Filomat 33:1 (2019), 307–317
https://doi.org/10.2298/FIL1901307K

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Recursive and Combinational Formulas for Permanents of General k-tridiagonal Toeplitz Matrices

Ahmet Zahid Küçük[a], Mehmet Özen[a], Halit İnce[a]

[a] *Department of Mathematics, Sakarya University,Turkey*

Abstract. This study presents recursive relations on the permanents of $k-tridiagonal\ Toeplitz$ matrices which are obtained by the reduction of the matrices to the other matrices whose permanents are easily calculated. These recursive relations are achieved by writing the permanents with bandwidth k in terms of the permanents with a bandwith smaller than k. Based on these recursive relations, an algorithm is given to calculate the permanents of $k-tridiagonal\ Toeplitz$ matrices. Furthermore, explicit combinational formulas, which are obtained using these recurrences, for the permanents are also presented.

1. Introduction

The permanent, which is a function of a matrix, is given by a definition resembling to that of the determinant. The permanent was also called as alternate determinant by Hammond in [1].

If T is an $n \times n$ square matrix then the permanent of T is defined by

$$Per(T) = \sum_{\sigma \epsilon S_n} \prod_{i=1}^{n} a_{i,\sigma_i} \tag{1}$$

where S_n be the symmetric group which consists of all permutations of $\{1, 2, 3, \ldots, n\}$, and σ be the element of this group where $\sigma = (\sigma_1, \sigma_2, ..., \sigma_n)$ [2].

The difference of the permanent definition from the determinant definition is that the factor, which determines the sign of permutations, is not present in the permanent. [2-3] are recommended for those who wish to review the essential properties of the permanent. Applications of the permanent are mostly found in linear algebra, combinatorics and probability theory. We recommend references [4-9] for the applications of the permanents in the area of mathematics. These applications are also encountered except the area of mathematics. For example, in quantum physics, it is shown that all expectation values of polar domain operators can be written with the permanent of a special matrix [10].

Consider the sequence of the permanents of matrices $\left\{T_n^{(k)}(a,b,c)\right\}_{n=1}^{\infty}$ defined by

$$P_n^{(k)} = Per\left(T_n^{(k)}(a,b,c)\right) = Per\left([t_{ij}]_{n\times n}\right) \tag{2}$$

2010 *Mathematics Subject Classification*. Primary 15A15; Secondary 15B05
Keywords. Permanent, k-tridiagonal matrix, Toeplitz matrix, Recurrence relation
Received: 28 March 2018; Revised: 06 February 2019; Accepted: 22 February 2019
Communicated by Predrag Stanimirović
Email addresses: azahidkucuk@gmail.com (Ahmet Zahid Küçük), ozen@sakarya.edu.tr (Mehmet Özen), ince@sakarya.edu.tr (Halit İnce)

where

$$t_{ij} = \begin{cases} a, \ i = j \\ b, \ i = j - k \\ c, \ i = j + k \\ 0, \ otherwise \end{cases} \tag{3}$$

for $n \geq k$ and $a, b, c \in \mathbb{C}$. Accordingly, $T_n^{(k)}$ matrices have the form as

$$T_n^{(k)} = \begin{bmatrix} a & 0 & \dots & 0 & b & & & & 0 \\ 0 & a & 0 & \dots & 0 & b & & & \\ \vdots & 0 & a & \ddots & & \ddots & \ddots & & \\ 0 & \vdots & \ddots & \ddots & \ddots & & \ddots & \ddots & \\ c & 0 & & \ddots & a & \ddots & & 0 & b \\ & c & \ddots & & \ddots & \ddots & \ddots & & 0 \\ & & \ddots & \ddots & & \ddots & a & 0 & \vdots \\ & & & \ddots & \ddots & & 0 & a & 0 \\ 0 & & & & c & 0 & \dots & 0 & a \end{bmatrix}_{n \times n} . \tag{4}$$

This $T_n^{(k)}(a, b, c)$ matrices are introduced as a family of tridiagonal matrices in [11]. According to [12], these matrices can be evaluated as banded matrices whose bandwidth is k. However, for matrices given by (3), it is more qualitative to use the $k-$ *tridiagonal* nomenclature, as in e.g. [26]. Thanks to this nomenclature, positions of the non-zero bands of this matrices can be determined definitely. On the other hand, the $T_n^{(k)}(a, b, c)$ matrices are also in Toeplitz structure. Toeplitz matrix, which is an important member of structured matrices [16], is in a form that each diagonal, which is parallel to the main diagonal, has the same element in its own band. Therefore, in this paper, we call the $T_n^{(k)}(a, b, c)$ matrices as $k-$ *tridiagonal Toeplitz* as in [17]. Throughout the paper, $1-$ *tridiagonal Toeplitz* matrices, which are the matrices of type $T_n^{(k)}(a, b, c)$ when $k = 1$, are called as *tridiagonal Toeplitz*. Note that the $k-$ *tridiagonal Toeplitz* matrix and the *tridiagonal* $k-$ *Toeplitz* matrix (see e.g. [18]) are different matrices.

