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An Open Problem of Lü, Li and Yang

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Abstract. In this paper, we use the idea of normal family to investigate the problem of entire functions that share two entire functions with one of their derivatives. In particular, we solve an open problem posed in the last section of [11]. Some examples have been exhibited to show that the conditions used in the paper are sharp.

1. Introduction, Definitions and Results

In this paper, by a meromorphic (resp. entire) function we shall always mean meromorphic (resp. entire) function in the whole complex plane $\mathbb{C}$. We denote by $n(r,\infty;f)$ the number of poles of f lying in $|z|<r$, the poles are counted with their multiplicities. We call the quantity

$$N(r,\infty;f)=\int\limits_0^r \frac{n(t,\infty;f)-n(0,\infty;f)}{t}dt+n(0,\infty;f)\log r$$

as the integrated counting function or simply the counting function of poles of f and

$$m(r,\infty;f)=\frac{1}{2\pi}\int\limits_0^{2\pi}\log^+|f(re^{i\theta})|d\theta$$

as the proximity function of poles of f, where $\log^+ x=\log x$, if $x\geq 1$ and $\log^+ x=0$, if $0\leq x<1$.

We use the notation $T(r,f)$ for the sum $m(r,\infty;f)+N(r,\infty;f)$ and it is called the Nevanlinna characteristic function of f. We adopt the standard notation $S(r,f)$ for any quantity satisfying the relation $\frac{S(r,f)}{T(r,f)}\to 0$ as $r\to\infty$ except possibly a set of finite linear measure.

For $a\in\mathbb{C}$, we write $N(r,a;f)=N(r,\infty;\frac{1}{f-a})$ and $m(r,a;f)=m(r,\infty;\frac{1}{f-a})$.

Again we denote by $\overline{n}(r,a;f)$ the number of distinct a points of f lying in $|z|<r$, where $a\in\mathbb{C}\cup\{\infty\}$. The quantity

$$\overline{N}(r,a;f)=\int\limits_0^r \frac{\overline{n}(t,a;f)-\overline{n}(0,a;f)}{t}dt+\overline{n}(0,a;f)\log r$$

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denotes the reduced counting function of a points of f (see, e.g., [6, 17]).

Let k be a positive integer and $a \in \mathbb{C} \cup \{\infty\}$. We use the notation $N_{k)}(r, a; f)$ to denote the counting function of a-points of f with multiplicity not greater than k and $N_{(k+1}(r, a; f)$ to represent the counting function of a-points of f with multiplicity greater than k respectively. Similarly $\overline{N}_{k)}(r, a; f)$ and $\overline{N}_{(k+1}(r, a; f)$ are their reduced functions respectively.

A meromorphic function $a(z)$ is said to be a small function of f if $T(r, a) = S(r, f)$, i.e., if $T(r, a) = o(T(r, f))$ as $r \to \infty$ except possibly a set of finite linear measure.

Let f and g be two non-constant meromorphic functions in the complex plane $\mathbb{C}$ and Q be a polynomial or a finite complex number. If $g(z) - Q(z) = 0$ whenever $f(z) - Q(z) = 0$, we write $f(z) = Q(z) \Rightarrow g(z) = Q(z)$.

Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions. Let $a(z)$ be a small function with respect to $f(z)$ and $g(z)$. If $f(z) - a(z)$ and $g(z) - a(z)$ have the same zeros with the same multiplicities then we say that $f(z)$ and $g(z)$ share $a(z)$ with CM (counting multiplicities) and if we do not consider the multiplicities then we say that $f(z)$ and $g(z)$ share $a(z)$ with IM (ignoring multiplicities).

Let $k \in \mathbb{N} \cup \{0\} \cup \{\infty\}$ and $a \in \mathbb{C} \cup \{\infty\}$. We denote by $E_k(a; f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \le k$ and $k + 1$ times if $m > k$. If for two meromorphic functions f and g we have $E_k(a; f) = E_k(a; g)$, then we say that f and g share a with weight k. We write f, g share (a, k) to mean that f, g share a with weight k.

Remark 1.1. *If f, g share (a, k), then z_0 is an a-point of f of multiplicity m $(\le k)$ if and only if it is an a-point of g of multiplicity m $(\le k)$ and z_0 is an a-point of f of multiplicity m $(> k)$ if and only if it is an a-point of g of multiplicity n $(> k)$, where m is not necessarily equal to n.*

If a is a small function we define that f and g share a IM or a CM or with weight l according as $f - a$ and $g - a$ share $(0, 0)$ or $(0, \infty)$ or $(0, l)$ respectively.

We recall that the order $\rho(f)$ of meromorphic function $f(z)$ is defined by

$$\rho(f) = \limsup_{r\to\infty} \frac{\log T(r, f)}{\log r}.$$

Furthermore when $f(z)$ is an entire function, we have

$$\rho(f) = \limsup_{r\to\infty} \frac{\log T(r, f)}{\log r} = \limsup_{r\to\infty} \frac{\log\log M(r, f)}{\log r},$$

where $M(r, f) = \max\limits_{|z|=r} |f(z)|$. Let f be an entire function. We know that f can be expressed by the power series $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$. We denote by

$$\mu(r, f) = \max_{n\in\mathbb{N}, |z|=r} \{|a_n z^n|\} \text{ and } \nu(r, f) = \sup\{n : |a_n| r^n = \mu(r, f)\}.$$

Let h be a meromorphic function in $\mathbb{C}$. Then h is called a normal function if there exists a positive real number M such that $h^{\#}(z) \le M \ \forall\ z \in \mathbb{C}$, where

$$h^{\#}(z) = \frac{|h'(z)|}{1 + |h(z)|^2}$$

denotes the spherical derivative of h.

Let $\mathcal{F}$ be a family of meromorphic functions in a domain $D \subset \mathbb{C}$. We say that $\mathcal{F}$ is normal in D if every sequence $\{f_n\}_n \subseteq \mathcal{F}$ contains a subsequence which converges spherically and uniformly on the compact subsets of D (see [15]).

Rubel and Yang was the first to study the entire functions that share values with their derivatives. In 1977 they proved the following important theorem.

