Filomat 33:1 (2019), 267–280
https://doi.org/10.2298/FIL1901267C

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Existence of Positive Solutions for a Class of Kirchhoff Type Systems Involving Critical Exponents

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Abstract. In this paper, we consider a class of Kirchhoff type systems involving critical exponents in bounded domains. Under appropriate conditions on the nonlinearities, we prove the existence and asymptotic behavior of positive solutions for the problem by using truncation argument combined with the mountain pass theorem and a variant of concentration compactness principle related to critical elliptic systems in [19].

1. Introduction

In this paper, we are concered with a class of Kirchhoff type systems involving critical exponents of the form

$$\begin{cases} -M\left(\int_\Omega |\nabla u|^2\,dx\right)\Delta u = \lambda F_u(x,u,v) + \frac{2\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^\beta \text{ in } \Omega, \\ -M\left(\int_\Omega |\nabla v|^2\,dx\right)\Delta v = \lambda F_v(x,u,v) + \frac{2\beta}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2}v \text{ in } \Omega, \\ u = v = 0 \text{ on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$, $N \geq 3$, $\alpha, \beta > 1$, $\alpha + \beta = 2^* = \frac{2N}{N-2}$, $\nabla F = (F_u, F_v)$ is the gradient of the C^1 function $F : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ with respect to $(u, v) \in \mathbb{R}^2$, and λ is a positive parameter, and $M : [0, +\infty) \to \mathbb{R}$ is an increasing and continuous function satisfying the condition:

(M_0) there exists $m_0 > 0$ such that $M(t) \geq m_0 = M(0)$ for all $t \in [0, +\infty)$.

Since problem (1) contains integrals over Ω, it is no longer a pointwise identity; therefore it is often called nonlocal problem. This problem models several physical and biological systems, where u describes a process which depends on the average of itself, such as the population density, see [9]. Moreover, problem (1) is related to the stationary version of the Kirchhoff equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0 \tag{2}$$

2010 *Mathematics Subject Classification*. Primary 35J50; Secondary 35D30, 35J60
Keywords. Kirchhoff type systems; Critical exponents; Variational methods
Received: 19 March 2018; Accepted: 02 February 2019
Communicated by Marko Nedeljkov
This research is supported by Vietnam National Foundation for Science and Technology Development (NAFOSTED) (Grant N.101.02.2017.04).
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presented by Kirchhoff in 1883, see [22]. This equation is an extension of the classical d'Alembert's wave equation by considering the effects of the changes in the length of the string during the vibrations. The parameters in (2) have the following meanings: L is the length of the string, h is the area of the cross-section, E is the Young modulus of the material, ρ is the mass density, and P_0 is the initial tension.

In recent years, Kirchhoff type equations have been studied in many papers, we refer to some interesting papers [2, 5, 7, 10–12, 15, 26], in which the authors have used different methods to get the existence of solutions. In [27, 30], Z. Zhang et al. studied the existence of nontrivial solutions and sign-changing solutions for (1). The study of systems of Kirchhoff type equations can be found in [6, 8, 13, 14, 29, 32]. Critical problems involving nonlocal operators creat many difficulties in applying variational methods. These come from the fact that the embedding $H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$ is not compact and thus the Palais-Smale condition fails. To overcome the difficulties brought, many authors used the concentration compactness principle due to Lions [24, 25], we refer to [3, 16, 17, 23, 28, 31] for more details. In a recent paper [19], D.S. Kang has established a variant of concentration compactness principle related to critical elliptic systems, which is based on the ideads by P.L. Lions [24, 25]. This result is very useful for the study of the existence of solutions for critical elliptic systems, see further the papers [20, 21] for the local case. In this paper, motivated by [16, 19–21] and the ideas introduced in [1], we study the existence of positive solutions for Kirchhoff type system (1) with critical growth. It should be noticed that we don't need any conditions on the Kirchhoff function $M(t)$ except for the boundedness from below on $[0, +\infty)$ as stated in (M_0). So, our situation introduced here is different from those presented in [3, 17, 18]. We also refer the interested readers to some results [23, 28, 31] in which the authors considered the problem in the special case $M(t) = a + bt$, $a > 0$ and $b \geq 0$. By the condition (M_0), the Kirchhoff function $M(t)$ may be unbounded. For this reason, in order to apply the concentration compactness principle by D.S. Kang [19], we need a truncation on $M(t)$ as in (3). Then, an existence result for system (1) is established by using the mountain pass theorem due to A. Ambrosetti and P.H. Rabinowitz [4].

We assume that $F \in C^1(\Omega \times \mathbb{R}^2, \mathbb{R})$, $\nabla F = (F_u, F_v)$, $F_u, F_v : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ are continuous functions satisfying the following conditions

(F_0) $F(x, s, t) = F_s(x, s, t) = F_t(x, s, t) = 0$ a.e. $x \in \Omega$ for all $s \leq 0$ or $t \leq 0$;

(F_1) $\lim_{|(s,t)| \to 0} \frac{|\nabla F(x,s,t)|}{|(s,t)|} = 0$ uniformly in $x \in \Omega$;

(F_2) $\lim_{|(s,t)| \to +\infty} \frac{|\nabla F(x,s,t)|}{|(s,t)|^{q-1}} = 0$ uniformly in $x \in \Omega$, where $q \in (2, 2^*)$;

(F_3) There exists $\theta \in (2, 2^*)$ such that

$$0 < \theta F(x, s, t) \leq F_s(x, s, t)s + F_t(x, s, t)t, \quad \forall x \in \Omega, \quad s, t > 0.$$

Let $H_0^1(\Omega)$ be the usual Sobolev space with respect to the norm $\|u\| = \left(\int_\Omega |\nabla u|^2 \, dx\right)^{\frac{1}{2}}$. Set $H = H_0^1(\Omega) \times H_0^1(\Omega)$. Then H is a Hilbert space with respect to the inner product

$$(w_1, w_2)_H = \int_\Omega (\nabla u_1 \nabla u_2 + \nabla v_1 \nabla v_2) \, dx, \quad \forall w_1 = (u_1, v_1), \quad w_2 = (u_2, v_2) \in H$$

and the norm

$$\|w\|_H = \left(\int_\Omega |\nabla u|^2 + |\nabla v|^2 \, dx\right)^{\frac{1}{2}}, \quad w = (u, v) \in H.$$

Denote by S_r the best constant in the embedding $H \hookrightarrow L^r(\Omega) \times L^r(\Omega)$, that is, $S_r \|w\|_{L^r(\Omega) \times L^r(\Omega)} \leq \|w\|_H$ for all $w \in H$.

Definition 1.1. *We say that $(u, v) \in H$ is a weak solution of system (1) if*

$$M\left(\|u\|^2\right) \int_\Omega \nabla u \nabla \varphi \, dx + M\left(\|v\|^2\right) \int_\Omega \nabla v \nabla \psi \, dx - \lambda \int_\Omega (F_u(x, u, v)\varphi + F_v(x, u, v)\psi) \, dx$$
$$- \frac{2\alpha}{\alpha + \beta} \int_\Omega |u|^{\alpha - 2} u |v|^\beta \varphi \, dx - \frac{2\beta}{\alpha + \beta} \int_\Omega |u|^\alpha |v|^{\beta - 2} v \psi \, dx = 0$$

for all $(\varphi, \psi) \in H$.