In [19], a recursive formula has been given for the permanent of general *tridiagonal Toeplitz* matrix. In addition, recursive relations have been obtained for the permanents of general *tridiagonal Toeplitz* matrix by expressing it in terms of the permutation matrices. In [20], the recursive relations for the permanent of a *tridiagonal* matrix which has bands consisting of 1, 1, and $\{a_i\}$ that is a finite sequence of real or complex numbers have been given. In [21], a relationship has been achieved between the permanents of two different general *tridiagonal* matrices whose elements on the main diagonal are with opposite signs to each other. In [22], for the permanent of general $2-$ *tridiagonal Toeplitz* matrix, two distinct recursive formulas which are separated according to the order of the matrix have been given. In [23-25], the permanents of various special *tridiagonal* matrices have been studied. The common feature of these studies is that both the elements used in the matrices are selected from the well-known number sequences, such as Fibonacci and Lucas, and the results obtained are also associated with number sequences. Besides, in [25], the permanents of some *tridiagonal Toeplitz* matrices have been expressed by Binet's formulas. In [26], an algorithm based on the LU factorization for the permanents of general $k-$ *tridiaonal Toeplitz* matrices are presented. In [27], the permanents of a special type of $k-$ *tridiaonal Toeplitz* matrix with complex entries have been expressed in terms of the Chebyshev polynomials. In [28], some calculation algorithms have been given for the permanents of general $k-$ *tridiagonal* matrices. However, it has also been stated in that study that these algorithms work just for matrices with integer elements. In [14], a significant representation has been given for the permanent of general $k-$ *tridiagonal* $k-$ *Toeplitz* matrix by utilizing direct sum operation for permanents and the permutation matrices. In [13] and [15], the permanents of a special kind of *Toeplitz* matrix have been studied and some recursive formulizations or algorithms have been obtained.

Although the following two studies are for the determinants, they include the outcomes that resemble to the our findings. In [11], some recursive relationships for the determinants of general $k-$*tridiagonal Toeplitz* matrices have been obtained. In [29], some recursive representations have been presented for determinants of general pentadiagonal matrices.

In this study, we aimed to obtain recursive and explicit formulizations for the permanents of the general $k-$*tridiagonal Toeplitz* matrices defined by (3). Our motivation in this work has three basic pillars that are found in [11], [27], and [19]. In that regard, we have worked in a wide range on the permanents of the matrices $T_n^{(k)}(a,b,c)$ whose determinants was studied in [11]. Recursive relations given for the permanents of a special matrix in [27] have been obtained for the permanents of the $T_n^{(k)}(a,b,c)$ matrices which are the generalized form of the matrices in [27]. Moreover, it has been examined whether recursive formulas for the permanents of $T_n^{(k)}(a,b,c)$ matrices for $k>1$ could be obtained in such a structure similar to the formula given for the permanent of $T_n^{(1)}(a,b,c)$ in [19].

2. Preliminaries

The most important reason why there are fewer studies on the permanent function than that of the determinant function is that every operation cannot be performed on the permanent. For example, the permanent function is unchanging when a row is multiplied by a scalar and added to another row [2]. It is not anticipated that the result of a permanent which has the same two rows is always zero [7]. Also, the basic multiplicative rule of determinants given by $det(AB) = det(A)det(B)$ does not hold for the permanent [2].

All these results greatly restrict the methods that can be used to calculate the permanent. For this reason, as in the calculation of the determinant, it is a common method to reduce the matrix to another matrix whose permanent can be computed easily. One of the tools that can be used to reducing matrix is the Laplace expansion. This method, also known as cofactor expansion, can also be utilized in permanent calculation [7]. Laplace expansion method has been used in this study intensively because it allows us to select the elements of the matrices also from complex numbers. Some methods, for example the Contraction [6], cannot be used for the permanent of matrices whose elements are complex numbers.

Now, let's present a Lemma which is the motivation of our study, before we proceed to the main results. Minc, in [19], has proved the following lemma:

Lemma 2.1. *Let the permanent of tridiagonal Toeplitz matrix be*

$$P_n^{(1)} = Per\left(\begin{bmatrix} a & b & & & \\ c & a & b & & \\ & \ddots & \ddots & \ddots & \\ & & c & a & b \\ & & & c & a \end{bmatrix}_{n\times n}\right). \tag{5}$$

Then for $n \geq 2$

$$P_n^{(1)} = aP_{n-1}^{(1)} + bcP_{n-2}^{(1)} \tag{6}$$

with initial conditions are $P_0^{(1)} = 1$, *and* $P_1^{(1)} = a$.

Let us move from here to see whether recursive relations like (6), which is a difference equation, can also be produced for $P_n^{(k)}$ in case of $k>1$.