Theorem 1.2. *[14] Let a and b be complex numbers such that $b \ne a$ and let f be a non-constant entire function. If f and f' share the values a and b CM, then $f \equiv f'$.*

From then on, this result has undergone various extensions and improvements (see [17]). In 1980 G. G. Gundersen improved Theorem 1.2 and obtained the following result.

Theorem 1.3. *[4] Let f be a non-constant meromorphic function, a and b be two distinct finite values. If f and f' share the values a and b CM, then $f \equiv f'$.*

Mues and Steinmetz [13] generalized Theorem 1.2 from sharing values CM to IM and obtained the following result.

Theorem 1.4. *[13] Let a and b be complex numbers such that $b \neq a$ and let f be a non-constant entire function. If f and f' share the values a and b IM, then $f \equiv f'$.*

In 1996, Brück [1] discussed the possible relation between f and f' when an entire function f and it's derivative f' share only one finite value CM. In this direction an interesting problem still open is the following conjecture proposed by Brück [1].

Conjecture 1.5. *Let f be a non-constant entire function. Suppose*

$$\rho_1(f) := \limsup_{r\to\infty} \frac{\log\log T(r,f)}{\log r}$$

is not a positive integer or infinite. If f and f' share one finite value a CM, then

$$\frac{f'-a}{f-a} = c \tag{1.1}$$

for some non-zero constant c.

The Conjecture 1.5 had been proved by Brück [1] for the case $a = 0$ and $N(r, 0; f') = S(r, f)$. From the differential equations

$$\frac{f'-a}{f-a} = e^{z^n} \text{ and } \frac{f'-a}{f-a} = e^{e^z}, \tag{1.2}$$

we see that when $\rho_1(f)$ is a positive integer or infinite, the conjecture does not hold.

Gundersen and Yang [5] proved that the Conjecture 1.5 is true when f is of finite order. Further Chen and Shon [2] proved that the Conjecture 1.5 is also true when f is of infinite order with $\rho_1(f) < \frac{1}{2}$. Recently Cao [3] proved that the Conjecture 1.5 is also true when f is of infinite order with $\rho_1(f) = \frac{1}{2}$. But the case $\rho_1(f) > \frac{1}{2}$ is still open.

It is now interesting to know what happens if f is replaced by f^n in the Conjecture 1.5. From (1.2) we see that the Conjecture 1.5 does not hold when $n = 1$. Thus we have to discuss the problem only when $n \geq 2$.

Yang and Zhang [16] proved that the Conjecture 1.5 holds for the function f^n without imposing the order restriction on f if n is relatively large. Actually they proved the following result.

Theorem 1.6. *[16] Let f be a non-constant entire function, $n(\geq 7)$ be an integer and let $F = f^n$. If F and F' share 1 CM, then $F \equiv F'$ and f assumes the form*

$$f(z) = ce^{\frac{1}{n}z},$$

where c is a non-zero constant.

Improving all the results obtained in [16], Zhang [19] proved the following theorem.

Theorem 1.7. *[19] Let f be a non-constant entire function, n, k be positive integers and $a(z)(\not\equiv 0, \infty)$ be a meromorphic small function of f. Suppose $f^n - a$ and $(f^n)^{(k)} - a$ share the value 0 CM and $n > k + 4$, then $f^n \equiv (f^n)^{(k)}$ and f assumes the form*

$$f(z) = ce^{\frac{\lambda}{n}z},$$

where c is a non-zero constant and $\lambda^k = 1$.

In 2009, Zhang and Yang [20] further improved the above result in the following manner.

Theorem 1.8. *[20] Let f be a non-constant entire function, n, k be positive integers and $a(z)(\not\equiv 0, \infty)$ be a meromorphic small function of f. Suppose $f^n - a$ and $(f^n)^{(k)} - a$ share the value 0 CM and $n > k + 1$. Then conclusion of Theorem 1.7 holds.*

In 2010, Zhang and Yang [21] further improved the above result in the following manner.

Theorem 1.9. *[21] Let f be a non-constant entire function, n and k be positive integers. Suppose f^n and $(f^n)^{(k)}$ share 1 CM and $n \geq k + 1$. Then conclusion of Theorem 1.7 holds.*

Using the theory of normal families, in 2011, Lü and Yi [10] proved the following theorem.

Theorem 1.10. *[10] Let f be a transcendental entire function, n, k be two positive integers with $n \geq k + 1$, $F = f^n$ and $Q \not\equiv 0$ be a polynomial. If $F - Q$ and $F^{(k)} - Q$ share the value 0 CM, then $F \equiv F^{(k)}$ and $f(z) = ce^{wz/n}$, where c and w are non-zero constants such that $w^k = 1$.*

Remark 1.11. *Following example shows that the hypothesis of the transcendental of f in Theorem 1.10 is necessary.*

Example 1.12. *[10] Let $f(z) = z$ and $n = 2, k = 1$. Then*

$$\frac{(f^2)' - Q}{f^2 - Q} = 2$$

and $(f^2)' - Q$, $f^2 - Q$ share 0 CM, but $(f^2)' \not\equiv f^2$, where $Q(z) = 2z^2 - 2z$.

Remark 1.13. *It is easy to see that the condition $n \geq k + 1$ in Theorem 1.10 is sharp by the following example.*

Example 1.14. *Let $f(z) = e^{e^z} \int_0^z e^{-e^t}(1 - e^t)t\, dt$ and $n = 1, k = 1$. Then*

$$\frac{f'(z) - z}{f(z) - z} = e^z$$

and $f'(z) - z$, $f(z) - z$ share 0 CM, but $f' \not\equiv f$.

Now observing the above theorem, Lü, Li and Yang [11] asked the following question:

Question 1.15. *What can be said "if $f^n - Q_1$ and $(f^n)^{(k)} - Q_2$ share the value 0 CM" ? where Q_1 and Q_2 are polynomials, and $Q_1Q_2 \not\equiv 0$.*

Lü, Li and Yang [11] solved the Question 1.15 for $k = 1$ by giving the transcendental entire solutions of the equation

$$F' - Q_1 = Re^{\alpha}(F - Q_2), \tag{1.3}$$

where $F = f^n$, R is a rational function and α is an entire function and they obtained the following results.