The main result of this paper can be stated as follows.

Theorem 1.2. *Assume that* (M_0), (F_0)-(F_3) *are satisfied. Then, there exists* $\lambda^* > 0$ *such that for all* $\lambda \geq \lambda^*$, *system (1) has a positive solution. Moreover, if* (u_λ, v_λ) *is a solution of system (1) then* $\lim_{\lambda\to+\infty} \|(u_\lambda, v_\lambda)\|_H = 0$.

2. Proof of the main result

Here we are assuming, without loss of generality, that the Kirchhoff function $M(t)$ is unbounded. Contrary case, the truncation on $M(t)$ is not necessary. Since we are intending to work with $N \geq 3$, we shall make a truncation on M as follows. From (M_0), given $a \in \mathbb{R}$ such that $m_0 < a < \frac{\theta}{2}m_0$, there exists $t_0 > 0$ such that $M(t_0) = a$. We set

$$M_a(t) := \begin{cases} M(t), & 0 \leq t \leq t_0, \\ a, & t \geq t_0. \end{cases} \tag{3}$$

From (M_0) we get

$$M_a(t) \leq a, \quad \forall t \geq 0. \tag{4}$$

As we shall see, the proof of Theorem 1.2 is based on a careful study of the solutions of the following auxiliary problem

$$\begin{cases} -M_a\left(\int_\Omega |\nabla u|^2\,dx\right)\Delta u = \lambda F_u(x,u,v) + \frac{2\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^\beta \text{ in } \Omega, \\ -M_a\left(\int_\Omega |\nabla v|^2\,dx\right)\Delta v = \lambda F_v(x,u,v) + \frac{2\beta}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2}v \text{ in } \Omega, \\ u = v = 0 \text{ on } \partial\Omega, \end{cases} \tag{5}$$

where $f, N, \alpha, \beta, \lambda$ are as in Section 1. We shall prove the following auxiliary result.

Theorem 2.1. *Assume that* (M_0), (F_0)-(F_3) *are satisfied. Then, there exists* $\lambda_0 > 0$ *such that for all* $\lambda \geq \lambda_0$ *and all* $a \in (m_0, \frac{\theta}{2}m_0)$, *system (5) has a positive solution.*

From the hypothesis (F_0) and we intend to find positive solutions, we recall that $(u, v) \in H$ is a weak solution of system (5) if

$$\begin{aligned} &M_a\left(\|u\|^2\right)\int_\Omega \nabla u\nabla\varphi\,dx + M_a\left(\|v\|^2\right)\int_\Omega \nabla v\nabla\psi\,dx - \lambda\int_\Omega (F_u(x,u,v)\varphi + F_v(x,u,v)\psi)\,dx \\ &-\frac{2\alpha}{\alpha+\beta}\int_\Omega u_+^{\alpha-1}v_+^\beta\varphi\,dx - \frac{2\beta}{\alpha+\beta}\int_\Omega u_+^\alpha v_+^{\beta-1}v\psi\,dx = 0 \end{aligned}$$

for all $(\varphi, \psi) \in H$, where $u_+ = \max\{0, u\}$ and $v_+ = \max\{0, v\}$. Hence, we shall look for positive solutions of (5) by finding critical points of the C^1- functional $I_{a,\lambda} : H \to \mathbb{R}$ given by the formula

$$I_{a,\lambda}(u,v) = \frac{1}{2}\widehat{M}_a\left(\|u\|^2\right) + \frac{1}{2}\widehat{M}_a\left(\|v\|^2\right) - \lambda\int_\Omega F(x,u,v)\,dx - \frac{2}{\alpha+\beta}\int_\Omega (u_+)^\alpha(v_+)^\beta\,dx,$$

where $\widehat{M}(t) = \int_0^t M(s)\,ds$. Note that

$$\begin{aligned} I'_{a,\lambda}(u,v)(\varphi,\psi) &= M_a\left(\|u\|^2\right)\int_\Omega \nabla u\nabla\varphi\,dx + M_a\left(\|v\|^2\right)\int_\Omega \nabla v\nabla\psi\,dx \\ &-\lambda\int_\Omega (F_u(x,u,v)\varphi + F_v(x,u,v)\psi)\,dx - \frac{2\alpha}{\alpha+\beta}\int_\Omega u_+^{\alpha-1}v_+^\beta\varphi\,dx - \frac{2\beta}{\alpha+\beta}\int_\Omega u_+^\alpha v_+^{\beta-1}v\psi\,dx \end{aligned}$$

for all $(\varphi, \psi) \in H$. Moreover, if the critical point is nontrival, by maximum principle, we conclude that it is a positive solution of the system.

We say that a sequence $\{w_n\} \subset H$ is a Palais-Smale sequence for the functional $I_{a,\lambda}$ at level $c \in \mathbb{R}$ if

$$I_{a,\lambda}(w_n) \to c \text{ and } I'_{a,\lambda}(w_n) \to 0 \text{ in } H^*,$$

where H^* is the dual space of H. If every Palais-Smale sequence of $I_{a,\lambda}$ has a strong convergent subsequence, then one says that $I_{a,\lambda}$ satisfies the Palais-Smale condition ((PS) condition for short).

In our arguments, we need the following useful result which helps us to overcome the nonlocal case.

Lemma 2.2. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 3$ be a domain (not necessary bounded). For $k, l > 0$, $\alpha, \beta > 1$, $\alpha + \beta \leq 2^*$, let us denote*

$$S^{k,l}_{\alpha,\beta} = \inf_{(u,v)\in H\setminus\{0\}} \frac{\int_\Omega (k|\nabla u|^2 + l|\nabla v|^2)\,dx}{\left(\int_\Omega |u|^\alpha |v|^\beta\,dx\right)^{\frac{2}{\alpha+\beta}}}. \tag{6}$$

Then we have

$$S^{k,l}_{\alpha,\beta} = k^{\frac{\alpha}{\alpha+\beta}} l^{\frac{\beta}{\alpha+\beta}} \left[\left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} + \left(\frac{\beta}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}}\right] S_{\alpha+\beta}, \tag{7}$$

where S is the best Sobolev constant, that is,

$$S_{\alpha+\beta} = \inf_{u\in H^1_0(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2\,dx}{\left(\int_\Omega |u|^{\alpha+\beta}\,dx\right)^{\frac{2}{\alpha+\beta}}}. \tag{8}$$

Proof. The proof of Lemma 2.2 is similar to [1, Theorem 5]. For the reader's convenience, we present it here in details.

We first observe that $S_{\alpha+\beta}$ is well defined since the embedding $H^1_0(\Omega) \hookrightarrow L^{\alpha+\beta}(\Omega)$ $(2 < \alpha + \beta \leq 2^*)$ is continuous. Moreover, since

$$|u|^\alpha |v|^\beta \leq \frac{\alpha}{\alpha+\beta}|u|^{\alpha+\beta} + \frac{\beta}{\alpha+\beta}|v|^{\alpha+\beta} \leq |u|^{\alpha+\beta} + |v|^{\alpha+\beta}$$

and

$$k|\nabla u|^2 + l|\nabla v|^2 \geq \min\{k, l\}\left(|\nabla u|^2 + |\nabla v|^2\right)$$

the number $S^{k,l}_{\alpha,\beta}$ is well defined.

