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Iterative Scheme for Finding Solutions of the General Split Feasibility Problem and the General Constrained Minimization Problems

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Abstract. Inspired by the works of [11], [6], and [14], we introduce a method to solve solution of the general split feasibility problem. In the last section, we give the general constrained minimization problem and a lemma to show the relationship between these problems. The method utilized to solve this problem is presented. Our results expand some results of Ceng, Ansari and Yao [2] and modify the results of Xu [17].

1. Introduction

Given closed convex subset $C \subseteq H_1, Q \subseteq H_2$ of Hilbert space H_1, H_2 and let $A : H_1 \to H_2$ be a bounded linear operator. The split feasibility problem (SFP) is to find a point $x \in C$ and $Ax \in Q$. This problem was introduced by Censor and Elfving [5].

Such models were successfully developed for instance in radiation therapy treatment planning, sensor networks, resolution enhancement.

In 2012, Ceng, Ansari and Yao [2] introduced the following lemma to solve SFP;

Lemma 1.1. *Given $x^* \in H_1$, the following statements are equivalent.*
i) x^* solves the SFP;
ii) $x^* = P_C\left(I - \lambda A^*\left(I - P_Q\right)A\right)x^*$, where A^* is adjoint of A;
iii) x^* solves the variational inequality problem (VIP) of finding $x^* \in C$ such that $\langle y - x^*, \nabla g(x^*)\rangle \geq 0$, for all $y \in C$ and $\nabla g = A^*\left(I - P_Q\right)A$.

Many authors use this lemma to prove their results, see for example, [3], [8].

Let $p, q \in \mathbb{N}$. For each $1 \leq i \leq p$, let C_i be a nonempty closed convex subset of a real Hilbert space H_1. For each $1 \leq j \leq q$, let Q_j be a nonempty closed convex subset of another real Hilbert space H_2 and let $A_j : H_1 \to H_2$ be a bounded linear operator. Suppose that K is another nonempty closed convex subset of H_1. *The constrained multiple-set split convex feasibility problem (MSCFP)* raised by Masad and Reich [11] is finding a point $x^* \in K$ such that

$$x^* \in \bigcap_{i=1}^{p} C_i \text{ and } A_j x^* \in Q_i,\ 1 \leq j \leq q. \tag{1}$$

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The MSCFP introduced by Censor et al. [6] and Xu [14] is a special case of (1), which is formulated as finding $x^* \in H_1$ such that

$$x^* \in \bigcap_{i=1}^{p} C_i \text{ and } Ax^* \in \bigcap_{j=1}^{q} Q_i, \tag{2}$$

where A is a bounded linear operator from H_1 to H_2. If $p = q = 1$, (2) is reduced to SFP. Let $A, B : H_1 \to H_2$ be bounded linear operators. Inspired by (1), (2) and SFP, we introduce *the general split feasibility problem* which is to find a point $x^* \in C$ and $Ax^*, Bx^* \in Q$. The set of this solution is denoted by $\Gamma = \{x \in C : Ax, Bx \in Q\}$.

By applying Mann 's iterative algorithm with *SFP*, Xu [17] proved the best following result;

Theorem 1.2. *Assume that SFP is consistent and* $\gamma \in \left(0, \frac{2}{\|A\|^2}\right)$. *Let* $\{x_n\}$ *be defined by the following averaged CQ algorithm:*

$$x_{n+1} = (1 - \alpha_n) x_n + \alpha_n P_C \left(I - \gamma A^* \left(I - P_Q\right) A\right) x_n,$$

for all $n \geq 0$ *where* $\{\alpha_n\}$ *is a sequence in a interval* $\left[0, \frac{4}{2 + \gamma \|A\|^2}\right]$ *satisfying the condition*

$$\sum_{n=1}^{\infty} \left(\frac{4}{2 + \gamma \|A\|^2} - \alpha_n\right) = \infty.$$

Then $\{x_n\}$ *converges weakly to a solution of SFP.*

The such theorem is used as a model for proving some result to solve the split feasibility problem, see for example, [2, 3, 7].

In the next section, we prove the important lemma as a tool for proving the theorem that solves the general split feasibility problem.

The purpose of this research, we introduce a new method for solving the general split feasibility problem and apply our main theorem to prove the theorem related to the general constrained minimization problem in the last section. Our results expand some results of Ceng, Ansari and Yao [2] and modify the results of Xu [17].

2. Preliminaries

In order to prove our main theorem. Therefore, these tools are needed.

Throughout this research, we uses the symbol " $\longrightarrow$ " and " $\rightharpoonup$ " represent strong and weak convergence, respectively. Let C be a subset of a real Hilbert space H. A mapping $T : C \to C$ is called *α-contractive* if there exists $\alpha \in [0, 1]$ such that $\|Tx - Ty\| \leq \alpha\|x - y\|$ for all $x, y \in C$. A mapping T is calll *nonexpansive* if $\alpha = 1$. The fixed point problem of T is to find a point $x^* \in C$ such that $Tx^* = x^*$. The set of all fixed point of T is denoted by $F(T)$. A mapping $A : C \to H$ is called α-inverse strongly monotone if there exists $\alpha > 0$ such that $\alpha\|Ax - Ay\|^2 \leq \langle Ax - Ay, x - y\rangle$ for all $x, y \in C$.