Theorem 1.16. *[11] Let f be a transcendental entire function and let $F = f^n$ be a solution of equation (1.3), $n \geq 2$ be an integer, then $\frac{Q_1}{Q_2}$ is a polynomial and*

$$f' \equiv \frac{Q_1}{nQ_2} f.$$

Theorem 1.17. *[11] Let f be a transcendental entire function, $n \geq 2$ be an integer. If $f^n - Q$ and $(f^n)' - Q$ share 0 CM, where $Q \not\equiv 0$ is a polynomial, then*

$$f(z) = ce^{z/n},$$

where c is a non-zero constant.

Also in the same paper Lü, Li and Yang [11] posed the following conjecture.

Conjecture 1.18. *Let f be a transcendental entire function, n, k be two positive integers. If $f^n - Q_1$ and $(f^n)^{(k)} - Q_2$ share 0 CM and $n \geq k+1$, then $(f^n)^{(k)} \equiv \frac{Q_2}{Q_1} f^n$. Further, if $Q_1 = Q_2$, then $f(z) = ce^{wz/n}$, where Q_1, Q_2 are polynomials with $Q_1Q_2 \not\equiv 0$ and c, w are non-zero constants such that $w^k = 1$.*

Again Lü, Li and Yang [11] asked the following question.

Question 1.19. *What can be said if the condition in Conjecture 1.18 "f^n" be replaced by "$P(f)$" where $P(z) = \sum\limits_{i=0}^{n} a_i z^i$?*

In 2016, Majumder [12] proved that Conjecture 1.18 is true and obtained the following result.

Theorem 1.20. *[12] Let f be a transcendental entire function, n and k be two positive integers. If $f^n - Q_1$ and $(f^n)^{(k)} - Q_2$ share 0 CM and $n \geq k+1$, then $(f^n)^{(k)} \equiv \frac{Q_2}{Q_1} f^n$. Further, if $Q_1 = Q_2$, then $f(z) = ce^{\frac{\lambda}{n}z}$, where Q_1, Q_2 are polynomials with $Q_1Q_2 \not\equiv 0$ and c, λ are non-zero constants such that $\lambda^k = 1$.*

One of our objective to write this paper is to give an affirmative answer of Question 1.19. Now observing Theorem 1.20 the following questions are inevitable.

Question 1.21. *What happens if "$f^n(z) - P_1(z)e^{Q(z)}$ and $(f^n(z))^{(k)} - P_2(z)e^{Q(z)}$ share the value 0 CM, where $P_i(z)(\not\equiv 0)(i = 1, 2)$ and $Q(z)$ are polynomials in Theorem 1.20 ?*

Question 1.22. *Can "CM" sharing in Theorem 1.20 be reduced to finite weight sharing ?*

In this paper, taking the possible answer of Questions 1.21 and 1.22 into back ground we obtain our main result as follows.

Theorem 1.23. *Let $f(z)$ be a transcendental entire function, n and k be two positive integers. Let $\alpha_i = P_i e^Q (i = 1, 2)$, where $P_1(z)(\not\equiv 0)$, $P_2(z)(\not\equiv 0)$ and $Q(z)$ are polynomials such that $2\max\{\deg(Q), 1 + \deg(P_2) - \deg(P_1)\} < \rho(f)$. Suppose $P(z) = \sum\limits_{i=0}^{n} a_i z^i$, where $a_i \in \mathbb{C}$, $a_n \neq 0$. If $P(f) - \alpha_1$ and $\left(P(f)\right)^{(k)} - \alpha_2$ share $(0, 1)$ and if the zeros of $P(f)$ are of multiplicities at least $k+1$, then $\left(P(f)\right)^{(k)} \equiv \frac{\alpha_2}{\alpha_1} P(f)$ and α_1, α_2 reduce to polynomials. Furthermore if $P_1 \equiv P_2$, then $P(z)$ assumes the form $P(z) = a_n(z-d)^n$, where $d \in \mathbb{C}$ and $f(z)$ assumes the form $f(z) = d + ce^{\frac{\lambda}{n}z}$, where c is a non-zero constant and $\lambda^k = 1$.*

Remark 1.24. *If Q is a constant, then Theorem 1.23 still holds without the assumption that $2\max\{\deg(Q), 1 + \deg(P_2) - \deg(P_1)\} < \rho(f)$.*

Remark 1.25. *It is easy to see that the condition "zeros of $P(f)$ are of multiplicities at least $k+1$" in Theorem 1.23 is sharp by the following examples.*

Example 1.26. *Let*

$$P(z) = z^2 + 2 \quad \text{and} \quad f(z) = e^{(z-1)^2}.$$

Let $\alpha_1(z) = 3$ and $\alpha_2(z) = 4(z-1)$. Clearly all the zeros of $P(f)$ are simple. Note that

$$P(f(z)) - 4 = e^{2(z-1)^2} - 1$$

and

$$P(f(z))' - 4(z-1) = 4(z-1)\left[e^{2(z-1)^2} - 1\right].$$

According to Remark 1.1, $P(f) - \alpha_1$ and $P(f)' - \alpha_2$ share $(0, 1)$, but $P(f)' \not\equiv \frac{\alpha_2}{\alpha_1} P(f)$.

Example 1.27. *Let*

$$P(z) = z \ \text{ and } \ f(z) = z^2 + \frac{1}{2}e^{(z-1)^2}.$$

Let $\alpha_1(z) = z^2 + \frac{1}{2}$ *and* $\alpha_2(z) = 3z - 1$. *Clearly all the zeros of* $P(f)$ *are simple. Note that*

$$P(f(z)) - (z^2 + \frac{1}{2}) = \frac{1}{2}\Big[e^{(z-1)^2} - 1\Big]$$

and

$$P(f(z))' - (3z - 1) = (z - 1)\Big[e^{(z-1)^2} - 1\Big].$$

Obviously $P(f) - \alpha_1$ *and* $P(f)' - \alpha_2$ *share* $(0, 1)$, *but* $P(f)' \not\equiv \frac{\alpha_2}{\alpha_1}P(f)$.