Consider $\{y_n\}$ a minimizing sequence for $S_{\alpha+\beta}$. Let $s, t > 0$ be chosen later. Taking $u_n = sy_n$ and $v_n = ty_n$ in the quotient (6) we have that

$$\frac{ks^2 + lt^2}{(s^\alpha t^\beta)^{\frac{2}{\alpha+\beta}}} \cdot \frac{\int_\Omega |y_n|^2\,dx}{\left(\int_\Omega |y_n|^{\alpha+\beta}\,dx\right)^{\frac{2}{\alpha+\beta}}} \geq S^{k,l}_{\alpha,\beta}. \tag{9}$$

Observe that

$$\frac{ks^2 + lt^2}{(s^\alpha t^\beta)^{\frac{2}{\alpha+\beta}}} = k\left(\frac{s}{t}\right)^{\frac{2\beta}{\alpha+\beta}} + l\left(\frac{s}{t}\right)^{\frac{-2\alpha}{\alpha+\beta}}$$

and define the function

$$g(x) = kx^{\frac{2\beta}{\alpha+\beta}} + lx^{\frac{-2\alpha}{\alpha+\beta}}, \quad x > 0.$$

A simple computation shows that the minimum of the function g at point $x = \sqrt{\frac{l\alpha}{k\beta}}$ with minimum value

$$g\left(\sqrt{\frac{l\alpha}{k\beta}}\right) = k^{\frac{\alpha}{\alpha+\beta}} l^{\frac{\beta}{\alpha+\beta}} \left[\left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} + \left(\frac{\beta}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}}\right].$$

Choosing s and t in (9) such that $\frac{s}{t} = \sqrt{\frac{l\alpha}{k\beta}}$ we deduce that

$$k^{\frac{\alpha}{\alpha+\beta}} l^{\frac{\beta}{\alpha+\beta}} \left[\left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} + \left(\frac{\beta}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}}\right] \frac{\int_\Omega |y_n|^2\,dx}{\left(\int_\Omega |y_n|^{\alpha+\beta}\,dx\right)^{\frac{2}{\alpha+\beta}}} \geq S^{k,l}_{\alpha,\beta}$$

and hence,

$$k^{\frac{\alpha}{\alpha+\beta}} l^{\frac{\beta}{\alpha+\beta}} \left[\left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} + \left(\frac{\beta}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}}\right] S_{\alpha+\beta} \geq S^{k,l}_{\alpha,\beta}. \tag{10}$$

To complete the proof, let (u_n, v_n) be a minimizing sequence for $S^{k,l}_{\alpha,\beta}$. Define $z_n = s_n v_n$ for some $s_n > 0$ such that

$$\int_\Omega |u_n|^{\alpha+\beta}\,dx = \int_\Omega |z_n|^{\alpha+\beta}\,dx. \tag{11}$$

By Young's inequality,

$$|u_n|^\alpha |z_n|^\beta \leq \frac{\alpha}{\alpha+\beta}|u_n|^{\alpha+\beta} + \frac{\beta}{\alpha+\beta}|z_n|^{\alpha+\beta}$$

and (11) it implies

$$\left(\int_\Omega |u_n|^\alpha |z_n|^\beta\,dx\right)^{\frac{2}{\alpha+\beta}} \leq \left(\int_\Omega |u_n|^{\alpha+\beta}\,dx\right)^{\frac{2}{\alpha+\beta}} = \left(\int_\Omega |z_n|^{\alpha+\beta}\,dx\right)^{\frac{2}{\alpha+\beta}}. \tag{12}$$

Using (12), we have

$$\begin{aligned}
\frac{\int_\Omega (k|\nabla u_n|^2 + l|\nabla v_n|^2)\,dx}{\left(\int_\Omega |u_n|^\alpha |v_n|^\beta\,dx\right)^{\frac{2}{\alpha+\beta}}} &= s_n^{\frac{2\beta}{\alpha+\beta}} \frac{\int_\Omega (k|\nabla u_n|^2 + l|\nabla v_n|^2)\,dx}{\left(\int_\Omega |u_n|^\alpha |z_n|^\beta\,dx\right)^{\frac{2}{\alpha+\beta}}} \\
&\geq k s_n^{\frac{2\beta}{\alpha+\beta}} \frac{\int_\Omega |\nabla u_n|^2\,dx}{\left(\int_\Omega |u_n|^{\alpha+\beta}\,dx\right)^{\frac{2}{\alpha+\beta}}} + l s_n^{\frac{2\beta}{\alpha+\beta}}.s_n^{-2} \frac{\int_\Omega |\nabla z_n|^2\,dx}{\left(\int_\Omega |z_n|^{\alpha+\beta}\,dx\right)^{\frac{2}{\alpha+\beta}}} \\
&\geq g(s_n) S_{\alpha+\beta}
\end{aligned}$$

and hence,

$$\frac{\int_\Omega (k|\nabla u_n|^2 + l|\nabla v_n|^2)\,dx}{\left(\int_\Omega |u_n|^\alpha |v_n|^\beta\,dx\right)^{\frac{2}{\alpha+\beta}}} \geq g\left(\sqrt{\frac{l\alpha}{k\beta}}\right) S_{\alpha+\beta}. \tag{13}$$

Passing to the limit in (13) we obtain

$$S^{k,l}_{\alpha,\beta} \geq k^{\frac{\alpha}{\alpha+\beta}} l^{\frac{\beta}{\alpha+\beta}} \left[\left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} + \left(\frac{\beta}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}}\right] S_{\alpha+\beta}. \tag{14}$$

From (10) and (14), the proof of Lemma 2.2 is completed. □

From Lemma 2.2, using the arguments as those presented by D.S. Kang [19, Theorem 1] we obtain the following result.

Lemma 2.3. *Assume that $\Omega \subset \mathbb{R}^N$, $N \geq 3$ is a bounded smooth domain, $k, l > 0$, $\alpha, \beta > 1$ and $\alpha + \beta = 2^* = \frac{2N}{N-2}$. Let $\{(u_n, v_n)\} \rightharpoonup (u, v)$ in H, $\{k|\nabla u_n|^2 + l|\nabla v_n|^2\} \rightharpoonup k|\nabla u|^2 + l|\nabla v|^2 + \mu$ and $|u_n|^\alpha |v_n|^\beta \rightharpoonup |u|^\alpha |v|^\beta + \nu$ as $n \to \infty$, where μ and ν are nonnegative bounded measures on $\mathbb{R}^N$. Then there exist an at most countable set J and families $\{x_j\}_{j \in J} \subset \mathbb{R}^N$ and $\{\mu_j\}_{j\in J}, \{\nu_j\}_{j\in J} \subset [0, +\infty)$ such that*

$$\mu \geq \sum_{j \in J} \mu_j \delta_{x_j}, \quad \nu = \sum_{j\in J} \nu_j \delta_{x_j}, \quad \nu_j^{\frac{2}{2^*}} S_{\alpha,\beta}^{k,l} \leq \mu_j, \quad \forall j \in J,$$

where δ_{x_j} is the Dirac mass at $x_j \in \Omega$ and $S_{\alpha,\beta}^{k,l}$ is given by Lemma 2.2.