The variational inequality problem (VIP) is a well known problem. That is to find a point $\varpi_* \in C$ such that

$$\langle y - \varpi_*, G\varpi_*\rangle \geq 0, \text{ for all } y \in C, \tag{3}$$

where $G : C \longrightarrow H$ is a mapping. The set of all solutions of (3) is denoted by $VI(C, G)$.

The variational inequality problem has been applied in various fields such as industry, finance, economics, social, ecology, regional, pure and applied sciences; see, [9],[10].

Let C be a closed convex subset of a real Hilbert space H and let P_C be the metric projection of H onto C i.e., for $x \in H$, $P_C x$ satisfies the property

$$\|x - P_C x\| = \min_{y \in C} \|x - y\|.$$

The following lemma is a property of P_C.

Lemma 2.1. *(See [13]) Given $x \in H$ and $y \in C$. Then $P_C x = y$ if and only if there holds the inequality* $\langle x - y, y - z\rangle \geq 0, \quad \forall z \in C.$

Lemma 2.2. *(See [12]) Let H be a Hilbert space, let C be nonempty closed convex subset of H and let A be a mapping of C into H. Let $u \in C$. Then for $\lambda > 0$,*

$$u \in VI(C, A) \Leftrightarrow u = P_C(I - \lambda A)u$$

where P_C is the metric projection of H onto C.

Lemma 2.3. *(See [16]) Let $\{s_n\}$ be a sequence of nonnegative real number satisfying*

$$s_{n+1} = (1 - \alpha_n)s_n + \alpha_n\beta_n, \quad \forall n \geq 0$$

where $\{\alpha_n\}, \{\beta_n\}$ satisfy the conditions

$$(1) \quad \{\alpha_n\} \subset [0, 1], \ \sum_{n=1}^{\infty} \alpha_n = \infty;$$

$$(2) \quad \limsup_{n\to\infty} \beta_n \leq 0 \text{ or } \sum_{n=1}^{\infty} |\alpha_n\beta_n| < \infty.$$

Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.4. *(See [15].) Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} = (1 - \alpha_n)s_n + \delta_n, \quad \forall n \geq 0$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence such that

$$(1) \quad \sum_{n=1}^{\infty} \alpha_n = \infty,$$

$$(2) \quad \limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \leq 0 \text{ or } \sum_{n=1}^{\infty} |\delta_n| < \infty.$$

Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.5. *Let H_1 and H_2 be real Hilbert spaces and C, Q be nonempty closed convex subsets of H_1 and H_2, respectively. Let $A, B : H_1 \to H_2$ be bounded linear operators with A^*, B^* are adjoint of A and B, respectively with $\Gamma \neq \emptyset$. Then the followings are equivalent.*

i) $x^* \in \Gamma$,

ii) $P_C\left(I - a\left(\frac{A^*\left(I - P_Q\right)A}{2} + \frac{B^*\left(I - P_Q\right)B}{2}\right)\right)x^* = x^*, \ \forall a > 0,$

where L_A, L_B are spectal redius of A^*A and B^*B, respectively with $a \in (0, \frac{2}{L})$ and $L = \max\{L_A, L_B\}$.

Proof. Let the conditions holds

i) $\Rightarrow$ *ii)* Let $x^* \in \Gamma$, we have $x^* \in C$ and $Ax^*, Bx^* \in Q$. It implies that

$$\left(I - P_Q\right)Ax^* = 0 = \left(I - P_Q\right)Bx^*.$$

Then

$$\frac{A^*\left(I-P_Q\right)Ax^*}{2}=\frac{B^*\left(I-P_Q\right)Bx^*}{2}=0.$$

It follow that

$$P_C\left(I-a\left(\frac{A^*\left(I-P_Q\right)A}{2}+\frac{B^*\left(I-P_Q\right)B}{2}\right)\right)x^*=x^*.$$

ii) $\Rightarrow$ *i)* Let $P_C\Big(I-a\left(\frac{A^*\left(I-P_Q\right)A}{2}+\frac{B^*\left(I-P_Q\right)B}{2}\right)\Big)x^*=x^*$ and let $w\in\Gamma$, we have $w\in C$ and $Aw,Bw\in Q$. From *i)* $\Rightarrow$ *ii)*, we have

$$P_C\Big(I-a\left(\frac{A^*\left(I-P_Q\right)A}{2}+\frac{B^*\left(I-P_Q\right)B}{2}\right)\Big)w=w.$$