Example 1.28. *Let*

$$P(z) = z \ \text{ and } \ f(z) = z^2 + e^{\frac{1}{2}z^2}.$$

Let $\alpha_1(z) = z^2 + 1$ *and* $\alpha_2(z) = 3z$. *Clearly all the zeros of* $P(f)$ *are simple. Note that*

$$P(f(z)) - (z^2 + 1) = e^{\frac{1}{2}z^2} - 1$$

and

$$P(f(z))' - 3z = z\Big[e^{\frac{1}{2}z^2} - 1\Big].$$

Obviously $P(f) - \alpha_1$ *and* $P(f)' - \alpha_2$ *share* $(0, 1)$, *but* $P(f)' \not\equiv \frac{\alpha_2}{\alpha_1}P(f)$.

Example 1.29. *Let*

$$P(z) = z \ \text{ and } \ f(z) = e^{3z} + \frac{2z}{3} + \frac{2}{9}.$$

Clearly all the zeros of $P(f)$ *are simple. Note that* $P(f(z))' - z = 3(P(f)(z) - z)$. *Then* $P(f) - \alpha_1$ *and* $P(f)' - \alpha_2$ *share* 0 *CM, but* $P(f)' \not\equiv \frac{\alpha_2}{\alpha_1}P(f)$, *where* $\alpha_1(z) = \alpha_2(z) = z$.

Example 1.30. *Let*

$$P(z) = z \ \text{ and } \ f(z) = e^{c_1 z} + c_2,$$

where $c_1, c_2 \in \mathbb{C} \setminus \{0\}$ *such that* $c_1 \neq 1$. *Clearly all the zeros of* $P(f)$ *are simple. Then* $P(f) - \alpha_1$ *and* $P(f)' - \alpha_2$ *share* 0 *CM, but* $P(f)' \not\equiv \frac{\alpha_2}{\alpha_1}P(f)$, *where* $\alpha_1(z) = \alpha_2(z) = c_3$ *and* $c_1c_2 = c_3(c_1 - 1)$.

Example 1.31. *Let*

$$P(z) = z \ \text{ and } \ f(z) = \frac{1}{2}e^{\frac{1}{2}z} + z.$$

Note that all the zeros of $P(f)$ *are simple. Clearly* $P(f) - \alpha_1$ *and* $P(f)' - \alpha_2$ *share* 0 *CM, but* $P(f)' \not\equiv \frac{\alpha_2}{\alpha_1}P(f)$, *where* $\alpha_1(z) = \alpha_2(z) = 2 - z$.

Remark 1.32. *By the following example, it is easy to see that the hypothesis of the transcendental of* f *in Theorem 1.23 is necessary.*

Example 1.33. *Let*

$$P(z) = z^2 \ \text{ and } \ f(z) = z.$$

Note that zeros of $P(f)$ *are of multiplicities* 2. *Let* $\alpha_1(z) = 2z^2 + z$ *and* $\alpha_2(z) = 2z^2 + 4z$. *Clearly* $P(f) - \alpha_1$ *and* $P(f)' - \alpha_2$ *share* 0 *CM, but* $P(f)' \not\equiv \frac{\alpha_2}{\alpha_1}P(f)$.

2. Lemmas

In this section we present the lemmas which will be needed in the sequel.

Lemma 2.1. *[17] Let f be an entire function of finite order and k be a positive integer, then*

$$m\Big(r, \frac{f^{(k)}}{f}\Big) = O(\log r) \text{ as } r \to \infty.$$

Lemma 2.2. *([7], Theorem 3.2) Let f be a transcendental entire function and $0 < \delta < \frac{1}{4}$. Suppose that at the point z with $|z| = r$ the inequality*

$$|f(z)| > M(r, f)\nu(r, f)^{-\frac{1}{4}+\delta}$$

holds, then there exists a set $F \subset \mathbb{R}^+$ of finite logarithmic measure, i.e., $\int_F \frac{1}{t}dt < +\infty$, such that

$$f^{(m)}(z) = \Big(\frac{\nu(r, f)}{z}\Big)^m (1 + o(1))f(z)$$

holds for all $m \geq 0$ and $r \notin F$.

Lemma 2.3. *[9] Let $\{f_n\}$ be a family of functions meromorphic (analytic) on the unit disc Δ. If $a_n \to a$, $|a| < 1$ and $f_n^{\#}(a_n) \to \infty$, then there exists*

(a) a subsequence of f_n (which we still write as $\{f_n\}$);
(b) points $z_n \to z_0$, $|z_0| < 1$;
(c) positive numbers $\rho_n \to 0$

such that $f_n(z_n + \rho_n\xi) = g_n(\xi) \to g(\xi)$ locally uniformly, where g is a non-constant meromorphic (entire) function on $\mathbb{C}$ such that

$$\rho_n \leq \frac{M}{f_n^{\#}(a_n)},$$

where M is a constant which is independent of n.

Lemma 2.4. *[18] Let f be a meromorphic function in the complex plane, $\rho(f) > 2$. Then for each $0 < \mu < \frac{\rho(f)-2}{2}$, there exist points $a_n \to \infty (n \to \infty)$ such that*

$$\lim_{n\to\infty} \frac{f^{\#}(a_n)}{|a_n|^{\mu}} = +\infty.$$

Lemma 2.5. *[8] Let f be a meromorphic function of infinite order on $\mathbb{C}$. Then there exist points $z_n \to \infty$ such that for every $N > 0$, $f^{\#}(z_n) > |z_n|^N$, if n is sufficiently large.*

3. Proof of the theorem

Proof. Let $F = \frac{H}{\alpha_1}$ and $G = \frac{H^{(k)}}{\alpha_2}$, where $H = P(f)$. Now we consider following two cases.
Case 1. Suppose $\rho(f) < \infty$. Clearly $\rho(\alpha_i) = \deg(Q)$ for $i = 1, 2$ and $\rho(f) = \rho(H)$. Since $\deg(Q) < \rho(f)$, it follows that $\rho(\alpha_i) < \rho(f)$ and so $\rho(\alpha_i) < \rho(H)$ for $i = 1, 2$. Note that $\rho\big(\frac{H}{\alpha_1}\big) \leq \max\{\rho(H), \rho(\alpha_1)\} = \rho(H)$. Since $\rho(\alpha_1) < \rho(H)$, it follow that $\rho(H) = \rho\big(\frac{H}{\alpha_1}.\alpha_1\big) \leq \max\{\rho\big(\frac{H}{\alpha_1}\big), \rho(\alpha_1)\} = \rho\big(\frac{H}{\alpha_1}\big)$. Consequently $\rho(H) = \rho\big(\frac{H}{\alpha_1}\big) = \rho(F)$. Therefore

$$\deg(Q) < \rho(f) = \rho(H) = \rho\Big(\frac{H}{\alpha_1}\Big) = \rho(F) < \infty.$$

Note that $\rho(H^{(k)}) = \rho(H) < \infty$. Clearly $\rho(G) \leq \max\{\rho(H^{(k)}), \rho(\alpha_2)\} < \infty$. Following two sub-cases are immediately.