3. Main Results

3.1. Recursive Formulizations

Theorem 3.1. *Let the permanent of* 2 − *tridiagonal Toeplitz matrix be*

$$P_n^{(2)} = Per\left(\begin{bmatrix} a & 0 & b & & & & 0 \\ 0 & a & 0 & b & & & \\ c & 0 & a & 0 & b & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & c & 0 & a & 0 & b \\ & & & c & 0 & a & 0 \\ 0 & & & & c & 0 & a \end{bmatrix}_{n\times n}\right). \tag{7}$$

Then for $n \geq 3$

$$P_n^{(2)} = aP_{n-1}^{(2)} + abcP_{n-3}^{(2)} + b^2c^2P_{n-4}^{(2)}. \tag{8}$$

with initial conditions are $P_{-1}^{(2)} = 0$, $P_0^{(2)} = 1$, $P_1^{(2)} = a$, *and* $P_2^{(2)} = a^2$. *This formula provides for* $a, b, c \in \mathbb{C}$.

Proof. Using the Laplace expansion for the $P_n^{(2)}$ with respect to its $1st$ column, we obtain

$$P_n^{(2)} = aP_{n-1}^{(2)} + cPer\left(\begin{bmatrix} 0 & b & 0 & & & & 0 \\ a & 0 & b & 0 & & & \\ c & 0 & a & 0 & b & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & c & 0 & a & 0 & b \\ & & & c & 0 & a & 0 \\ & & & & c & 0 & a \end{bmatrix}_{n-1\times n-1}\right). \tag{9}$$

Now, we expand the permanent appearing in the second term of the right hand side of equality (9) with respect to its $1st$ row. So, we obtain

$$P_n^{(2)} = aP_{n-1}^{(2)} + bcPer\left(\begin{bmatrix} a & b & 0 & & & & & & 0 \\ c & a & 0 & b & & & & & \\ 0 & 0 & a & 0 & b & & & & \\ & c & 0 & a & 0 & b & & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & & & c & 0 & a & 0 & b & \\ & & & & c & 0 & a & 0 & \\ 0 & & & & & c & 0 & a & \end{bmatrix}_{n-2\times n-2}\right). \tag{10}$$

We again expand the permanent appearing in the second term of the right hand side of equality (10) with respect to its $1st$ row. So, we obtain

$$P_n^{(2)} = aP_{n-1}^{(2)} + abcP_{n-3}^{(2)} + b^2cPer\left(\begin{bmatrix} c & 0 & b & & & & & 0 \\ 0 & a & 0 & b & & & & \\ 0 & 0 & a & 0 & b & & & \\ & c & 0 & a & 0 & b & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & & c & 0 & a & 0 & b \\ & & & & c & 0 & a & 0 \\ & & & & & c & 0 & a \end{bmatrix}_{n-3\times n-3}\right). \tag{11}$$

Finally, we expand the permanent appearing in the third term of the right hand side of equality (11) with respect to its $1st$ column. So, we obtain

$$P_n^{(2)} = aP_{n-1}^{(2)} + abcP_{n-3}^{(2)} + b^2c^2Per\left(\begin{bmatrix} a & 0 & b & & & 0 \\ 0 & a & 0 & b & & \\ c & 0 & a & 0 & b & \\ & \ddots & \ddots & \ddots & \ddots & \\ & & c & 0 & a & 0 \\ & & & c & 0 & a \end{bmatrix}_{n-4\times n-4}\right) \tag{12}$$

which is written as

$$P_n^{(2)} = aP_{n-1}^{(2)} + abcP_{n-3}^{(2)} + b^2c^2P_{n-4}^{(2)}. \tag{13}$$

So the proof is completed. □

Theorem 3.2. *Let the permanent of* $3-$ *tridiagonal Toeplitz matrix be*

$$P_n^{(3)} = Per\left(\begin{bmatrix} a & 0 & 0 & b & & & & & 0 \\ 0 & a & 0 & 0 & b & & & & \\ 0 & 0 & a & 0 & 0 & b & & & \\ c & 0 & 0 & a & 0 & 0 & b & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & c & 0 & 0 & a & 0 & 0 & b \\ & & & c & 0 & 0 & a & 0 & 0 \\ & & & & c & 0 & 0 & a & 0 \\ 0 & & & & & c & 0 & 0 & a \end{bmatrix}_{n\times n}\right). \tag{14}$$

Then for $n \geq 4$

$$P_n^{(3)} = aP_{n-1}^{(3)} + bcP_{\lceil \frac{n}{3} \rceil - 2}^{(1)} P_{n-\lceil \frac{n}{3} \rceil}^{(2)} \tag{15}$$

where $P_s^{(3)} = a^s$ *for* $0 \leq s \leq 3$. *This formula provides for* $a, b, c \in \mathbb{C}$.