Then, we have

$$\begin{aligned}
\|x^*-w\|^2 &\le \left\|x^*-w-a\Big(\frac{A^*\left(I-P_Q\right)Ax^*}{2}+\frac{B^*\left(I-P_Q\right)Bx^*}{2}\Big)\right\|^2\\
&= \|x^*-w\|^2-2a\langle x^*-w,\frac{A^*\left(I-P_Q\right)Ax^*}{2}+\frac{B^*\left(I-P_Q\right)Bx^*}{2}\rangle\\
&\quad +a^2\left\|\frac{A^*\left(I-P_Q\right)Ax^*}{2}+\frac{B^*\left(I-P_Q\right)Bx^*}{2}\right\|^2\\
&\le \|x^*-w\|^2-a\langle Ax^*-Aw,\left(I-P_Q\right)Ax^*\rangle-a\langle Bx^*-Bw,\left(I-P_Q\right)Bx^*\rangle\\
&\quad +\frac{a^2}{2}\|A^*(I-P_Q)Ax^*\|^2+\frac{a^2}{2}\|B^*(I-P_Q)Bx^*\|^2\\
&\le \|x^*-w\|^2-a\langle Ax^*-P_QAx^*,\left(I-P_Q\right)Ax^*\rangle-a\langle P_QAx^*-Aw,\left(I-P_Q\right)Ax^*\rangle\\
&\quad -a\langle Bx^*-P_QBx^*,\left(I-P_Q\right)Bx^*\rangle-a\langle P_QBx^*-Bw,\left(I-P_Q\right)Bx^*\rangle\\
&\quad +\frac{a^2L}{2}\|(I-P_Q)Ax^*\|^2+\frac{a^2L}{2}\|(I-P_Q)Bx^*\|^2\\
&\le \|x^*-w\|^2-a(1-\frac{aL}{2})\|(I-P_Q)Ax^*\|^2-(1-\frac{aL}{2})\|(I-P_Q)Bx^*\|^2.
\end{aligned}$$

It implies that $Ax^*=P_QAx^*,Bx^*=P_QBx^*\in Q$.
It follows that

$$x^*=P_C\left(I-a\left(\frac{A^*\left(I-P_Q\right)A}{2}+\frac{B^*\left(I-P_Q\right)B}{2}\right)\right)x^*=P_Cx^*\in C.$$

Hence $x^*\in\Gamma$. □

Example 2.6. *Let $H_1=\mathbb{R}^2,H_2=\mathbb{R}$ and let $C=\overline{H}(a,a_1-a_2)=\{x=(x_1,x_2)\in H_1:a_1x_1+a_2x_2=a_1-a_2$ for all $a=(a_1,a_2)\in H_2$ and $Q=[-2,3]\subseteq H_1$. Defined mappings $A,B:H_1\to H_2$ by $Ax=x_1,Bx=x_2$ for all $x=(x_1,x_2)\in H_1$. It is obvious that* $(1,-1)\in\Gamma$.

Next, we will show that $P_C\Big(I - \lambda\Big(\frac{A^*(I-P_Q)A}{2} + \frac{B^*(I-P_Q)B}{2}\Big)\Big)(1,-1) = (1,-1)$.

From the definition of A, B, *we can defined adjoint operators* $A^*, B^* : H_2 \to H_1$ *of* A, B *by* $A^*z = (z, 0), B^*z = (0, z)$ *for all* $z \in H_2$. *From the definition of* C, *We can defined metric projection* $P_C : H_1 \to C$ *by*

$$P_C z = (z_1, z_2) - \Big(\frac{a_1 z_1 + a_2 z_2 - (a_1 - a_2)}{\sqrt{a_1^2 + a_2^2}}\Big)(a_1, a_2), \tag{4}$$

for all $z = (z_1, z_2) \in H_1$.

From A, A^*, B *and* B^*, *we have*

$$\frac{A^*(I-P_Q)A(1,-1)}{2} = \frac{A^*0}{2} = (0,0), \tag{5}$$

and

$$\frac{B^*(I-P_Q)B(1,-1)}{2} = \frac{B^*0}{2} = (0,0). \tag{6}$$

From (4) (5) and (6), we have

$$\begin{aligned} P_C\Big(I - \lambda\Big(\frac{A^*(I-P_Q)A}{2} + \frac{B^*(I-P_Q)B}{2}\Big)\Big) &= P_C(1,-1) \\ &= (1,-1) - \Big(\frac{a_1 - a_2 - (a_1 - a_2)}{\sqrt{a_1^2 + a_2^2}}\Big)(a_1, a_2) \\ &= (1,-1). \end{aligned}$$

