only one positive periodic solution of system (8), which is globally attractive.

Example 5.2. *In this example, all of the parameters in system (8) is selected similarly as in Example 5.1 with exception of* $\tau = 10$. *By simple calculation, we obtain* $\sigma_1 \approx -2 \times 10^3 < 0$ *and* $\sigma_2 \approx -10^2 < 0$. *So the conditions* (A) *and* (B) *do not hold and the calculation in Theorem 4.3 may not be true. In order to demonstrate this case, we also take the initial values as* $(x(s), y(s)) = (1.2, 5)$, $(x(s), y(s)) = (3, 27)$ *for* $s \in [-10, 0]$. *Its integral curves and orbits are shown in Figs. 7-9, respectively.*

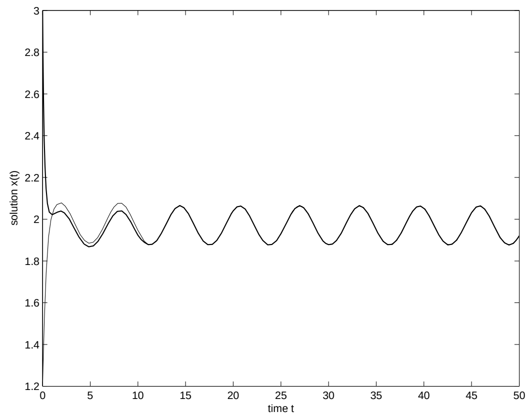

Fig.7. The integral curves of prey.

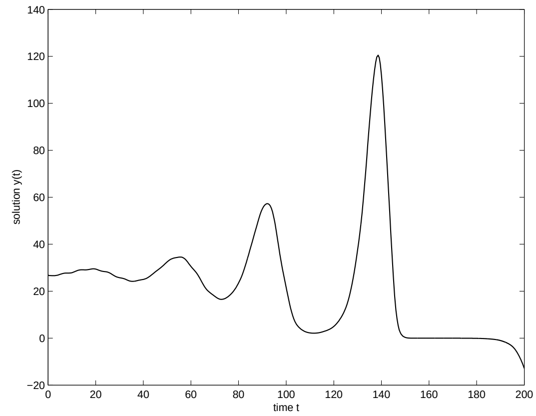

Fig.8. The integral curves of predator.

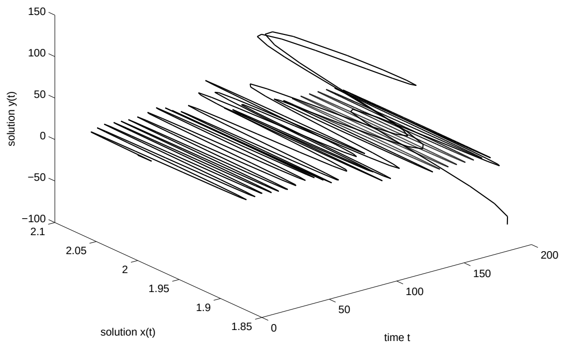

Fig.9. The orbits of predator-prey-time.

From Fig.8 and Fig.9, we see that the solution of predator is not positive periodic and the system (8) has no globally attractive positive periodic solution.

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