Proof. We will prove this theorem using Laplace expansion. While doing this, we will not write the matrices obtained after each expansion step clearly as in the proof of the Theorem 3.1. Instead, we will give the steps applied as a procedure.
First step:
The procedure starts by expanding the permanent of $P_n^{(3)}$ along the its $1st$ row. Then, we obtain

$$P_n^{(3)} = aP_{n-1}^{(3)} + bP\left(T_{n-1}^{*}\right). \tag{16}$$

Here, the $P\left(T_{n-1}^{*}\right)$ notation represents the permanent of the submatrix with $(n-1)\times(n-1)$ order that emerges after the first step but doesn't fit the form (3).
Second step:
$P\left(T_{n-1}^{*}\right)$ is expanded along the its $1st$ column. Then, we obtain

$$P_n^{(3)} = aP_{n-1}^{(3)} + bcP\left(T_{n-2}^{\blacktriangle}\right). \tag{17}$$

Here, $P\left(T_{n-2}^{\blacktriangle}\right)$ notation represents the permanent of the sub–matrix with $(n-2)\times(n-2)$ order that emerges

after the second step but doesn't fit the form (3).
Remaining steps:
The following steps of the procedure will be continued by expanding $P\left(T^{\blacktriangle}_{n-2}\right)$ and all submatrices which will be derived from $P\left(T^{\blacktriangle}_{n-2}\right)$ along the $(3+2i)th$ rows where $i = 1, 2, 3, ..., \lceil \frac{n}{3} \rceil - 2$. Accordingly, first, $P\left(T^{\blacktriangle}_{n-2}\right)$ is expanded by the its $5th$ row. In the next step, permanents of $(n-3) \times (n-3)$ ordered matrices emerged after the previous step are expanded by their $7th$ rows. Then, permanents of $(n-4) \times (n-4)$ ordered matrices emerged after the previous step are expanded by their $9th$ row and so forth. The overall expansion process is repeated $\lceil \frac{n}{3} \rceil$ times in total until the procedure ends. One can obviously see that all the permanents of $(n - \lceil \frac{n}{3} \rceil) \times (n - \lceil \frac{n}{3} \rceil)$ ordered matrices emerged after the last expansion step of the procedure are in $2-$ *tridiagonal Toeplitz* form. The coefficients of these permanents shown as $P^{(2)}_{n-\lceil \frac{n}{3} \rceil}$ constitute a polynomial with variables a, b, and c. This polynomial corresponds to $P^{(1)}_{\lceil \frac{n}{3} \rceil - 2}$. The aforesaid coefficients can be seen in an illustrative example given below. Thus, the equation (17) becomes

$$P^{(3)}_n = aP^{(3)}_{n-1} + bcP^{(1)}_{\lceil \frac{n}{3} \rceil - 2} P^{(2)}_{n-\lceil \frac{n}{3} \rceil}. \tag{18}$$

□

Recursive relation given for $P^{(3)}_n$ will be generalized to $P^{(k)}_n$ by the following theorem.

Theorem 3.3. *The permanent of general $k-$ tridiagonal Toeplitz matrix satisfies the following recursive equation:*

$$P^{(k)}_n = aP^{(k)}_{n-1} + bcP^{(1)}_{\lceil \frac{n}{k} \rceil - 2} P^{(k-1)}_{n-\lceil \frac{n}{k} \rceil} \tag{19}$$

where $n \geq k+1$ and $P^{(k)}_s = a^s$ for $0 \leq s \leq k$. This formula provides for $a, b, c \in \mathbb{C}$.

Proof. This theorem is proved in analogy to the proof of Theorem 3.2. The *first step* and the *second step* in the procedure given in the proof of Theorem 3.2 are applied to $P^{(k)}_n$ in the same way. The subsequent steps which are conjoined under heading of the *Remaining steps* in that procedure will be applied as by expanding the $(k + i(k-1))$ th rows where $i = 1, 2, 3, ..., \left\lceil \frac{n}{k} \right\rceil - 2$. The procedure will end for $P^{(k)}_n$ when the number of total steps are $\left\lceil \frac{n}{k} \right\rceil$. □

Now we give the following recursive algorithm based on Theorem 3.1, 3.2, and 3.3.