Remark 2.7. *The result of this example is guaranteed by Lemma 2.5.*

3. Main results

Theorem 3.1. *Let* H_1 *and* H_2 *be real Hilbert spaces and let* C, Q *be nonempty closed convex subsets of* H_1 *and* H_2, *respectively. Let* $A, B : H_1 \to H_2$ *be bounded linear operators with* A^*, B^* *are adjoint of* A *and* B, *respectively and* $L = \max\{L_A, L_B\}$, *where* L_A *and* L_B *are special radius of* A^*A *and* B^*B *and let* $D : C \to H_1$ *be d-inverse strongly monotone. Assume that* $\Gamma \cap VI(C, D) \neq \emptyset$. *Let the sequence* $\{x_n\}$ *generated by* $x_1 \in C$ *and*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n P_C (I - \lambda D) x_n + \gamma_n P_C \left(aI - a\left(\frac{A^*\left(I-P_Q\right)A}{2} + \frac{B^*\left(I-P_Q\right)B}{2}\right)\right) x_n, \tag{7}$$

for all $n \in \mathbb{N}$, *where* $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subseteq (0,1)$ *with* $\alpha_n + \beta_n + \gamma_n = 1$ *and* $f : C \to C$ *is* α*-contractive mapping with* $\alpha \in (0,1)$. *Suppose that the following conditions hold;*

i) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$,

ii) $c \leq \beta_n, \gamma_n \leq d$, *for some* $c, d > 0$,

iii) $\lambda \in (0, 2d), a \in \left(0, \frac{2}{L}\right)$,

iv) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}|, \sum_{n=1}^{\infty} |\beta_n - \beta_{n-1}| < \infty$.

Then the sequence $\{x_n\}$ *converges strongly to* $x_0 = P_{\Gamma\cap VI(C,D)} f(x_0)$.

Proof. Putting $\nabla g = \frac{A^*\left(I-P_Q\right)A}{2} + \frac{B^*\left(I-P_Q\right)B}{2}$. First, we show that ∇g is $\frac{1}{L}$-inverse strongly monotone. Let $x, y \in C$. Since $\nabla g = \frac{A^*\left(I-P_Q\right)A}{2} + \frac{B^*\left(I-P_Q\right)B}{2}$, we have

$$\begin{aligned}
\|\nabla g(x) - \nabla g(y)\|^2 &= \|\frac{A^*\left(I-P_Q\right)Ax}{2} + \frac{B^*\left(I-P_Q\right)Bx}{2} - \frac{A^*\left(I-P_Q\right)Ay}{2} - \frac{B^*\left(I-P_Q\right)By}{2}\|^2 \\
&= \|\frac{A^*\left(I-P_Q\right)Ax}{2} - \frac{A^*\left(I-P_Q\right)Ay}{2} + \frac{B^*\left(I-P_Q\right)Bx}{2} - \frac{B^*\left(I-P_Q\right)By}{2}\|^2 \\
&\le \frac{1}{2}\|A^*\left(I-P_Q\right)Ax - A^*\left(I-P_Q\right)Ay\|^2 + \frac{1}{2}\|B^*\left(I-P_Q\right)Bx - B^*\left(I-P_Q\right)By\|^2 \\
&\le \frac{L}{2}\|\left(I-P_Q\right)Ax - \left(I-P_Q\right)Ay\|^2 + \frac{L}{2}\|\left(I-P_Q\right)Bx - \left(I-P_Q\right)By\|^2. \qquad (8)
\end{aligned}$$

From property of P_C, we have

$$\begin{aligned}
\|(I-P_Q)Ax - (I-P_Q)Ay\|^2 &= \langle (I-P_Q)Ax - (I-P_Q)Ay, (I-P_Q)Ax - (I-P_Q)Ay\rangle \\
&= \langle (I-P_Q)Ax - (I-P_Q)Ay, Ax - Ay - (P_QAx - P_QAy)\rangle \\
&= \langle A^*(I-P_Q)Ax - A^*(I-P_Q)Ay, x-y\rangle \\
&\quad -\langle (I-P_Q)Ax - (I-P_Q)Ay, P_QAx - P_QAy\rangle \\
&= \langle A^*(I-P_Q)Ax - A^*(I-P_Q)Ay, x-y\rangle \\
&\quad -\langle (I-P_Q)Ax, P_QAx - P_QAy\rangle \\
&\quad +\langle (I-P_Q)Ay, P_QAx - P_QAy\rangle \\
&\le \langle A^*(I-P_Q)Ax - A^*(I-P_Q)Ay, x-y\rangle. \qquad (9)
\end{aligned}$$

By using the same method as (9), we have

$$\|(I-P_Q)Bx - (I-P_Q)By\|^2 \le \langle B^*(I-P_Q)Bx - B^*(I-P_Q)By, x-y\rangle. \qquad (10)$$

Substitute (9), (10) into (8), we have

$$\begin{aligned}
\|\nabla g(x) - \nabla g(y)\|^2 &\le \frac{L}{2}\langle A^*(I-P_Q)Ax - A^*(I-P_Q)Ay, x-y\rangle \\
&\quad +\frac{L}{2}\langle B^*(I-P_Q)Bx - B^*(I-P_Q)By, x-y\rangle \\
&= L\langle \frac{A^*(I-P_Q)Ax}{2} + \frac{B^*(I-P_Q)Bx}{2} - \left(\frac{A^*(I-P_Q)Ay}{2} + \frac{B^*(I-P_Q)By}{2}\right), x-y\rangle \\
&= L\langle \nabla g(x) - \nabla g(y), x-y\rangle.
\end{aligned}$$