Lemma 2.4. *For all $\lambda > 0$, there exist positive constants ρ and r such that $I_{a,\lambda}(w) \geq r > 0$ for all $w \in H$ with $\|w\|_H = \rho$.*

Proof. From (F_1) for each $\epsilon > 0$, there exists $\delta > 0$ such that

$$|\nabla F(x, s, t)| < \epsilon |(s, t)| \text{ for all } |(s, t)| < \delta \text{ and all } x \in \Omega. \tag{15}$$

On the other hand, from (F_2), there exists $C_1 > 0$ such that

$$|\nabla F(x, s, t)| \leq C_1(1 + |(s, t)|^{q-1}) \text{ for all } (s, t) \in \mathbb{R}^2 \text{ and all } x \in \Omega. \tag{16}$$

From (15) and (16), for each $\epsilon > 0$, there exists a constant $C_\epsilon > 0$ such that

$$|\nabla F(x, s, t)| \leq \epsilon |(s, t)| + C_\epsilon |(s, t)|^{q-1} \text{ for all } (s, t) \in \mathbb{R}^2 \text{ and all } x \in \Omega. \tag{17}$$

Hence,

$$\begin{aligned} |F(x, s, t)| &= |F(x, s, t) - F(x, 0, 0)| \\ &= |\nabla F(x, \theta_1 s, \theta_2 t) \cdot (s, t)| \\ &\leq |\nabla F(x, \theta_1 s, \theta_2 t)||(s, t)| \\ &\leq \epsilon |(s, t)|^2 + C_\epsilon |(s, t)|^q \end{aligned} \tag{18}$$

for all $(s, t) \in \mathbb{R}^2$ and all $x \in \Omega$, where $\theta_i \in (0, 1)$ and $\overline{C}_\epsilon$ is a positive constant.

From (18) and (M_0), for all $w = (u, v) \in H$, we get

$$\begin{aligned} I_{a,\lambda}(w) &= \frac{1}{2}\widehat{M}_a\left(\|u\|^2\right) + \frac{1}{2}\widehat{M}_a\left(\|v\|^2\right) - \lambda \int_\Omega F(x, u, v)\, dx - \frac{2}{\alpha+\beta}\int_\Omega (u_+)^\alpha (v_+)^\beta\, dx \\ &\geq \frac{m_0}{2}\|u\|^2 + \frac{m_0}{2}\|v\|^2 - \lambda \int_\Omega \left(\epsilon|(u, v)|^2 + C_\epsilon |(u, v)|^q\right) dx \\ &\quad - \frac{2}{2^*(S_{\alpha,\beta}^{1,1})^{\frac{2^*}{2}}}\left(\int_\Omega \left(|\nabla u|^2 + |\nabla v|^2\right) dx\right)^{\frac{2^*}{2}} \\ &\geq \frac{m_0}{2}\|w\|_H^2 - \lambda \epsilon S_2^{-2}\|w\|_H^2 - \lambda C_\epsilon S_q^{-q}\|w\|_H^q - \frac{2}{2^*(S_{\alpha,\beta}^{1,1})^{\frac{2^*}{2}}}\|w\|_H^{2^*}. \end{aligned}$$

For $\lambda > 0$, let $\epsilon = \frac{m_0 S_2^2}{4\lambda}$, we get

$$\begin{aligned} I_{a,\lambda}(w) &\geq \frac{m_0}{4}\|w\|_H^2 - \lambda C_\epsilon S_q^{-q}\|w\|_H^q - \frac{2}{2^*(S_{\alpha,\beta}^{1,1})^{\frac{2^*}{2}}}\|w\|_H^{2^*} \\ &= \|w\|_H^2 \left(\frac{m_0}{4} - \lambda C_\epsilon S_q^{-q}\|w\|_H^{q-2} - \frac{2}{2^*(S_{\alpha,\beta}^{1,1})^{\frac{2^*}{2}}}\|w\|_H^{2^*-2}\right). \end{aligned}$$

Since $2 < q < 2^*$, there exist positive constants ρ and r such that $I_{a,\lambda}(w) \geq r > 0$ for all $w \in H$ with $\|w\|_H = \rho$. $\square$

Lemma 2.5. *For all $\lambda > 0$, there exists $e \in H$ with $I_{a,\lambda}(e) < 0$ and $\|e\|_H > \rho$.*

Proof. Fix $s, t > 0$ and $x \in \Omega$. For $\tau > \tau_0 > 0$, from (F_3), we get

$$\begin{aligned}\frac{d}{d\tau}F(x,\tau s,\tau t) &= sF_s(x,\tau s,\tau t) + tF_t(x,\tau s,\tau t)\\ &= \frac{1}{\tau}\left(\tau sF_s(x,\tau s,\tau t) + \tau tF_t(x,\tau s,\tau t)\right)\\ &\geq \frac{\theta}{\tau}F(x,\tau s,\tau t),\end{aligned}$$

which gives

$$\frac{\frac{d}{d\tau}F(x,\tau s,\tau t)}{F(x,\tau s,\tau t)} \geq \frac{\theta}{\tau}.$$

Hence, integrating the above inequality from τ_0 to τ implies that

$$\ln F(x,\tau s,\tau t) - \ln F(x,\tau_0 s,\tau_0 t) \geq \ln \tau^\theta - \ln \tau_0^\theta$$

or

$$\frac{F(x,\tau s,\tau t)}{\tau^\theta} \geq \frac{\tau_0^\theta}{F(x,\tau_0 s,\tau_0 t)} = \gamma(x,s,t) > 0, \quad \forall s,t > 0 \text{ and } x \in \Omega.$$

Thus, we obtain

$$F(x,\tau s,\tau t) \geq \gamma(x,s,t)\tau^\theta, \quad \forall s,t > 0 \text{ and } x \in \Omega. \tag{19}$$

Fix $w_0 = (u_0, v_0)$, $u_0, v_0 \in C_0^\infty(\Omega)\backslash\{0\}$ with $u_0 > 0$ and $v_0 > 0$ and $\|u_0\| = \|v_0\| = 1$. Using (4) and (19), for all $\tau > 0$ large enough, we have

$$\begin{aligned}I_{a,\lambda}(\tau u_0,\tau v_0) &= \frac{1}{2}\widehat{M}_a\left(\|\tau u_0\|^2\right) + \frac{1}{2}\widehat{M}_a\left(\|\tau v_0\|^2\right) - \lambda\int_\Omega F(x,\tau u_0,\tau v_0)\,dx\\ &\quad - \frac{2}{\alpha+\beta}\int_\Omega (\tau u_0)^\alpha(\tau v_0)^\beta\,dx\\ &\leq a\tau^2 - \lambda\tau^\theta\int_\Omega \gamma(x,u_0,v_0)\,dx - \frac{2\tau^{2^*}}{\alpha+\beta}\int_\Omega u_0^\alpha v_0^\beta\,dx.\end{aligned}$$

Since $\theta > 2$, the result follows by considering $e = \tau_* w_0$ for some $\tau_* > 0$ large enough. $\square$

Using a version of the Mountain pass theorem due to Ambrosetti and Rabinowitz [?], without (PS) condition, there exists a sequence $\{w_n\} \subset H$ such that

$$I_{a,\lambda}(w_n) \to c_{a,\lambda}, \quad I'_{a,\lambda}(w_n) \to 0 \text{ as } n \to \infty,$$

where

$$c_{a,\lambda} = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I_{a,\lambda}(\gamma(t))$$

and

$$\Gamma = \left\{\gamma \in C([0,1],H) : \gamma(0) = 0,\ I_{a,\lambda}(\gamma(1)) < 0\right\}.$$