Algorithm 1 A recursive algorithm to calculate the permanent $P_n^{(k)}$

INPUT: Order of the matrix n, bandwidth of the matrix k.
OUTPUT: Permanent of the matrix $T_n^{(k)}$.
procedure $\textsc{perm}(k, n)$
  if $n = -1$ **then**
    return 0
  else if $k \geq n$ **then**
    return a^n
  else if $k = 1$ **then**
    return $a\textsc{perm}(1, n-1) + bc\textsc{perm}(1, n-2)$
  else if $k = 2$ **then**
    if $n \geq 3$ **then**
      return $a\textsc{perm}(2, n-1) + abc\textsc{perm}(2, n-3) + b^2c^2\textsc{perm}(2, n-4)$
    end if
  else if $k \geq 3$ **then**
    return
    $a\textsc{perm}(k, n-1) + bc\textsc{perm}(1, \left\lceil \frac{n}{k} \right\rceil - 2)\textsc{perm}(k-1, n - \left\lceil \frac{n}{k} \right\rceil)$
  end if
end procedure

3.2. *An Illustrative Example*

The procedure for the permanent of the matrix that forms in the case of $k = 3$ and $n = 19$ is as follows:

*Step*1:

$P_{19}^{(3)}$ is expanded along the its 1*st* row. Then, we obtain

$$P_{19}^{(3)} = aP_{18}^{(3)} + bP\left(T_{18}^{*}\right). \tag{20}$$

Here, the $P\left(T_{18}^{*}\right)$ notation represents the permanent of order 18 *by* 18 that emerges after the first step but doesn't fit to the form shown in (3).

*Step*2 :

$P\left(T_{18}^{*}\right)$ is expanded along the its 1*st* column. Then, we obtain

$$P_{19}^{(3)} = aP_{18}^{(3)} + bcP\left(T_{17}^{\blacktriangle}\right). \tag{21}$$

Here, the $P\left(T_{n-2}^{\blacktriangle}\right)$ notation represents the permanent of order 17 *by* 17 that emerges after the second step but doesn't fit to the form shown in (3).

The procedure will be continued by expanding $P\left(T_{17}^{\blacktriangle}\right)$ and all matrices which will be derived from $P\left(T_{17}^{\blacktriangle}\right)$ along the $(3+2i)th$ rows where $i = 1, 2, 3, ..., \lceil \frac{19}{3} \rceil - 2$. Accordingly, expansion will be performed along the 5*th*, 7*th*, 9*th*, 11*th* and 13*th* rows. Emergent permanents after every new step will not be shown clearly; instead, their coefficients on behalf of the permanents will be written. The zeros inside the coefficients represent the terms where the values of the permanents are zero. The remaining steps are as follows.

*Step*3 :

$P\left(T_{17}^{\blacktriangle}\right)$ is expanded by its 5*th* row. Coefficients of the generated permanents of order 16 *by* 16 are bca and b^2c, respectively.

*Step*4 :

All emergent permanents after the previous step are expanded by their 7*th* rows. So, the coefficients of generated permanents of order 15 *by* 15 are bca^2, b^2ca, b^2c^2, and b^3c, respectively.

*Step*5 :
All emergent permanents after the previous step are expanded by their 9*th* rows. So, the coefficients of generated permanents of order 14 *by* 14 are bca^3, b^2ca^2, b^2c^2a, b^3ca, b^2c^2a, b^3c^2, and b^4c, respectively.

*Step*6 :
All emergent permanents after the previous step are expanded by their 11*th* rows. So, the coefficients of generated permanents of order 13 *by* 13 are bca^4, b^2ca^3, $b^2c^2a^2$, b^3ca^2, $b^2c^2a^2$, b^3c^2a, b^4ca, $b^2c^2a^2$, b^3c^2a, b^3c^3, b^4c^2, and b^5c, respectively.

*Step*7 :
All emergent permanents after the previous step are expanded by their 13*th* rows. So, the coefficients of generated permanents of order 12 *by* 12 are bca^5, $b^2c^2a^3$, $b^2c^2a^3$, 0, $b^2c^2a^3$, b^3c^3a, 0, $b^2c^2a^3$, b^3c^3a, b^3c^3a, 0, and 0, respectively.

The procedure ends in this step since the total number of steps is equal to $\left\lceil \frac{19}{3} \right\rceil$. It will be seen that all permanents emerged after the last step have been reduced to the form of $P_{12}^{(2)}$. Thus, achieved equality is

$$P_{19}^{(3)} = aP_{18}^{(3)} + bc(a^5 + 4a^3bc + 3ab^2c^2)P_{12}^{(2)}. \tag{22}$$

The coefficient of $bcP_{12}^{(2)}$ can be arranged as

$$a\left(\binom{5}{0}\left(a^2\right)^2 + \binom{4}{1}\left(a^2\right)^1 bc + \binom{3}{2}(bc)^2\right). \tag{23}$$

This statement is equal to $P_5^{(1)}$ according to the Theorem 3.4. So,

$$P_{19}^{(3)} = aP_{18}^{(3)} + bcP_5^{(1)}P_{12}^{(2)} \tag{24}$$

is obtained.