Sub-case 1.1. Suppose Q is a constant. In that case α_1 and α_2 reduce to polynomials.

By the given condition we see that F and G share $(1,1)$ except for the zeros of $\alpha_i(z)$ for $i = 1,2$ and so $\overline{N}(r,1;F) = \overline{N}(r,1;G) + O(\log r)$. Let

$$\Phi = \frac{F'(F-G)}{F(F-1)} = \frac{F'}{F-1}\Big[1 - \frac{P_1}{P_2}.\frac{H^{(k)}}{H}\Big]. \tag{3.1}$$

We now consider following two sub-cases.

Sub-case 1.1.1. $\Phi \not\equiv 0$. Clearly $F \not\equiv G$. Now by Lemma 2.1 we get $m(r,\infty;\Phi) = O(\log r)$. Let z_0 be a zero of F of multiplicity $p_0(\geq k+1)$ such that $\alpha_i(z_0) \neq 0$ for $i = 1,2$. Clearly z_0 is a zero of H of multiplicity p_0. Therefore z_0 is a zero of $H^{(k)}$ of multiplicity $p_0 - k$ and so z_0 must be a zero of G of multiplicity $p_0 - k$. Clearly from (3.1), we get

$$\Phi(z) = O((z - z_0)^{p_0 - k - 1}).$$

Consequently $\Phi(z)$ is holomorphic at z_0. Let z_1 be a common zero of $F - 1$ and $G - 1$ such that $\alpha_i(z_1) \neq 0$, where $i = 1,2$. Suppose z_1 is a zero of $F - 1$ of multiplicity p_1. Since F and G share $(1,1)$ except for the zeros of α_1 and α_2 respectively, it follows that z_1 must be a zero of $G - 1$ of multiplicity q_1. Then in some neighbourhood of z_1, we get by Taylor's expansion

$$F(z) - 1 = a_{p_1}(z - z_1)^{p_1} + a_{p_1+1}(z - z_1)^{p_1+1} + \ldots, a_{p_1} \neq 0$$

$$G(z) - 1 = b_{q_1}(z - z_1)^{q_1} + b_{q_1+1}(z - z_1)^{q_1+1} + \ldots, b_{q_1} \neq 0.$$

Clearly $F'(z) = p_1 a_{p_1}(z - z_1)^{p_1 - 1} + (p_1 + 1)a_{p_1+1}(z - z_1)^{p_1} + \ldots$. Note that

$$F(z) - G(z) = \begin{cases} a_{p_1}(z - z_1)^{p_1} + \ldots, & \text{if } p_1 < q_1 \\ -b_{q_1}(z - z_1)^{q_1} - \ldots, & \text{if } p_1 > q_1 \\ (a_{p_1} - b_{p_1})(z - z_1)^{p_1} + \ldots, & \text{if } p_1 = q_1. \end{cases}$$

Clearly from (3.1) we get

$$\Phi(z) = O\Big((z - z_1)^{t-1}\Big), \tag{3.2}$$

where $t \geq \min\{p_1, q_1\}$. Now from (3.2), it follows that Φ is holomorphic at z_1. From these and the hypotheses of Theorem 1.23 we see that $N(r,\infty;\Phi) = O(\log r)$. Consequently $T(r,\Phi) = O(\log r)$. This shows that $\Phi(z)$ is a rational function.

Let $p_1 \geq 2$. Since F and G share $(1,1)$ except for the zeros of $\alpha_1(z)$ and $\alpha_2(z)$ respectively, it follows that $q_1 \geq 2$. Therefore from (3.2) we see that

$$\overline{N}_{(2}(r,1;F) \leq N(r,0;\Phi) \leq T(r,\Phi) + O(1) = O(\log r) \text{ as } r \to \infty,$$

i.e., $\overline{N}_{(2}(r,1;F) = O(\log r)$ as $r \to \infty$. Since F and G share $(1,1)$ except for the zeros of $\alpha_1(z)$ and $\alpha_2(z)$ respectively, it follows that $\overline{N}_{(2}(r,1;G) = O(\log r)$ as $r \to \infty$. This shows that $F - 1$ and $G - 1$ have finitely many multiple zeros, i.e.,

$$N_{(2}(r,1;F) = O(\log r) \text{ as } r \to \infty \text{ and } N_{(2}(r,1;G) = O(\log r) \text{ as } r \to \infty.$$

Therefore

$$N_{(2}(r,\alpha_1;H) = O(\log r) \text{ as } r \to \infty \text{ and } N_{(2}(r,\alpha_2;H^{(k)}) = O(\log r) \text{ as } r \to \infty.$$

This shows that $H - \alpha_1$ and $H^{(k)} - \alpha_2$ have finitely many multiple zeros. Since $H - \alpha_1$ and $H^{(k)} - \alpha_2$ share $(0,1)$, then there exists a polynomial γ, such that

$$\frac{H^{(k)} - \alpha_2}{H - \alpha_1} = \beta e^{\gamma}, \tag{3.3}$$

where $\beta(\not\equiv 0)$ is a rational function. Now from (3.3) we get

$$\gamma = \log \frac{1}{\beta} \frac{H^{(k)} - \alpha_2}{H - \alpha_1} = \log \frac{1}{\beta} \frac{\frac{H^{(k)}}{H} - \frac{\alpha_2}{H}}{1 - \frac{\alpha_1}{H}},$$

where $\log g$ is the principle branch of the logarithm. By Lemma 2.2, we have

$$\frac{H^{(k)}(z_r)}{H(z_r)} = \Big(\frac{\nu(r,H)}{z_r}\Big)^k (1 + o(1)), \tag{3.4}$$