So, we have ∇g is $\frac{1}{L}$-inverse strongly monotone. From the definition of ∇g , we have

$$\begin{aligned}
\|P_C(I-a\nabla g)x - P_C(I-a\nabla g)y\|^2 &\le \|x-y-a(\nabla g(x) - \nabla g(y))\|^2 \\
&= \|x-y\|^2 - 2a\langle x-y, \nabla g(x) - \nabla g(y)\rangle + a^2\|\nabla g(x) - \nabla g(y)\|^2 \\
&\le \|x-y\|^2 - \frac{2a}{L}\|\nabla g(x) - \nabla g(y)\|^2\rangle + a^2\|\nabla g(x) - \nabla g(y)\|^2 \\
&= \|x-y\|^2 - a\left(\frac{2}{L} - a\right)\|\nabla g(x) - \nabla g(y)\|^2 \\
&\le \|x-y\|^2, \qquad (11)
\end{aligned}$$

for all $x, y \in C$. By using the same method as (11), we have

$$\|P_C(I-\lambda D)x - P_C(I-\lambda D)y\| \le \|x-y\|, \qquad (12)$$

for all $x, y \in C$.
From the definition of x_n, (11) and (12), we have

$$\begin{aligned}\|x_{n+1} - z\| &\leq \alpha_n \left\|f(x_n) - z\right\| + \beta_n \|P_C(I - \lambda D)x_n - z\| + \gamma_n \left\|P_C(I - a\nabla g)x_n - z\right\| \\ &\leq \alpha_n\Big(\alpha\|x_n - z\| + \|f(z) - z\|\Big) + \beta_n \|P_C(I - \lambda D)x_n - z\| + \gamma_n \left\|P_C(I - a\nabla g)x_n - z\right\| \\ &\leq (1 - \alpha_n(1 - \alpha))\|x_n - z\| + \alpha_n\|f(z) - z\| \\ &\leq \max\left\{\|x_1 - z\|, \frac{\left\|f(z) - z\right\|}{1 - \alpha}\right\},\end{aligned}$$

for all $n \in \mathbb{N}$ and $z \in \Gamma \cap VI(C, D)$. By induction, we conclude that the sequence $\{x_n\}$ is bounded.
From (7), we have

$$\begin{aligned}\|x_{n+1} - x_n\| &\leq |\alpha_n - \alpha_{n-1}| \left\|f(x_{n-1})\right\| + \alpha_n\alpha\|x_n - x_{n-1}\| + \left|\beta_n - \beta_{n-1}\right| \|P_C(I - \lambda D)x_{n-1}\| \\ &\quad + \beta_n\|P_C(I - \lambda D)x_n - P_C(I - \lambda D)x_{n-1}\| + \left|\gamma_n - \gamma_{n-1}\right| \|P_C(1 - a\nabla g)x_{n-1}\| \\ &\quad + \gamma_n\|P_C(1 - a\nabla g)x_n - P_C(1 - a\nabla g)x_{n-1}\| \\ &\leq (1 - \alpha_n(1 - \alpha))\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| \left\|f(x_{n-1})\right\| + \left|\beta_n - \beta_{n-1}\right| \|P_C(I - \lambda D)x_{n-1}\| \\ &\quad + \left|\gamma_n - \gamma_{n-1}\right| \|P_C(1 - a\nabla g)x_{n-1}\|.\end{aligned}$$

From the conditions $i), iv)$ and Lemma 2.4, we have

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{13}$$

We can rewrite (7) by

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) E_n x_n, \tag{14}$$

where $E_n = \frac{\beta_n}{1 - \alpha_n} P_C(I - \lambda D) + \frac{\gamma_n}{1 - \alpha_n} P_C(I - a\nabla g)$ for all $n \in \mathbb{N}$.
Since $P_C(I - \lambda D)$ and $P_C(I - a\nabla g)$ are nonexpansive mappings, we have E_n is a nonexpansive mappings, for all $n \in \mathbb{N}$.
It is easy to see that

$$F(P_C(I - \lambda D)) \cap F(P_C(I - a\nabla g) \subseteq F(E_n), \tag{15}$$

for all $n \in \mathbb{N}$.
From Lemma 2.2 and 2.5, we have

$$F(P_C(I - \lambda D)) \cap F(P_C(I - a\nabla g) = \Gamma \cap VI(C, D) \neq \emptyset.$$

Let $z_0 \in F(E_n)$, for all $n \in \mathbb{N}$ and $z \in \Gamma \cap VI(C, D)$, we have

$$\begin{aligned}\|z_0 - z\|^2 &\leq \frac{\beta_n}{1 - \alpha_n}\|P_C(1 - \lambda D)z_0 - z\|^2 + \frac{\gamma_n}{1 - \alpha_n}\|P_C(I - a\nabla g)z_0 - z\|^2 \\ &\quad - \frac{\beta_n\gamma_n}{(1 - \alpha_n)^2}\|P_C(1 - \lambda D)z_0 - P_C(I - a\nabla g)z_0\|^2 \\ &\leq \|z_0 - z\|^2 - \frac{\beta_n\gamma_n}{(1 - \alpha_n)^2}\|P_C(1 - \lambda D)z_0 - P_C(I - a\nabla g)z_0\|^2.\end{aligned}$$