3.3. Explicit Formulizations for $k = 1$ and $k = 2$

Theorem 3.4. *The permanent of the general tridiagonal matrix can be formulized as*

$$P_n^{(1)} = a^{\frac{1-(-1)^n}{2}} \sum_{r=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n-r}{r} \left(a^2\right)^{\left\lfloor \frac{n}{2} \right\rfloor - r} (bc)^r. \tag{25}$$

Proof. It can be proved by induction on n.
Equality (25) for $n = 2$

$$P_2^{(1)} = a^{\frac{1-(-1)^2}{2}} \sum_{r=0}^{1} \binom{2-r}{r} \left(a^2\right)^{1-r} (bc)^r = \binom{2}{0}\left(a^2\right)^1 + \binom{1}{1} bc = a^2 + bc. \tag{26}$$

On the other hand,

$$P_2^{(1)} = Per\left(\begin{bmatrix} a & b \\ c & a \end{bmatrix}_{2\times 2}\right) = a^2 + bc. \tag{27}$$

Therefore, if $n = 2$, equality (25) holds. Suppose that equality (25) holds for n. It should also be shown that equality (25) holds for $(n + 1)$, that is

$$P_{n+1}^{(1)} = a^{\frac{1-(-1)^{n+1}}{2}} \sum_{r=0}^{\left\lfloor \frac{n+1}{2} \right\rfloor} \binom{n+1-r}{r} \left(a^2\right)^{\left\lfloor \frac{n+1}{2} \right\rfloor - r} (bc)^r. \tag{28}$$

First, let's write $P^{(1)}_{n+1}$ by using equality (6). Then, we obtain

$$P^{(1)}_{n+1} = a.P^{(1)}_{n} + bcP^{(1)}_{n-1}. \tag{29}$$

If the right hand side of equality (29) is rewritten by using equality (25) which is supposed, then

$$P^{(1)}_{n+1} = a^{1+\frac{1-(-1)^n}{2}} \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-r}{r} \left(a^2\right)^{\lfloor \frac{n}{2} \rfloor - r} (bc)^r + bca^{\frac{1-(-1)^{n-1}}{2}} \sum_{r=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-r}{r} \left(a^2\right)^{\lfloor \frac{n-1}{2} \rfloor - r} (bc)^r . \tag{30}$$

is obtained. Upper bounds of the summations in equality (30) should be considered with respect to the case of n which are odd or even. But, we will continue the rest of the proof for only one case of n.
Let n be odd. Then, $\left\lfloor \frac{n}{2} \right\rfloor = \left\lfloor \frac{n-1}{2} \right\rfloor = \frac{n-1}{2}$. Accordingly, equality (30) turns into

$$P^{(1)}_{n+1} = a^2 \sum_{r=0}^{\frac{n-1}{2}} \binom{n-r}{r} a^{n-1-2r} (bc)^r + bc \sum_{r=0}^{\frac{n-1}{2}} \binom{n-1-r}{r} a^{n-1-2r} (bc)^r . \tag{31}$$

If equality (31) is written explicitly, we obtain

$$P^{(1)}_{n+1} = \binom{n}{0} a^{n+1} + \left(\binom{n-1}{1} + \binom{n-1}{0}\right) a^{n-1}bc + \left(\binom{n-2}{2} + \binom{n-2}{1}\right) a^{n-3} (bc)^2 + \ldots + \left(\binom{\frac{n+1}{2}}{\frac{n-1}{2}} + \binom{\frac{n+1}{2}}{\frac{n-3}{2}}\right) a^2 (bc)^{\frac{n-1}{2}} + (bc)^{\frac{n+1}{2}} . \tag{32}$$

If equality (32) is re-arranged using the following property of the combination

$$\binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r}$$

then

$$P^{(1)}_{n+1} = \left(a^2\right)^{\frac{n+1}{2}} + \binom{n}{1} \left(a^2\right)^{\frac{n-1}{2}} bc + \binom{n-1}{2} \left(a^2\right)^{\frac{n-3}{2}} (bc)^2 + \ldots + \binom{\frac{n+3}{2}}{\frac{n-1}{2}} a^2 (bc)^{\frac{n-1}{2}} + (bc)^{\frac{n+1}{2}} \tag{33}$$

is obtained. Finally, equality (33) can be written in the following form

$$P^{(1)}_{n+1} = a^{\frac{1-(-1)^{n+1}}{2}} \sum_{r=0}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n+1-r}{r} \left(a^2\right)^{\lfloor \frac{n+1}{2} \rfloor - r} (bc)^r . \tag{34}$$

Hence, the proof is completed for n is odd. It is easy to perform the proof similarly for n is even. □

Theorem 3.5. *The permanent of general* $2-$ *tridiagonal matrix can be formulized as*

$$P^{(2)}_{n} = a^n + bc \sum_{i=0}^{n-3} a^{n-3-i} P^{(1)}_{\lfloor \frac{i}{2} \rfloor} P^{(1)}_{\lfloor \frac{i+1}{2} \rfloor + 1} \tag{35}$$

where $P^{(1)}_0 = 1$, $P^{(1)}_1 = a$, *and* $P^{(1)}_2 = a^2$.