Lemma 2.6. *It holds that*

$$\lim_{\lambda\to+\infty} c_{a,\lambda} = 0.$$

Proof. Since the functional $I_{a,\lambda}$ has the Mountain pass geometry, it follows that there exists $t_\lambda > 0$ verifying $I_{a,\lambda}(t_\lambda w_0) = \max_{t\geq 0} I_{a,\lambda}(tw_0)$, where $w_0 = (u_0, v_0)$ is the function given by Lemma 2.5. Hence, $\frac{d}{dt}I_{a,\lambda}(t_\lambda w_0)(t_\lambda w_0) = 0$ or

$$\begin{aligned} 0 &= M_a\left(\|t_\lambda u_0\|^2\right)\int_\Omega |\nabla t_\lambda u_0|^2\,dx + M_a\left(\|t_\lambda v_0\|^2\right)\int_\Omega |\nabla t_\lambda v_0|^2\,dx \\ &\quad - \lambda\int_\Omega (F_u(x, t_\lambda u_0, t_\lambda v_0)t_\lambda u_0 + F_v(x, t_\lambda u_0, t_\lambda v_0)t_\lambda v_0)\,dx - 2t_\lambda^{2^*}\int_\Omega u_0^\alpha v_0^\beta\,dx. \end{aligned}$$

Hence,

$$2t_\lambda^2 M_a(|t_\lambda|^2) = \lambda\int_\Omega (F_u(x, t_\lambda u_0, t_\lambda v_0)t_\lambda u_0 + F_v(x, t_\lambda u_0, t_\lambda v_0)t_\lambda v_0)\,dx + 2t_\lambda^{2^*}\int_\Omega u_0^\alpha v_0^\beta\,dx. \tag{20}$$

From (4), (20) and (F_3), we get

$$a \geq t_\lambda^{2^*-2}\int_\Omega u_0^\alpha v_0^\beta\,dx,$$

which implies that $\{t_\lambda\}$ is bounded. Thus, there exist a sequence $\lambda_n \to +\infty$ and $\bar{t} \geq 0$ such that $t_{\lambda_n} \to \bar{t}$ as $n \to \infty$. Consequently, there is $C_2 > 0$ such that

$$t_{\lambda_n}^2 M_a(t_{\lambda_n}^2) \leq C_2, \quad \forall n \in \mathbb{N},$$

and $\forall n \in \mathbb{N}$,

$$\lambda_n\int_\Omega (F_u(x, t_{\lambda_n}u_0, t_{\lambda_n}v_0)t_{\lambda_n}u_0 + F_v(x, t_{\lambda_n}u_0, t_{\lambda_n}v_0)t_{\lambda_n}v_0)\,dx + 2t_{\lambda_n}^{2^*}\int_\Omega u_0^\alpha v_0^\beta\,dx \leq C_2. \tag{21}$$

If $\bar{t} > 0$, by (17) and the Dominated Convergence Theorem,

$$\begin{aligned} &\lim_{n\to\infty}\int_\Omega (F_u(x, t_{\lambda_n}u_0, t_{\lambda_n}v_0)t_{\lambda_n}u_0 + F_v(x, t_{\lambda_n}u_0, t_{\lambda_n}v_0)t_{\lambda_n}v_0)\,dx \\ &= \int_\Omega (F_u(x, \bar{t}u_0, \bar{t}v_0)\bar{t}u_0 + F_v(x, \bar{t}u_0, \bar{t}v_0)\bar{t}v_0)\,dx \end{aligned}$$

and thus (21) leads to

$$\lim_{n\to\infty}\left(\lambda_n\int_\Omega (F_u(x, t_{\lambda_n}u_0, t_{\lambda_n}v_0)t_{\lambda_n}u_0 + F_v(x, t_{\lambda_n}u_0, t_{\lambda_n}v_0)t_{\lambda_n}v_0)\,dx + 2t_{\lambda_n}^{2^*}\int_\Omega u_0^\alpha v_0^\beta\,dx\right) = +\infty,$$

which is an absurd. Thus, we conclude that $\bar{t} = 0$. Now, let us consider the path $\gamma_*(t) = te$ for $t \in [0, 1]$, which belongs to Γ, to get the following estimate

$$0 < c_{a,\lambda} \leq \max_{t\in[0,1]} I_{a,\lambda}(\gamma_*(t)) = I_{a,\lambda}(t_\lambda w_0) \leq \widehat{M}_a(t_\lambda^2).$$

In this way,

$$\lim_{\lambda\to+\infty}\widehat{M}(t_\lambda^2) = 0,$$

which leads to $\lim_{\lambda\to+\infty} c_{a,\lambda} = 0$. □

Lemma 2.7. *Let $\{w_n\} = \{(u_n, v_n)\} \subset H$ be a sequence such that*

$$I_{a,\lambda}(w_n) \to c_{a,\lambda}, \quad I'_{a,\lambda}(w_n) \to 0 \text{ as } n \to \infty. \tag{22}$$

Then $\{w_n\}$ is bounded.

Proof. Assuming by contradiction that $\{w_n\}$ is not bounded in H, up to a subsequence, we may assume that $\|w_n\|_H \to +\infty$ as $n \to \infty$. By (22), (M_0), (F_3) and $\frac{2a}{m_0} < \theta < 2^*$, we deduce that for n large enough,

$$\begin{aligned}
&1 + c_{a,\lambda} + \|w_n\|_H \\
&\geq I_{a,\lambda}(w_n) - \frac{1}{\theta} I'_{a,\lambda}(w_n)(w_n) \\
&= \frac{1}{2}\widehat{M}_a(\|u_n\|^2) + \frac{1}{2}\widehat{M}_a(\|v_n\|^2) - \lambda \int_\Omega F(x, u_n, v_n)\, dx - \frac{2}{2^*}\int_\Omega |u_n|^\alpha |v_n|^\beta\, dx \\
&\quad - \frac{1}{\theta} M_a(\|u_n\|^2) \int_\Omega |\nabla u_n|^2\, dx - \frac{1}{\theta} M_a(\|v_n\|^2)\int_\Omega |\nabla v_n|^2\, dx \\
&\quad + \frac{\lambda}{\theta}\int_\Omega (F_u(x, u_n, v_n)u_n + F_v(x, u_n, v_n)v_n)\, dx + \frac{2}{\theta}\int_\Omega |u_n|^\alpha |v_n|^\beta\, dx \\
&\geq \left(\frac{m_0}{2} - \frac{a}{\theta}\right)(\|u_n\|^2 + \|v_n\|^2) - \frac{\lambda}{\theta}\int_\Omega (F_u(x, u_n, v_n)u_n + F_v(x, u_n, v_n)v_n - \theta F(x, u_n, v_n))\, dx \\
&\quad + \left(\frac{2}{\theta} - \frac{2}{2^*}\right)\int_\Omega |u_n|^\alpha |v_n|^\beta\, dx \\
&\geq \left(\frac{m_0}{2} - \frac{a}{\theta}\right)\|w_n\|_H^2 - C_3,
\end{aligned}$$

where C_3 is a positive constant. Therefore, the sequence $\{w_n\}$ is bounded in H. □