From condition $iii)$, we can conclude that $P_C(1 - \lambda D)z_0 = P_C(I - a\nabla g)z_0$.
Since $z_0 \in F(E_n)$, for all $n \in \mathbb{N}$, we have

$$z_0 = \frac{\beta_n}{1 - \alpha_n} P_C(I - \lambda D)z_0 + \frac{\gamma_n}{1 - \alpha_n} P_C(I - a\nabla g)z_0 = P_C(I - \lambda D)z_0 = P_C(I - a\nabla g)z_0.$$

So, we get

$$z_0 \in F(P_C(I-\lambda D)) \cap F(P_C(I-a\nabla g) = \Gamma \cap VI(C,D).$$

It follows that

$$F(E_n) \subseteq F(P_C(I-\lambda D)) \cap F(P_C(I-a\nabla g).$$

Then

$$F(E_n) = F(P_C(I-\lambda D)) \cap F(P_C(I-a\nabla g), \tag{16}$$

for all $n \in \mathbb{N}$. From (14), we have

$$x_{n+1} - x_n = \alpha_n(f(x_n) - x_n) + (1-\alpha_n)(E_n x_n - x_n). \tag{17}$$

From (13) and condition i), we have

$$\lim_{n\to\infty} \|E_n x_n - x_n\| = 0. \tag{18}$$

Since the sequence $\{x_n\}$ is bounded in a real Hilbert space H_1, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges weakly to w, where $w \in C$.
From the condition ii) we may assume that $\beta_{n_k} \to \beta$ and $\gamma_{n_k} \to \gamma$ as $k \to \infty$ with $\beta, \gamma \in [c,d]$.
It follows that

$$1 = \lim_{k\to\infty}\left(\frac{\beta_{n_k}}{1-\alpha_{n_k}} + \frac{\gamma_{n_k}}{1-\alpha_{n_k}}\right) = \beta + \gamma.$$

Putting $E = \beta P_C(I-\lambda D) + \gamma P_C(I-a\nabla g)$. It is easy to see that E is a nonexpansive mapping. By using method as $F(E_n) = F(P_C(I-\lambda D)) \cap F(P_C(I-a\nabla g)$, we have

$$F(E) = F(P_C(I-\lambda D)) \cap F(P_C(I-a\nabla g). \tag{19}$$

From the definition of E_n and E, we have

$$E_{n_k}x_{n_k} - Ex_{n_k} = \left(\frac{\beta_{n_k}}{1-\alpha_{n_k}} - \beta\right) P_C(I-\lambda D)x_{n_k} + \left(\frac{\gamma_{n_k}}{1-\alpha_{n_k}} - \gamma\right) P_C(I-a\nabla g)x_{n_k}.$$

From $\lim_{k\to\infty} \beta_{n_k} = \beta$, $\lim_{k\to\infty} \gamma_{n_k} = \gamma$ and condition i), we have

$$\lim_{k\to\infty} \|E_{n_k}x_{n_k} - Ex_{n_k}\| = 0. \tag{20}$$

From (18) and (20), we have

$$\lim_{k\to\infty} \|x_{n_k} - Ex_{n_k}\| = 0. \tag{21}$$

Assume that $w \notin \Gamma \cap VI(C,D)$. From (19), Lemma 2.2 and 2.5, we have $w \notin F(E)$. From Opial's conditions and (21), we have

$$\begin{aligned}
\liminf_{k\to\infty} \|x_{n_k} - w\| &< \liminf_{k\to\infty} \|x_{n_k} - Ew\| \\
&\leq \liminf_{k\to\infty} \left(\|x_{n_k} - Ex_{n_k}\| + \|Ex_{n_k} - Ew\|\right) \\
&\leq \liminf_{k\to\infty} \|x_{n_k} - w\|.
\end{aligned}$$

This is a contradiction. Then $w \in \Gamma \cap VI(C, D)$.
Since the sequence $\{x_n\}$ is bounded, we have

$$\limsup_{n\to\infty}\langle f(x_0) - x_0, x_n - x_0\rangle = \lim_{k\to\infty}\langle f(x_0) - x_0, x_{n_k} - x_0\rangle = \langle f(x_0) - x_0, w - x_0\rangle \leq 0, \tag{22}$$