possibly outside a set of finite logarithmic measure E, where $|H(z_r)| = M(r,H)$. Observing that H is a transcendental function, so $\frac{\alpha_i}{H}|_{z_r} \to 0$ as $r \to \infty$ for $i = 1,2$. Since H is of finite order, so we have $\log \nu(r,H) = O(\log r)$. Therefore we get

$$|\gamma(z_r)| = \Big|\log \frac{1}{\beta} \frac{\frac{H^{(k)}}{H} - \frac{\alpha_2}{H}}{1 - \frac{\alpha_1}{H}}\Big| = O(\log r),$$

for $|z_r| = r \notin E$. This shows that γ is a constant. Without loss of generality we assume that

$$H^{(k)} - \alpha_2 \equiv \beta(H - \alpha_1), \text{ i.e., } H^{(k)} \equiv \beta H + \alpha_2 - \alpha_1\beta. \tag{3.5}$$

We now consider following two sub-cases.

Sub-case 1.1.1.1. Suppose H has infinitely many zeros.

Let $\{c_n\}_{n=1}^{\infty}$ be the zeros of H and not the zeros of α_1 and α_2. Putting c_n into (3.5) yields $\beta(c_n) = \frac{\alpha_2(c_n)}{\alpha_1(c_n)}$. In this case we must have $\beta \equiv \frac{\alpha_2}{\alpha_1}$, which implies that $F \equiv G$. Therefore we arrive at a contradiction.

Sub-case 1.1.1.2. Suppose H has finitely many zeros.

In this case $H = a_n f^n$ and $\beta \not\equiv \frac{\alpha_2}{\alpha_1}$. Since H is of finite order, we can take $H(z) = P_3(z)e^{P_4(z)}$, where P_3 is a non-zero polynomial and $P_4(z)$ is a non-constant polynomial. Then $H^{(k)}(z) = \Big[P_3(z)P_4^{(k)}(z) + P_5(z)\Big]e^{P_4(z)}$, where $P_5 = P_4^{(k-1)}P_3' + \overline{P}(P_3'', P_4')$ and $\overline{P}(P_3'', P_4')$ is differential polynomial in P_3'' and P_4'. Now from (3.5) we have

$$\Big[P_3(z)P_4^{(k)}(z) + P_5(z)\Big]e^{P_4(z)} \equiv \beta(z)P_3(z)e^{P_4(z)} + \alpha_2(z) - \beta(z)\alpha_1(z).$$

Consequently $P_3P_4^{(k)} + P_5 \equiv \beta P_3$ and $\alpha_2 - \beta\alpha_1 \equiv 0$. Thus we have $\beta \equiv \frac{\alpha_2}{\alpha_1}$, which is a contradiction.

Sub-case 1.1.2. $\Phi \equiv 0$. Since $H(z)$ is a transcendental entire function, it follows that $F' \not\equiv 0$. Therefore $F \equiv G$, i.e.,

$$(P(f))^{(k)} \equiv \frac{\alpha_2}{\alpha_1} P(f).$$

Further if $P_1 \equiv P_2$, then

$$P(f) \equiv \big(P(f)\big)^{(k)}. \tag{3.6}$$

Suppose that the roots of $P(z) = 0$ are $d_1, d_2, \ldots, d_m$ of multiplicities $l_1, l_2, \ldots, l_m$. Then $l_1 + l_2 + \ldots + l_m = n$. Since zeros of $P(f)$ are of multiplicities at least $k+1$, from (3.6) we have $P(f) \neq 0$ and so

$$P(f) = a_n(f - d_1)^{l_1}(f - d_2)^{l_2} \ldots (f - d_m)^{l_m} \neq 0. \tag{3.7}$$

Since f is a transcendental entire function, it has at most one Picard exceptional value and so from (3.7) we have $d_1 = d_2 = \ldots = d_m$. Then there exists a complex constant d satisfying $P(f) = a_n(f-d)^n$. Therefore from (3.6) we have

$$(f - d)^n = \Big(\big(f - d\big)^n\Big)^{(k)}.$$

By the given condition, it follows that $f - d \neq 0$ and so f assumes the form $f(z) = d + ce^{\frac{\lambda}{n}z}$, where c is a non-zero constant and $\lambda^k = 1$.

Sub-case 1.2. Suppose Q is non-constant. Let $\mu_1 = 2\max\{\deg(Q), 1 + \deg(P_2) - \deg(P_1)\} \geq 2$ and $\mu_2 = \frac{\mu_1 - 2}{2}$.

Since $\mu_1 < \rho(f)$, we have $0 \leq \mu_2 < \frac{\rho(f)-2}{2}$. Let $0 < \varepsilon < \frac{\rho(f)-\mu_1}{2}$. Then $0 \leq \mu_2 < \mu_2 + \varepsilon < \frac{\rho(f)-2}{2}$. Let $\mu = \mu_2 + \varepsilon$. Now by Lemma 2.4, for $0 < \mu < \frac{\rho(f)-2}{2}$ there exists a sequence $\{w_n\}_n$ such that $w_n \to \infty$ $(n \to \infty)$ and

$$\lim_{n\to\infty} \frac{F^{\#}(w_n)}{|w_n|^{\mu}} = +\infty. \tag{3.8}$$

Since P_1 is a polynomial, for all $z \in \mathbb{C}$ satisfying $|z| \geq r_1$, we have

$$0 \leftarrow \left|\frac{P_1'(z)}{P_1(z)}\right| \leq \frac{M_1}{|z|} < 1, \quad P_1(z) \neq 0. \tag{3.9}$$