Proof. [Proof of Theorem 2.1] From Lemma 2.4, we have

$$\lim_{\lambda \to +\infty} c_{a,\lambda} = 0. \tag{23}$$

Therefore, there exists $\lambda_0 > 0$ such that

$$c_{a,\lambda} < \left(\frac{2}{\theta} - \frac{2}{2^*}\right)\left(\frac{S^{k,l}_{\alpha,\beta}}{2}\right)^{\frac{N}{2}}, \tag{24}$$

for all $\lambda \geq \lambda_0$, where $S^{k,l}_{\alpha,\beta}$ is given by (7). Now, fix $\lambda \geq \lambda_0$ and let us show that system (5) admits a positive solution. From Lemmas 2.4 and 2.5, there exists a bounded sequence $\{w_n\} = \{(u_n, v_n)\} \subset H$ verifying

$$I_{a,\lambda}(w_n) \to c_{a,\lambda}, \quad I'_{a,\lambda}(w_n) \to 0 \text{ as } n \to \infty. \tag{25}$$

Since $\|w_n\|_H^2 = \|u_n\|^2 + \|v_n\|^2$, the sequences $\{u_n\}$ and $\{v_n\}$ are bounded in $H^1_0(\Omega)$. Hence, up to subsequences, there are $\alpha_0, \beta_0 \geq 0$ such that

$$\|u_n\| \to \alpha_0, \quad \|v_n\| \to \beta_0 \text{ as } n \to \infty.$$

Since $M_a(t)$ is a continuous function, we obtain

$$M_a(\|u_n\|^2) \to M_a(\alpha_0^2) > 0, \quad M_a(\|v_n\|^2) \to M_a(\beta_0^2) > 0 \text{ as } n \to \infty. \tag{26}$$

Assume that $\{w_n\} = \{(u_n, v_n)\}$ converges weakly to $w = (u, v) \in H$. We claim that $\|u_n\|^2 \to \|u\|^2$ and $\|v_n\|^2 \to \|v\|^2$ as $n \to \infty$, which imply that $u_n \to u$ and $v_n \to v$ in $H^1_0(\Omega)$ as $n \to \infty$, and then we conclude that $w = (u, v)$ is a nontrival solution of problem (5). Indeed, assuming the claim and using the fact that $I_{a,\lambda}$ is C^1 we obtain

$$I_{a,\lambda}(w_n) \to I_{a,\lambda}(w) \text{ and } I'_{a,\lambda}(w_n) \to I'_{a,\lambda}(w)$$

and so letting $n \to \infty$,

$$I_{a,\lambda}(w) = c_{a,\lambda} \text{ and } I'_{a,\lambda}(w) = 0.$$

This means that the result follows.

In order to prove the claim, choosing $k = M_a(\alpha_0^2) > 0$ and $l = M_a(\beta_0^2) > 0$, taking a subsequence, we may assume that

$$\begin{aligned} M_a(\alpha_0^2)|\nabla u_n|^2 + M_a(\beta_0^2)|\nabla v_n|^2 &\rightharpoonup M_a(\alpha_0^2)|\nabla u|^2 + M_a(\beta_0^2)|\nabla v|^2 + \mu, \\ |u_n|^\alpha |v_n|^\beta &\rightharpoonup |u|^\alpha |v|^\beta + \nu, \end{aligned} \tag{27}$$

in the weak*−sense of measures, where μ and ν are nonnegative bounded measures on $\mathbb{R}^N$. Using Lemma 2.3, we obtain an at most countable index set J, two families $\{x_j\}_{j\in J} \subset \mathbb{R}^N$ and $\{\nu_j\}_{j\in J} \subset [0, +\infty)$ such that

$$\nu = \sum_{j\in J} \nu_j \delta_{x_j}, \quad \mu \geq \sum_{j\in J} \mu_j \delta_{x_j}, \quad S_{\alpha,\beta}^{k,l} \nu_j^{\frac{2}{2^*}} \leq \mu_j, \quad \forall j \in J, \tag{28}$$

where δ_{x_j} is the Dirac mass at $x_j \in \Omega$ and

$$S_{\alpha,\beta}^{k,l} = M_a^{\frac{\alpha}{\alpha+\beta}}(\alpha_0^2) M_a^{\frac{\beta}{\alpha+\beta}}(\beta_0^2) \left[\left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} + \left(\frac{\beta}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}} \right] S_{\alpha+\beta}. \tag{29}$$

Now, we claim that $J = \emptyset$. Arguing by contradiction, assume that $J \neq \emptyset$ and fix $j \in J$. Consider $\phi_j \in C_0^\infty(\Omega, [0, 1])$ such that $\phi_j \equiv 1$ on $B_1(0)$, $\phi_j \equiv 0$ on $\Omega \backslash B_2(0)$ and $|\nabla \phi_j|_\infty \leq 2$. Defining $\phi_{j,\epsilon}(x) = \phi_j(\frac{x-x_j}{\epsilon})$, where $\epsilon > 0$ we deduce that the sequence $\{\phi_{j,\epsilon} w_n\} = \{(\phi_{j,\epsilon} u_n, \phi_{j,\epsilon} v_n)\}$ is bounded in the space H. It then follows from (25) that $I'_{a,\lambda}(w_n)(\phi_{j,\epsilon} w_n) \to 0$ as $n \to \infty$, that is,

$$\begin{aligned} & M_a\left(\|u_n\|^2\right) \int_\Omega u_n \nabla u_n \nabla \phi_{j,\epsilon}\, dx + M_a\left(\|v_n\|^2\right) \int_\Omega v_n \nabla v_n \nabla \phi_{j,\epsilon}\, dx \\ &= -M_a\left(\|u_n\|^2\right) \int_\Omega \phi_{j,\epsilon} |\nabla u_n|^2\, dx - M_a\left(\|v_n\|^2\right) \int_\Omega \phi_{j,\epsilon} |\nabla v_n|^2\, dx \\ &\quad + \lambda \int_\Omega (F_u(x, u_n, v_n)\phi_{j,\epsilon} u_n + F_v(x, u_n, v_n)\phi_{j,\epsilon} v_n)\, dx \\ &\quad + 2 \int_\Omega |u_n|^\alpha |v_n|^\beta \phi_{j,\epsilon}\, dx + o_n(1) \\ &= -\left(M_a(\alpha_0^2) + o_n(1)\right) \int_\Omega \phi_{j,\epsilon} |\nabla u_n|^2\, dx - \left(M_a(\beta_0^2) + o_n(1)\right) \int_\Omega \phi_{j,\epsilon} |\nabla v_n|^2\, dx \\ &\quad + \lambda \int_\Omega (F_u(x, u_n, v_n)\phi_{j,\epsilon} u_n + F_v(x, u_n, v_n)\phi_{j,\epsilon} v_n)\, dx \\ &\quad + 2 \int_\Omega |u_n|^\alpha |v_n|^\beta \phi_{j,\epsilon}\, dx + o_n(1) \\ &= -M_a(\alpha_0^2) \int_\Omega \phi_{j,\epsilon} |\nabla u_n|^2\, dx - M_a(\beta_0^2) \int_\Omega \phi_{j,\epsilon} |\nabla v_n|^2\, dx \\ &\quad + \lambda \int_\Omega (F_u(x, u_n, v_n)\phi_{j,\epsilon} u_n + F_v(x, u_n, v_n)\phi_{j,\epsilon} v_n)\, dx \\ &\quad + 2 \int_\Omega |u_n|^\alpha |v_n|^\beta \phi_{j,\epsilon}\, dx + o_n(1). \end{aligned} \tag{30}$$