where $x_0 = P_{\Gamma\cap VI(C,D)} f(x_0)$.
From (7), we have

$$\begin{aligned}
\|x_{n+1} - x_0\|^2 &= \|\alpha_n f(x_n) + \beta_n P_C (I - \lambda D) x_n + \gamma_n P_C \left(I - a\left(\frac{A^*\left(I - P_Q\right)A}{2} + \frac{B^*\left(I - P_Q\right)B}{2}\right)\right) x_n - x_0\|^2 \\
&\leq \|\beta_n(P_C (I - \lambda D) x_n - x_0) + \gamma_n\left(P_C\left(I - a\left(\frac{A^*\left(I - P_Q\right)A}{2} + \frac{B^*\left(I - P_Q\right)B}{2}\right)\right) x_n - x_0\right)\|^2 \\
&\quad +2\alpha_n\langle f(x_n) - x_0, x_{n+1} - x_0\rangle \\
&\leq (1 - \alpha_n)^2\|x_n - x_0\|^2 + 2\alpha_n\langle f(x_0) - x_0, x_{n+1} - x_0\rangle + 2\alpha_n\alpha\|x_n - x_0\|\|x_{n+1} - x_0\| \\
&\leq (1 - \alpha_n)^2\|x_n - x_0\|^2 + 2\alpha_n\langle f(x_0) - x_0, x_{n+1} - x_0\rangle + \alpha_n\alpha\|x_n - x_0\|^2 + \alpha_n\alpha\|x_{n+1} - x_0\|^2.
\end{aligned}$$

It implies that

$$\|x_{n+1} - x_0\|^2 \leq \left(1 - \frac{2\alpha_n (1 - \alpha)}{1 - \alpha_n\alpha}\right)\|x_n - x_0\|^2 + \frac{2\alpha_n (1 - \alpha)}{1 - \alpha_n\alpha}\left(\frac{\alpha_n}{2(1 - \alpha)}\|x_n - x_0\|^2 + \frac{1}{1 - \alpha}\langle f(x_0) - x_0, x_{n+1} - x_0\rangle\right).$$

From Lemma 2.3, condition *i*) and (22), we obtain that the sequence $\{x_n\}$ converges strongly to $x_0 = P_{\Gamma\cap VI(C,D)} f(x_0)$. This complete the proof. □

Using Theorem 3.1 , we can solve split feasibility problem.

Theorem 3.2. *Let H_1 and H_2 be real Hilbert spaces and let C, Q be nonempty closed convex subsets of H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be bounded linear operator with A^* is adjoint of A where L is special radius of A^*A and let $D : C \to H_1$ be d-inverse strongly monotone. Assume that $\Gamma_A \cap VI(C, D) \neq \emptyset$, where $\Gamma_A = \{x \in C : Ax \in Q\}$. Let the sequence $\{x_n\}$ generated by $x_1 \in C$ and*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n P_C (I - \lambda D) x_n + \gamma_n P_C \left(I - a\left(A^*\left(I - P_Q\right)A\right)\right) x_n,$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subseteq (0, 1)$ with $a_n + \beta_n + \gamma_n = 1$ and $f : C \to C$ is α-contractive mapping with $\alpha \in (0, 1)$. Suppose that the following conditions hold;

i) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$,

ii) $c \leq \beta_n, \gamma_n \leq d$, *for some* $c, d > 0$,

iii) $\lambda \in (0, 2d), a \in \left(0, \frac{2}{L}\right)$,

iv) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}|, \sum_{n=1}^{\infty} |\beta_n - \beta_{n-1}| < \infty$.

Then the sequence $\{x_n\}$ converges strongly to $x_0 = P_{\Gamma_A\cap VI(C,D)} f(x_0)$.

Remark 3.3. *If we take $D \equiv 0$ in Theorem 3.2, we have*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n P_C \left(I - a\left(A^*\left(I - P_Q\right)A\right)\right) x_n, \tag{23}$$

for all $n \in \mathbb{N}$, which is modification iterative scheme $\{x_n\}$ in Theorem 1.2 and by Theorem 3.2, we have the sequence $\{x_n\}$ generated by (23) converges strongly to a solution of SFP under the sufficient conditions of Theorem 3.2.

4. Application

Let $C \subseteq H_1, Q \subseteq H_2$ of Hilbert space H_1, H_2 and let $A : H_1 \to H_2$ be a bounded linear operator.

Let $g : H_1 \to \mathbb{R}$ be a continuous differentiable function. The minimization problem;

$$\min_{x \in C} g(x) := \frac{1}{2} \|(I - P_Q)Ax\|^2, \tag{24}$$

is to find a point $x^* \in C$ such that $g(x^*) \leq g(x)$ for all $x \in C$.

From studying the minimization problem, we introduce *the general constrained minimization problem* as follows,

$$\min_{x \in C} g(x) := \frac{\left\|\left(I - P_Q\right) Ax\right\|^2}{4} + \frac{\left\|\left(I - P_Q\right) Bx\right\|^2}{4}. \tag{25}$$

The set of all solution of (25) is denoted by $\Gamma_g = \{x^* \in C : g(x^*) \leq g(x), \ \forall x \in C\}$.

The following results show the relationship between the general split feasibility problem and the general constrained minimization problem.