Let $r > r_1$ and $D = \{z : |z| \geq r\}$. Then F is analytic in D. Since $w_n \to \infty$ as $n \to \infty$, without loss of generality we may assume that $|w_n| \geq r + 1$ for all n. Let $D_1 = \{z : |z| < 1\}$ and

$$F_n(z) = F(w_n + z) = \frac{H(w_n + z)}{\alpha_1(w_n + z)}.$$

Since $|w_n + z| \geq |w_n| - |z|$, it follows that $w_n + z \in D$ for all $z \in D_1$. Also since $F(z)$ is analytic in D, it follows that $F_n(z)$ is analytic in D_1 for all n. Thus we have structured a family $(F_n)_n$ of holomorphic functions. Note that $F_n^{\#}(0) = F^{\#}(w_n) \to \infty$ as $n \to \infty$. Now it follows from Marty's criterion that $(F_n)_n$ is not normal at $z = 0$. Let $a_n = 0$, for all n and $a = 0$. Then $a_n \to a$ and $|a| < 1$. Also $F_n^{\#}(a_n) = F_n^{\#}(0) = F^{\#}(w_n) \to \infty$ as $n \to \infty$. Now we apply Lemma 2.3. Choosing an appropriate subsequence of $(F_n)_n$, if necessary, we may assume that there exist sequences $(z_n)_n$ and $(\rho_n)_n$ such that $|z_n| < r < 1$, $z_n \to 0$, $\rho_n \to 0$ and that the sequence $(g_n)_n$ defined by

$$g_n(\zeta) = F_n(z_n + \rho_n\zeta) = \frac{H(w_n + z_n + \rho_n\zeta)}{\alpha_1(w_n + z_n + \rho_n\zeta)} \to g(\zeta) \tag{3.10}$$

converges locally and uniformly in $\mathbb{C}$, where $g(\zeta)$ is a non-constant entire function, whose zeros are of multiplicities at least $k + 1$. Also

$$\rho_n \leq \frac{M}{F_n^{\#}(a_n)} = \frac{M}{F^{\#}(w_n)} \tag{3.11}$$

for a positive number M. Now from (3.8) and (3.11) we deduce that

$$\rho_n \leq \frac{M}{F^{\#}(w_n)} \leq M_1|w_n|^{-\mu} \tag{3.12}$$

for sufficiently large values of n, where M_1 is a positive constant. We now prove that

$$\rho_n^k \frac{H^{(k)}(w_n + z_n + \rho_n\zeta)}{\alpha_1(w_n + z_n + \rho_n\zeta)} \to g^{(k)}(\zeta). \tag{3.13}$$

From (3.10) we see that

$$\begin{aligned} \rho_n \frac{H'(w_n + z_n + \rho_n\zeta)}{\alpha_1(w_n + z_n + \rho_n\zeta)} &= g_n'(\zeta) + \rho_n \frac{\alpha_1'(w_n + z_n + \rho_n\zeta)}{\alpha_1^2(w_n + z_n + \rho_n\zeta)} H(w_n + z_n + \rho_n\zeta) \\ &= g_n'(\zeta) + \rho_n \frac{\alpha_1'(w_n + z_n + \rho_n\zeta)}{\alpha_1(w_n + z_n + \rho_n\zeta)} g_n(\zeta). \end{aligned} \tag{3.14}$$

Also we see that

$$\frac{\alpha_1'(w_n + z_n + \rho_n\zeta)}{\alpha_1(w_n + z_n + \rho_n\zeta)} = \frac{P_1'(w_n + z_n + \rho_n\zeta)}{P_1(w_n + z_n + \rho_n\zeta)} + Q'(w_n + z_n + \rho_n\zeta). \tag{3.15}$$

Observe that $\frac{P_1'(w_n+z_n+\rho_n\zeta)}{P_1(w_n+z_n+\rho_n\zeta)} \to 0$ as $n \to \infty$. Let $s = \deg(Q')$. Since $2\deg(Q) \leq \mu_1$, it follows that $0 \leq s \leq \mu_2 < \mu$. Therefore from (3.12) we have

$$\lim_{n\to\infty} \rho_n|w_n|^s \leq \lim_{n\to\infty} M_1|w_n|^{s-\mu} = 0. \tag{3.16}$$

Note that $|Q'(w_n + z_n + \rho_n\zeta)| = O(|w_n|^s)$ and so from (3.16) we have

$$\rho_n|Q'(w_n + z_n + \rho_n\zeta)| = O(\rho_n|w_n|^s) \to 0 \ \ (as \ \ n \to \infty). \tag{3.17}$$

From (3.15) and (3.17) we have

$$\rho_n \frac{\alpha_1'(w_n + z_n + \rho_n\zeta)}{\alpha_1(w_n + z_n + \rho_n\zeta)} \to 0 \ \ (as \ \ n \to \infty). \tag{3.18}$$

Now from (3.10), (3.14) and (3.18) we observe that

$$\rho_n \frac{H'(w_n + z_n + \rho_n\zeta)}{\alpha_1(w_n + z_n + \rho_n\zeta)} \to g'(\zeta).$$

Suppose

$$\rho_n^l \frac{H^{(l)}(w_n + z_n + \rho_n\zeta)}{\alpha_1(w_n + z_n + \rho_n\zeta)} \to g^{(l)}(\zeta).$$

Let

$$G_n(\zeta) = \rho_n^l \frac{H^{(l)}(w_n + z_n + \rho_n\zeta)}{\alpha_1(w_n + z_n + \rho_n\zeta)}.$$

Then $G_n(\zeta) \to g^{(l)}(\zeta)$. Note that

$$\begin{aligned} & \rho_n^{l+1} \frac{H^{(l+1)}(w_n + z_n + \rho_n\zeta)}{\alpha_1(w_n + z_n + \rho_n\zeta)} \\ = \quad & G_n'(\zeta) + \rho_n^{l+1} \frac{\alpha_1'(w_n + z_n + \rho_n\zeta)}{\alpha_1^2(w_n + z_n + \rho_n\zeta)} H^{(l)}(w_n + z_n + \rho_n\zeta) \\ = \quad & G_n'(\zeta) + \rho_n \frac{\alpha_1'(w_n + z_n + \rho_n\zeta)}{\alpha_1(w_n + z_n + \rho_n\zeta)} G_n(\zeta). \end{aligned} \tag{3.19}$$

Now from (3.18) and (3.19) we see that

$$\rho_n^{l+1} \frac{H^{(l+1)}(w_n + z_n + \rho_n\zeta)}{\alpha_1(w_n + z_n + \rho_n\zeta)} \to G_n'(\zeta),$$

i.e.,

$$\rho_n^{l+1} \frac{H^{(l+1)}(w_n + z_n + \rho_n\zeta)}{\alpha_1(w_n + z_n + \rho_n\zeta)} \to g_n^{(l+1)}(\zeta).$$