Since the support of $\phi_{j,\epsilon}$ is $B_{2\epsilon}(x_j)$, using the Hölder inequality and the boundedness of the sequence

$\{w_n\} = \{(u_n, v_n)\}$, we have

$$\begin{aligned}\left|\int_\Omega u_n \nabla u_n \nabla \phi_{j,\epsilon}\, dx\right| &= \left|\int_{B_{2\epsilon}(x_j)} u_n \nabla u_n \nabla \phi_{j,\epsilon}\, dx\right| \\ &\le \int_{B_{2\epsilon}(x_j)} |u_n \nabla u_n \nabla \phi_{j,\epsilon}|\, dx \\ &\le \left(\int_{B_{2\epsilon}(x_j)} |\nabla u_n|^2\, dx\right)^{\frac{1}{2}} \left(\int_{B_{2\epsilon}(x_j)} |u_n \nabla \phi_{j,\epsilon}|^2\, dx\right)^{\frac{1}{2}} \\ &\le C_3 \left(\int_{B_{2\epsilon}(x_j)} |u_n|^2 |\nabla \phi_{j,\epsilon}|^2\, dx\right)^{\frac{1}{2}} \\ &\le C_3 \left(\int_{B_{2\epsilon}(x_j)} |u_n|^{2^*}\, dx\right)^{\frac{1}{2^*}} \left(\int_{B_{2\epsilon}(x_j)} |\nabla \phi_{j,\epsilon}|^N\, dx\right)^{\frac{1}{N}} \\ &\le \overline{C}_3 \left(\int_{B_{2\epsilon}(x_j)} |u_n|^{2^*}\, dx\right)^{\frac{1}{2^*}} \\ &\to 0 \text{ as } n \to \infty \text{ and } \epsilon \to 0.\end{aligned} \tag{31}$$

From (31) and the fact that $M_a(\|u_n\|^2) \to M_a(\alpha_0^2)$ as $n \to \infty$ we have

$$\lim_{n\to\infty,\ \epsilon\to 0} M_a\left(\|u_n\|^2\right) \int_\Omega u_n \nabla u_n \nabla \phi_{j,\epsilon}\, dx = 0. \tag{32}$$

Similarly, we have

$$\lim_{n\to\infty,\ \epsilon\to 0} M_a\left(\|v_n\|^2\right) \int_\Omega v_n \nabla v_n \nabla \phi_{j,\epsilon}\, dx = 0. \tag{33}$$

On the other hand, by (F_2) and the boundedness of the sequences $\{u_n\}$, $\{v_n\}$ in $H_0^1(\Omega)$ we also have

$$\lim_{n\to\infty} \int_\Omega (F_u(x, u_n, v_n)\phi_{j,\epsilon} u_n + F_v(x, u_n, v_n)\phi_{j,\epsilon} v_n)\, dx = 0. \tag{34}$$

From (30)-(34), letting $n \to \infty$, we deduce that

$$2\int_\Omega d\nu \ge \int_\Omega \phi_{j,\epsilon} d\mu + o_\epsilon(1).$$

Letting $\epsilon \to 0$ and using the standard theory of Radon measures, we conclude that $2\nu_j \ge \mu_j$. Using (28) we have

$$\nu_j \ge \left(\frac{S^{k,l}_{\alpha,\beta}}{2}\right)^{\frac{N}{2}}, \tag{35}$$

where $S^{k,l}_{\alpha,\beta}$ is given by (29).

Now, we shall prove that (35) cannot occur, and therefore the set $J = \emptyset$. Indeed, arguing by contradiction, let us suppose that $\nu_j \ge \left(\frac{S^{k,l}_{\alpha,\beta}}{2}\right)^{\frac{N}{2}}$ for some $j \in J$. Since $\{w_n\} = \{(u_n, v_n)\}$ is a $(PS)_{c_{a,\lambda}}$ for the functional $I_{a,\lambda}$, from the conditions (F_3) and (M_0), and $m_0 < a < \frac{\theta}{2}m_0$ we have

$$\begin{aligned}
c_{a,\lambda} &= I_{a,\lambda}(w_n) - \frac{1}{\theta}I'_{a,\lambda}(w_n)(w_n) + o_n(1) \\
&\geq \frac{1}{2}\widehat{M}_a(\|u_n\|^2) + \frac{1}{2}\widehat{M}_a(\|v_n\|^2) - \frac{1}{\theta}M_a(\|u_n\|^2)\|u_n\|^2 - \frac{1}{\theta}M_a(\|v_n\|^2)\|v_n\|^2 \\
&\qquad + \left(\frac{2}{\theta} - \frac{2}{2^*}\right)\int_\Omega |u_n|^\alpha |v_n|^\beta\, dx + o_n(1) \\
&\geq \left(\frac{m_0}{2} - \frac{a}{\theta}\right)(\|u_n\|^2 + \|v_n\|^2) + \left(\frac{2}{\theta} - \frac{2}{2^*}\right)\int_\Omega |u_n|^\alpha |v_n|^\beta\, dx + o_n(1) \\
&\geq \left(\frac{2}{\theta} - \frac{2}{2^*}\right)\int_\Omega |u_n|^\alpha |v_n|^\beta\, dx + o_n(1) \\
&\geq \left(\frac{2}{\theta} - \frac{2}{2^*}\right)\int_\Omega |u_n|^\alpha |v_n|^\beta \phi_{j,\epsilon}\, dx + o_n(1).
\end{aligned} \tag{36}$$

Letting $n \to \infty$ in (36), we get

$$c_{a,\lambda} \geq \left(\frac{2}{\theta} - \frac{2}{2^*}\right)\int_\Omega \phi_{j,\epsilon} d\nu$$

and then

$$c_{a,\lambda} \geq \left(\frac{2}{\theta} - \frac{2}{2^*}\right)\left(\frac{S^{k,l}_{\alpha,\beta}}{2}\right)^{\frac{N}{2}}, \tag{37}$$

which contradicts (24). Thus, $J = \emptyset$ and it follows that

$$\lim_{n\to\infty}\int_\Omega |u_n|^\alpha |v_n|^\beta\, dx = \int_\Omega |u|^\alpha |v|^\beta\, dx. \tag{38}$$

We also have $u_n(x) \to u(x)$ and $v_n(x) \to v(x)$ a.e. $x \in \Omega$ as $n \to \infty$, so by the condition (F_2) and the Dominated Convergence Theorem, we deduce that

$$\lim_{n\to\infty}\int_\Omega (F_u(x, u_n, v_n)u_n - F_u(x, u, v)u)\, dx = 0. \tag{39}$$

Similarly,

$$\lim_{n\to\infty}\int_\Omega (F_v(x, u_n, v_n)v_n - F_v(x, u, v)v)\, dx = 0. \tag{40}$$

Combining (38)-(40) with the fact that $I'_{a,\lambda}(u_n, v_n)(u_n, 0) \to 0$ and $I'_{a,\lambda}(u_n, v_n)(0, v_n) \to 0$ as $n \to \infty$ we get

$$\lim_{n\to\infty} M_a\left(\|u_n\|^2\right)\|u_n\|^2 = \lambda\int_\Omega F_u(x, u, v)u\, dx + \frac{2\alpha}{\alpha+\beta}\int_\Omega |u|^\alpha |v|^\beta\, dx, \tag{41}$$

$$\lim_{n\to\infty} M_a\left(\|v_n\|^2\right)\|v_n\|^2 = \lambda\int_\Omega F_v(x, u, v)v\, dx + \frac{2\beta}{\alpha+\beta}\int_\Omega |u|^\alpha |v|^\beta\, dx. \tag{42}$$