Proof. It can be proved by induction on n.
Equality (35) for $n = 3$

$$P^{(2)}_{3} = a^3 + bc \sum_{i=0}^{0} a^{-i} P^{(1)}_{\lfloor \frac{i}{2} \rfloor} P^{(1)}_{\lfloor \frac{i+1}{2} \rfloor + 1} = a^3 + bcP^{(1)}_0 P^{(1)}_1 = a^3 + bca. \tag{36}$$

On the other hand,

$$P_3^{(2)} = Per\left(\begin{bmatrix} a & 0 & b \\ 0 & a & 0 \\ c & 0 & a \end{bmatrix}_{3\times 3}\right) = a^3 + bca. \tag{37}$$

Therefore, equality (35) holds for $n = 3$. Suppose that equality (35) holds for $n = r$. That is,

$$P_r^{(2)} = a^r + bc\sum_{i=0}^{r-3} a^{r-3-i} P^{(1)}_{\lfloor \frac{i}{2} \rfloor} P^{(1)}_{\lfloor \frac{i+1}{2} \rfloor + 1}. \tag{38}$$

It should also be shown that equality (35) holds for $n = r + 1$, i.e.

$$P_{r+1}^{(2)} = a^{r+1} + bc\sum_{i=0}^{r-2} a^{r-2-i} P^{(1)}_{\lfloor \frac{i}{2} \rfloor} P^{(1)}_{\lfloor \frac{i+1}{2} \rfloor + 1}. \tag{39}$$

First, let $k = 2$ and $n = r + 1$ in equality (19). Then, we have

$$P_{r+1}^{(2)} = aP_r^{(2)} + bcP^{(1)}_{\lceil \frac{r+1}{2} \rceil - 2} P^{(1)}_{r+1-\lceil \frac{r+1}{2} \rceil}. \tag{40}$$

Let's put equality (38) in to equality (40):

$$P_{r+1}^{(2)} = a\left(a^r + bc\sum_{i=0}^{r-3} a^{r-3-i} P^{(1)}_{\lfloor \frac{i}{2} \rfloor} P^{(1)}_{\lfloor \frac{i+1}{2} \rfloor + 1}\right) + bcP^{(1)}_{\lceil \frac{r+1}{2} \rceil - 2} P^{(1)}_{r+1-\lceil \frac{r+1}{2} \rceil}. \tag{41}$$

Since $\lceil \frac{r+1}{2} \rceil - 2 = \lfloor \frac{r-2}{2} \rfloor$ and $r + 1 - \lceil \frac{r+1}{2} \rceil = \lfloor \frac{r-1}{2} \rfloor + 1$ for every integer r, it can be written $P^{(1)}_{\lfloor \frac{r-2}{2} \rfloor} P^{(1)}_{\lfloor \frac{r-1}{2} \rfloor + 1}$ instead of $P^{(1)}_{\lceil \frac{r+1}{2} \rceil - 2} P^{(1)}_{r+1-\lceil \frac{r+1}{2} \rceil}$ which is the second term of the right hand side of (41). Accordingly, if equality (41) is rewritten explicitly then we have

$$P_{r+1}^{(2)} = a^{r+1} + bc\left(a^{r-2}P_0^{(1)}P_1^{(1)} + ... + aP^{(1)}_{\lfloor \frac{r-3}{2} \rfloor} P^{(1)}_{\lfloor \frac{r-2}{2} \rfloor + 1} + P^{(1)}_{\lfloor \frac{r-2}{2} \rfloor} P^{(1)}_{\lfloor \frac{r-1}{2} \rfloor + 1}\right). \tag{42}$$

Hence, equality (42) can be written as

$$P_{r+1}^{(2)} = a^{r+1} + bc\sum_{i=0}^{r-2} a^{r-2-i} P^{(1)}_{\lfloor \frac{i}{2} \rfloor} P^{(1)}_{\lfloor \frac{i+1}{2} \rfloor + 1} \tag{43}$$

which is desired. □

In fact, the formula given with the Theorem 3.5 also is an explicit formula for the permanent of $2-$ *tridiagonal Toeplitz* matrix. Because, $P^{(1)}_{\lfloor \frac{i}{2} \rfloor}$ and $P^{(1)}_{\lfloor \frac{i+1}{2} \rfloor + 1}$ multipliers in that formula have been expressed combinatorially through the Theorem 3.4.

4. Conclusions

Recursive relations which are given in this study make it possible to calculate the permanents of general $k-$ *tridiagonal Toeplitz* matrices with the help of permanents of $(k-i)-$ *tridiagonal Toeplitz* matrices where $i = 1, 2, ..., k-1$. By using these recursive relations, explicit formulas have been obtained for $P_n^{(1)}$ and $P_n^{(2)}$. In addition, an algorithm built through the Laplace expansion has been proposed for the permanents of general $k-$ *tridiagonal Toeplitz* matrices. It can be investigated whether this algorithm can be developed for more general matrices than the once we used.