Lemma 4.1. *Let H_1 and H_2 be real Hilbert space and C, Q be nonempty closed convex subsets of H_1 and H_2, respectively. Let $A, B : H_1 \to H_2$ be bounded linear operators with A^*, B^* are adjoint of A and B, respectively and let $g : H_1 \to \mathbb{R}$ be a continuous differentiable function defined by $g(x) = \frac{\left\|\left(I - P_Q\right) Ax\right\|^2}{4} + \frac{\left\|\left(I - P_Q\right) Bx\right\|^2}{4}$ for all $x \in H_1$. Assume that $\Gamma \neq \emptyset$. Then the followings are equivalent.*

i) $x^* \in \Gamma$,

ii) $x^* \in \Gamma_g$.

Proof. *ii)* $\Rightarrow$ *i)* Let $x^* \in \Gamma_g$ and let $\overline{x} \in \Gamma$, we get $\overline{x} \in C$ and $A\overline{x}, B\overline{x} \in Q$.

Since $x^* \in \Gamma_g$, we have

$$\frac{\|Ax^* - P_Q Ax^*\|^2}{4} + \frac{\|Bx^* - P_Q Bx^*\|^2}{4} \leq \frac{\|Ay - P_Q Ay\|^2}{4} + \frac{\|By - P_Q By\|^2}{4}, \tag{26}$$

for all $y \in C$.

Since $\overline{x} \in C$, we have

$$\frac{\|Ax^* - P_Q Ax^*\|^2}{4} + \frac{\|Bx^* - P_Q Bx^*\|^2}{4} \leq \frac{\|A\overline{x} - P_Q A\overline{x}\|^2}{4} + \frac{\|B\overline{x} - P_Q B\overline{x}\|^2}{4}. \tag{27}$$

Since $A\overline{x}, B\overline{x} \in Q$, we have $A\overline{x} = P_Q A\overline{x}$ and $B\overline{x} = P_Q B\overline{x}$.

From (27), we have

$$\frac{\|Ax^* - P_Q Ax^*\|^2}{4} + \frac{\|Bx^* - P_Q Bx^*\|^2}{4} = 0.$$

It implies that $Ax^* = P_Q Ax^* \in Q$ and $Bx^* = P_Q Bx^* \in Q$.

Since $x^* \in \Gamma_g$, we have $x^* \in C$.

Hence $x^* \in \Gamma$.

i) $\Rightarrow$ *ii)* Let $x^* \in \Gamma$, we have $x^* \in C$ and $Ax^*, Bx^* \in Q$. Then, we have

$$\frac{\left\|Ax^* - P_Q Ax^*\right\|^2}{4} + \frac{\left\|Bx^* - P_Q Bx^*\right\|^2}{4} = 0 \leq \frac{\left\|Ay - P_Q Ay\right\|^2}{4} + \frac{\left\|By - P_Q By\right\|^2}{4}$$

for all $y \in C$. It implies that $x^* \in \Gamma_g$. $\square$

Remark 4.2. *We observe that* $\nabla g = \frac{A^*\left(I - P_Q\right)A}{2} + \frac{B^*\left(I - P_Q\right)B}{2}$, *where* A^* *and* B^* *are adjoint of* A *and* B, *respectively and* ∇g *is a gradient of* g. *From Lemma 2.5 and 4.1, we have* $\Gamma_g = \Gamma = VI(C, \nabla g)$, *where* $\Gamma \neq \emptyset$.

Theorem 4.3. *Let* H_1 *and* H_2 *be real Hilbert spaces and let* C, Q *be nonempty closed convex subsets of* H_1 *and* H_2, *respectively. Let* $A, B : H_1 \to H_2$ *be bounded linear operators with* A^*, B^* *are adjoint of* A *and* B, *respectively and* $L = \max\{L_A, L_B\}$, *where* L_A *and* L_B *are special radius of* A^*A *and* B^*B. *Let the function* $g : H_1 \to \mathbb{R}$ *be differentiable continuous function defined by* $g(x) = \frac{\left\|\left(I - P_Q\right)Ax\right\|}{4} + \frac{\left\|\left(I - P_Q\right)Bx\right\|^2}{4}$ *and let* $D : C \to H_1$ *be d-inverse strongly monotone. Assume that* $\Gamma \cap VI(C, D) \neq \emptyset$. *Let the sequence* $\{x_n\}$ *generated by* $x_1 \in C$ *and*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n P_C(I - \lambda D)x_n + \gamma_n P_C(I - a\nabla g)x_n,$$

for all $n \in \mathbb{N}$, *where* $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subseteq (0, 1)$ *with* $a_n + \beta_n + \gamma_n = 1$ *and* $f : C \to C$ *is* α*-contractive mapping with* $\alpha \in (0, 1)$. *Suppose that the following conditions hold;*

i) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty,$

ii) $c \leq \beta_n, \gamma_n \leq d$, *for some* $c, d > 0$,

iii) $\lambda \in (0, 2d), a \in \left(0, \frac{2}{L}\right),$

iv) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}|, \sum_{n=1}^{\infty} |\beta_n - \beta_{n-1}| < \infty.$

Then the sequence $\{x_n\}$ *converges strongly to* $x_0 = P_{\Gamma_g \cap VI(C,D)} f(x_0)$.

Proof. From Theorem 3.1 and Lemma 4.1, we can conclude Theorem 4.3. □

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