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On the Number of Perfect Matchings of Generalized Theta Graphs and the Edge Cover Polynomials of Friendship Graphs

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Abstract. Let $k \geq 1$ and $q_1, \ldots, q_k$ be some positive integers. The generalized theta graph $\Theta_{q_1,\ldots,q_k}$ is the graph that is formed by taking a pair of vertices u and v, and joining them by k internally disjoint paths of lengths $q_1, \ldots, q_k$. Let $C\Theta C$ be the graph that obtained by attaching some cycles to at most two vertices with degree k of the generalized theta graph $\Theta_{q_1,\ldots,q_k}$. An edge covering of a graph is a set of edges such that every vertex of the graph is incident to at least one edge of the set. Let G be a simple graph with m edges. The edge cover polynomial of G is the polynomial $E(G, x) = \sum_{i=1}^{m} e(G, i)x^i$, where $e(G, i)$ is the number of edge coverings of G of size i. Let t be a positive integer and F_t be the friendship (or Dutch windmill) graph with $2t + 1$ vertices and $3t$ edges. In this paper we obtain the number of perfect matchings of $C\Theta C$ graphs and then study the edge cover polynomial of friendship graphs. We show that the friendship graphs are determined by their edge cover polynomials. We find that all non-zero roots of the edge cover polynomial of friendship graphs are simple. Finally we prove that the edge cover polynomials of friendship graphs are unimodal.

1. Introduction

Throughout this paper we will consider only simple graphs. In other words the graphs are finite and undirected, without loops and multiple edges. Let $G = (V(G), E(G))$ be a simple graph. The *order* and the *size* of G denote the number of vertices and the number of edges of G, respectively. The *complete graph*, the *cycle*, and the *path* of order n, are denoted by K_n, C_n and P_n, respectively. For every vertex $v \in V(G)$, the *degree* of v is the number of edges incident with v and is denoted by $deg_G(v)$. For simplicity we write $deg(v)$ instead of $deg_G(v)$. We denote the minimum degree of the vertices of G by $\delta(G)$. For two vertex-disjoint graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, the *disjoint union* of G_1 and G_2 is denoted by $G_1 + G_2$ is the graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$. The graph rG denotes the disjoint union of r copies of G. Let $S \subseteq V(G)$. By $G \setminus S$ we mean the graph that is obtained from G by removing the vertices of S. By $n_1, \ldots, n_k \geq t$ we mean that $n_1 \geq t, \ldots, n_k \geq t$.

A *matching* of G is a set of edges of G that no two of them have common vertex. If the size of a matching is r, then it is called an *r-matching*. A *perfect matching* of G is a matching with cardinality $\frac{n}{2}$, where n is the order of G. We denote the number of perfect matchings of G by $pm(G)$. For every graph G with no isolated vertex, an *edge covering* of G is a set of edges of G such that every vertex is incident with at least one edge of the set. In other words, an edge covering of a graph is a set of edges that together meet all vertices of the

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graph. A *minimum edge covering* is an edge covering of the smallest possible size. The *edge covering number* of G is the size of a minimum edge covering of G and denoted by $\rho(G)$. In this paper we let $\rho(G) = 0$, if G has some isolated vertices. For a detailed treatment of these parameters, the reader is referred to [14].

There are numerous polynomials associated with graphs. For example *independence polynomial* [6, 22, 23, 25, 27], *matching polynomial* [19, 21], *vertex-cover polynomial* [17], *edge elimination polynomial* [11], *domination polynomial* [2–4, 7, 9, 26], *chromatic polynomial* [13, 28], and *Tutte polynomial* [30].