This work can be evaluated as a distinct application in the matter of reducing matrices because the recursive relations in this study are achieved by expressing the permanents of the matrices with bandwidth k in terms of the permanents with a bandwidth smaller than k.

References

[1] J. Hammond, Question 6001, Educ. Times, 32 (1879) 179.
[2] R. A. Brualdi, H. J. Ryser, Combinatorial Matrix Theory, (1st edition), Cambridge University Press, New York, 1991.
[3] M. Marcus, H. Minc, Permanents, The American Mathematical Monthly, 72 (1965) 577–591.
[4] R. D. McMillan, The Permanent Function, Oklahoma State University Library, PhD Thesis No: 724964, (1969).
[5] D. C. Kozen, The Design and Analysis of Algorithms (pp.144–150), Springer, New York, 1992.
[6] R. A. Brualdi, P. M. Gibson, Convex polyhedra of doubly stochastic matrices I: Applications of the permanent function, The Journal of Combinatorial Theory (A) 22 (1977) 194–230.
[7] H. Minc, Permanents, in: Encyclopedia of Mathematics and its Applications vol.6, Cambridge University Press, 1978.
[8] R. B. Bapat, Multinomial probabilities, permanents and a conjecture of Karlin and Rinott, Proceedings of the American Mathematical Society, 102 (1988). 467–472.
[9] B. Codonetti, G. Resta, Computation of sparse circulant permanents via determinants, Linear Algebra and its Applications, 355 (2002) 15–34.
[10] S. Scheel, Permanents in Linear Optical Networks, arXiv preprint quant-ph/0406127, (2004).
[11] G. E. Bergun, V. E. Hoggatt, A family of tridiagonal matrices, Fibonacci Quart., 16 (1978) 285–288.
[12] W. F. Trench, Banded symmetric Toeplitz matrices: where linear algebra borrows from difference equations, Available at: http://ramanujan.math.trinity.edu/wtrench/research/papers/TRENCH_TN_12.PDF
[13] H. Minc, Recurrence formulas for permanents of (0,1)-circulants,. Linear Algebra and Its Applications, 71 (1985) 241–265.
[14] E. Kırklar, F. Yılmaz, A note on k-tridiagonal k-Toeplitz matrices, Alabama Journal of Mathematics 39 (2015).
[15] B. Codenotti, V. Crespi, G. Resta, On the permanent of certain (0, 1) Toeplitz matrices, Linear Algebra and its Applications, 267 (1997) 65–100.
[16] D. A. Bini, Matrix structures and applications, Les Cours du CIRM, 4.1 (2014) 1–45.
[17] J. Jia, T. Sogabe, M. El-Mikkawy, Inversion of k-tridiagonal Matrices with Toeplitz Structure, Computers and Mathematics with Applications, 65, (2013) 116–125.
[18] C. M. Fonseca, J. Petronilho, Explicit inverse of a tridiagonal k-Toeplitz matrix, Numerische Mathematik, 100 (2005) 457–482.
[19] H. Minc, On permanents of circulants, Pacific Journal of Mathematics, 42 (1972).
[20] D. Lehmer, Fibonacci and related sequences in periodic tridiagonal matrices, Fibonacci Quart., 13 (1975) 150–158.
[21] E. Kilic, D. Tasci, On the permanents of some tridiagonal matrices with applications to the Fibonacci and Lucas numbers, Rocky Mountain J. Math., 37 (2007) 1953–1969.
[22] E. Kılıç, On a constant-diagonals matrix, Applied Mathematics and Computation, 204 (2008) 184–190.
[23] I. Matousova, P. Trojovsky, On a sequence of tridiagonal matrices, whose permanents are related to Fibonacci and Lucas numbers, International Journal of Pure and Applied Mathematics, 105 (2015) 715–723.
[24] A. Sahin, K. Kaygısız, Determinantal and permanental representations of Fibonacci type numbers and polynomials, Rocky Mountain Journal of Mathematics, 46 (2016) 227–242.
[25] E. Kilic, D. Tasci, On the second order linear recurrences by tridiagonal matrices, Ars Combinatoria, 91 (2009) 11–18.
[26] M. Asci, D. Tasci, M. El-Mikkawy, On determinants and permanents of k-tridiagonal Toeplitz matrices, Util. Math., 89 (2012) 97–106.
[27] A. Z. Kucuk, M. Duz, Relationships between the permanents of a certain type of k-tridiagonal symmetric Toeplitz matrix and the Chebyshev polynomials, Journal of Applied Mathematics and Computational Mechanics, 16 (2017) 75–86.
[28] T. Sogabe, F. Yılmaz, A note on a fast breakdown-free algorithm for computing the determinants and the permanents of k-tridiagonal matrices, Applied Mathematics and Computation, 249 (2014) 98–102.
[29] J. Mikloško, A Recursive Computation of the Determinant of A Pentadiagonal Matrix, Journal of Computational and Applied Mathematics, 1 (1975) 73–78.