On the other hand, by (25), for $(\varphi, \psi) \in H$, $I'_{a,\lambda}(u_n, v_n)(\varphi, 0) \to 0$ and $I'_{a,\lambda}(u_n, v_n)(0, \psi) \to 0$ as $n \to \infty$, that is,

$$M_a\left(\|u_n\|^2\right)\int_\Omega \nabla u_n \nabla\varphi\, dx = \lambda\int_\Omega F_u(x, u_n, v_n)\varphi\, dx + \frac{2\alpha}{\alpha+\beta}\int_\Omega |u_n|^{\alpha-2}u_n|v_n|^\beta\varphi\, dx,$$

$$M_a\left(\|v_n\|^2\right)\int_\Omega \nabla v_n \nabla\psi\, dx = \lambda\int_\Omega F_u(x, u_n, v_n)\psi\, dx + \frac{2\beta}{\alpha+\beta}\int_\Omega |u_n|^\alpha |v_n|^{\beta-2}v_n\psi\, dx.$$

By (17) and (26), using Dominated Convergence Theorem, we reach

$$M_a\left(\alpha_0^2\right)\int_\Omega \nabla u\nabla\varphi\,dx = \lambda\int_\Omega F_u(x,u,v)\varphi\,dx + \frac{2\alpha}{\alpha+\beta}\int_\Omega |u|^{\alpha-2}u|v|^\beta\varphi\,dx,$$

$$M_a\left(\beta_0^2\right)\int_\Omega \nabla v\nabla\psi\,dx = \lambda\int_\Omega F_u(x,u,v)\psi\,dx + \frac{2\beta}{\alpha+\beta}\int_\Omega |u|^{\alpha}|v|^{\beta-2}v\psi\,dx, \quad (\varphi,\psi)\in H$$

and so

$$M_a\left(\alpha_0^2\right)\|u\|^2 = \lambda\int_\Omega F_u(x,u,v)u\,dx + \frac{2\alpha}{\alpha+\beta}\int_\Omega |u|^{\alpha}|v|^\beta\,dx, \tag{43}$$

$$M_a\left(\beta_0^2\right)\|v\|^2 = \lambda\int_\Omega F_v(x,u,v)v\,dx + \frac{2\beta}{\alpha+\beta}\int_\Omega |u|^{\alpha}|v|^\beta\,dx. \tag{44}$$

Now, using (41)-(44) we get

$$M_a\left(\|u_n\|^2\right)\|u_n\|^2 \to M_a\left(\alpha_0^2\right)\|u\|^2, \quad M_a\left(\|v_n\|^2\right)\|v_n\|^2 \to M_a\left(\beta_0^2\right)\|v\|^2 \text{ as } n\to\infty. \tag{45}$$

From (26) and (45) we conclude that $\|u_n\|^2 \to \|u\|^2$ and $\|v_n\|^2 \to \|v\|^2$ as $n\to\infty$. This completes the proof of Theorem 2.1. □

Now, we are in the position to prove Theorem 1.2.

Proof. [Proof of Theorem 1.2] Let λ_0 be as in Theorem 2.1 and, for $\lambda \geq \lambda_0$, let $w_\lambda = (u_\lambda, v_\lambda) \in H$ be the nontrivial solution of problem (5) found in Theorem 2.1. We claim that there exists $\lambda^* \geq \lambda_0$ such that $\|u_\lambda\|^2 \leq t_0$ and $\|v_\lambda\|^2 \leq t_0$, for all $\lambda \geq \lambda^*$. If this is the case, it follows from the definition of $M_a(t)$ that $M_a(\|u_\lambda\|^2) = M(\|u_\lambda\|^2)$. Thus, $w_\lambda = (u_\lambda, v_\lambda)$ is a weak solution of problem (1).

We argue by contradiction that, there is a sequence $\{\lambda_n\} \subset \mathbb{R}$ such that $\lambda_n \to +\infty$ as $n\to\infty$ and either $\|u_{\lambda_n}\|^2 \geq t_0$ or $\|v_{\lambda_n}\|^2 \geq t_0$. Then we have

$$\begin{aligned} c_{a,\lambda_n} &\geq \frac{1}{2}\widehat{M}_a(\|u_{\lambda_n}\|^2) + \frac{1}{2}\widehat{M}_a(\|v_{\lambda_n}\|^2) - \frac{1}{\theta}M_a(\|u_{\lambda_n}\|^2)\|u_{\lambda_n}\|^2 - \frac{1}{\theta}M_a(\|v_{\lambda_n}\|^2)\|v_{\lambda_n}\|^2 \\ &\geq \left(\frac{m_0}{2} - \frac{a}{\theta}\right)\left(\|u_{\lambda_n}\|^2 + \|v_{\lambda_n}\|^2\right) \\ &\geq \left(\frac{m_0}{2} - \frac{a}{\theta}\right)t_0, \end{aligned} \tag{46}$$

which is a contradiction since $\lim_{n\to\infty} c_{a,\lambda_n} = 0$ and $a \in (m_0, \frac{\theta}{2}m_0)$.

Finally, we shall prove that $\lim_{\lambda\to+\infty}\|u_\lambda\| = \lim_{\lambda\to+\infty}\|v_\lambda\| = 0$. Indeed, by (M_0) and the fact that $\|u_\lambda\|^2 \leq t_0$ and $\|v_\lambda\|^2 \leq t_0$, it follows that $M(\|u_\lambda\|^2) \leq M(t_0) = a$ and $M(\|v_\lambda\|^2) \leq M(t_0) = a$. Hence, using (M_0) and (F_4) we have

$$\begin{aligned} c_{a,\lambda} &\geq \frac{1}{2}\widehat{M}(\|u_\lambda\|^2) + \frac{1}{2}\widehat{M}(\|v_\lambda\|^2) - \frac{1}{\theta}M(\|u_\lambda\|^2)\|u_\lambda\|^2 - \frac{1}{\theta}M(\|v_\lambda\|^2)\|v_\lambda\|^2 \\ &\geq \frac{m_0}{2}\|u_\lambda\|^2 + \frac{m_0}{2}\|v_\lambda\|^2 - \frac{a}{\theta}\|u_\lambda\|^2 - \frac{a}{\theta}\|v_\lambda\|^2 \\ &= \left(\frac{m_0}{2} - \frac{a}{\theta}\right)\left(\|u_\lambda\|^2 + \|v_\lambda\|^2\right). \end{aligned} \tag{47}$$

Using Lemma 2.5 again we have that $\lim_{\lambda\to+\infty} c_{a,\lambda} = 0$. Therefore, it follows since $a \in (m_0, \frac{\theta}{2}m_0)$ that $\lim_{\lambda\to+\infty}\|u_\lambda\| = \lim_{\lambda\to+\infty}\|v_\lambda\| = 0$. The proof of Theorem 1.2 is now completed. □

Acknowledgments

The author would like to thank the referees for their suggestions and helpful comments which improved the presentation of the original manuscript.

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