Then by mathematical induction we get desired result (3.13).
Clearly $g^{(k)}(z) \not\equiv 0$, for otherwise $g(z)$ would be a polynomial of degree less than k and so $g(z)$ could not have zero of multiplicity at least $k + 1$.
Firstly we claim that $g = 1 \Rightarrow g^{(k)} = 0$. Suppose that $g(\eta_0) = 1$. Then by Hurwitz's Theorem there exists a sequence $(\eta_n)_n$, $\eta_n \to \eta_0$ such that (for sufficiently large n)

$$g_n(\eta_n) = \frac{H(w_n + z_n + \rho_n\eta_n)}{\alpha_1(w_n + z_n + \rho_n\eta_n)} = 1,$$

i.e., $H(w_n + z_n + \rho_n\eta_n) = \alpha_1(w_n + z_n + \rho_n\eta_n)$. By the assumption we have

$$H^{(k)}(w_n + z_n + \rho_n\eta_n) = \alpha_2(w_n + z_n + \rho_n\eta_n). \tag{3.20}$$

Note that

$$\left|\frac{\alpha_2(w_n + z_n + \rho_n\zeta)}{\alpha_1(w_n + z_n + \rho_n\zeta)}\right| = \left|\frac{P_2(w_n + z_n + \rho_n\zeta)}{P_1(w_n + z_n + \rho_n\zeta)}\right| = \begin{cases} O(1), & \text{if } \deg(P_2) \leq \deg(P_1) \\ O(|w_n|^t), & \text{if } \deg(P_2) > \deg(P_1), \end{cases} \tag{3.21}$$

where $t = \deg(P_2) - \deg(P_1) > 0$. By the given condition we see that $t \leq \mu_2$, i.e., $0 < t \leq \mu_2 < \mu$. Then from (3.12), we deduce that

$$\lim_{n\to\infty} \rho_n^k |w_n|^t \leq \lim_{n\to\infty} M_1^k |w_n|^{t-k\mu} = 0. \tag{3.22}$$

Since $\rho_n \to 0$ as $n \to \infty$, from (3.21) and (3.22) we have

$$\rho_n^k \left| \frac{\alpha_2(w_n + z_n + \rho_n\zeta)}{\alpha_1(w_n + z_n + \rho_n\zeta)} \right| \to 0 \ \ (as \ \ n \to \infty). \tag{3.23}$$

Now from (3.13), (3.20) and (3.23) we see that

$$g^{(k)}(\eta_0) = \lim_{n\to\infty} g^{(k)}(\eta_n) = \lim_{n\to\infty} \rho_n^k \frac{H^{(k)}(w_n + z_n + \rho_n\eta_n)}{\alpha_1(w_n + z_n + \rho_n\eta_n)} = \lim_{n\to\infty} \rho_n^k \frac{\alpha_2(w_n + z_n + \rho_n\eta_n)}{\alpha_1(w_n + z_n + \rho_n\eta_n)} = 0.$$

Thus $g(\eta) = 1 \Rightarrow g^{(k)}(\eta) = 0$. Finally we prove that $g^{(k)} = 0 \Rightarrow g = 1$. Now from (3.13) and (3.23) we see that

$$\rho_n^k \frac{H^{(k)}(w_n + z_n + \rho_n\zeta) - \alpha_2(w_n + z_n + \rho_n\zeta)}{\alpha_1(w_n + z_n + \rho_n\zeta)} \to g^{(k)}(\zeta). \tag{3.24}$$

Suppose that $g^{(k)}(\xi_0) = 0$. Then by (3.24) and Hurwitz's Theorem there exists a sequence $(\xi_n)_n$, $\xi_n \to \xi_0$ such that (for sufficiently large n) $H^{(k)}(w_n + z_n + \rho_n\xi_n) = \alpha_2(w_n + z_n + \rho_n\xi_n)$. By the given condition we have $H(w_n + z_n + \rho_n\xi_n) = \alpha_1(w_n + z_n + \rho_n\xi_n)$. Therefore from (3.10) we have

$$g(\xi_0) = \lim_{n\to\infty} \frac{H(w_n + z_n + \rho_n\xi_n)}{\alpha_1(w_n + z_n + \rho_n\xi_n)} = 1.$$

Thus $g^{(k)} = 0 \Rightarrow g = 1$. As a result we have (1) $g = 0 \Rightarrow g^{(k)} = 0$, (2) $g = 1 \Leftrightarrow g^{(k)} = 0$. From (1) and (2) one can easily deduce that $g \neq 0$. Also from (2) we see that zeros of $g - 1$ are of multiplicity at least $k + 1$. Now by the second fundamental theorem we have

$$\begin{aligned}
T(r, g) &\leq \overline{N}(r, 0; g) + \overline{N}(r, \infty; g) + \overline{N}(r, 1; g) + S(r, g) \\
&\leq \frac{1}{k+1} N(r, 1; g) + S(r, g) \\
&\leq \frac{1}{k+1} T(r, g) + S(r, g),
\end{aligned}$$

which is a contradiction.
Case 2. Suppose $\rho(f) = +\infty$. Then $\rho(H) = +\infty$ and since $\rho(\alpha_1) < +\infty$, $\rho(F) = +\infty$. Now by Lemma 2.5, there exist $\{w_n\}_n \to \infty (n \to \infty)$ such that for every $N > 0$, if n is sufficiently large

$$F^{\#}(w_n) > |w_n|^N. \tag{3.25}$$

From (3.11) and (3.25), we deduce that for every $N > 0$, if n is sufficiently large

$$\rho_n < M|w_n|^{-N}. \tag{3.26}$$

If we take $N > s$, then from (3.26) we deduce that $\lim_{n\to\infty} \rho_n |w_n|^s = 0$ and so (3.18) holds. Again if we take $N > t$, then from (3.26) we deduce that $\lim_{n\to\infty} \rho_n^k |w_n|^t = 0$ and so (3.23) holds. We omit the proof since the proof of Case 2 can be carried out in the line of proof of Sub-case 1.2. □

4. AN OPEN PROBLEM

Keeping other conditions intact can the sharing condition in Theorem 1.23 be relaxed to $(0, 0)$ so that the conclusion remains the same ?

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