By studying these polynomials one can obtain some properties of a graph. For instance the roots of these polynomials reflect some important information about the structure of graphs. Recently, authors have defined a new graph polynomial *edge cover polynomial* as following [5]. Let G be a graph of size m and with no isolated vertex. By $\mathcal{E}(G, i)$ we mean the family of all edge coverings of G with cardinality i. Let $e(G, i) = |\mathcal{E}(G, i)|$. The edge cover polynomial of G that is denoted by $E(G, x)$ is defined as

$$E(G, x) = \sum_{i=\rho(G)}^{m} e(G, i)x^i.$$

In fact the coefficient of x^i is the number of edge coverings of G with cardinality i. If G has some isolated vertices, then we let $E(G, x) = 0$. In addition we let $E(G, x) = 1$, when both order and size of G are zero. For example $E(K_4, x) = x^6 + 6x^5 + 15x^4 + 16x^3 + 3x^2$. See [5, 16] for more details. The edge cover polynomial has some interesting properties. One of the most nice results on the edge cover polynomial is that all roots of this polynomial are in the following set [16]

$$\left\{ z \in \mathbb{C} \; : \; |z| < \frac{(2 + \sqrt{3})^2}{1 + \sqrt{3}} \simeq 5.099 \right\}.$$

One of the most important problems on the graph polynomials is the following:

Problem. Which graphs are determined by their related graph polynomials?

Another interesting property on graph polynomials is unimodality. A polynomial $f(x) = \sum_{i=0}^{n} a_i x^i$ with real coefficients (or a sequence $(a_0, \ldots, a_n)$) is called *unimodal* if there is some $k \in \{0, \ldots, n\}$, such that

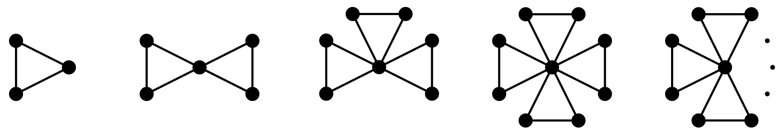
$$a_0 \leq \cdots \leq a_{k-1} \leq a_k \geq a_{k+1} \geq \cdots \geq a_n.$$

Also, $f(x)$ (or a sequence $(a_0, \ldots, a_n)$) is called *logarithmically concave* (or simply, *log-concave*), if for every $1 \leq i \leq n - 1$, $a_i^2 \geq a_{i-1}a_{i+1}$. $f(x)$ (or a sequence $(a_0, \ldots, a_n)$) is called *symmetric* (or *palindromic*) if $a_i = a_{n-i}$ for $i = 0, 1, \ldots, n$. It is known that any log-concave polynomial with positive coefficients (or a sequence of positive numbers) is also unimodal.

The unimodality problems of graph polynomials have always been of great interest to researchers in graph theory. For example in [28] it is conjectured that the chromatic polynomial of a graph is unimodal. Also there is a famous conjecture due to Paul Erdös et al. on the unimodality of the independence polynomial of trees [10]. The unimodality of graph polynomials, in particular the unimodality of independence polynomial, have been extensively studied, see [8, 10, 12, 24, 29, 32].

The friendship (or Dutch windmill or t-fan) graph F_t is a graph that can be constructed by coalescence t copies of the cycle graph C_3 of length 3 with a common vertex. By construction, the friendship graph F_t is isomorphic to the windmill graph $Wd(3, t)$, see [20]. The Friendship Theorem of Paul Erdös, Alfred Rényi and Vera T. Sós [18], states that graphs with the property that every two vertices have exactly one neighbor in common are exactly the friendship graphs. The Figure 1 shows some examples of friendship graphs.

There are many papers on the properties of friendship graphs. In [31] it is proved that the friendship graphs can be determined by their signless Laplacian spectrum. In [1] and [15] the authors have shown that the friendship graphs can be determined by their eigenvalue spectrum. Recently in [7] the authors have studied the domination polynomials of friendship graphs. These motivate us to study the edge cover polynomial of friendship graphs. In [5] it has been proved that the complete graphs, the cycles and the complete bipartite graphs can be determined by their edge cover polynomials. In this paper we show that the friendship graphs are determined by their edge cover polynomials. In addition we prove that the edge cover polynomials of friendship graphs are unimodal. About the structure of this paper, in the next section we define a new family of graphs that play an important role in the paper. We find the number of perfect matchings of these graphs. In the last section we investigate the edge cover polynomials of friendship graphs. In particular we show that if G is a simple graph such that $E(G, x) = E(F_t, x)$, then $G \cong F_t$. Finally we prove that the edge cover polynomials of friendship graphs are unimodal.

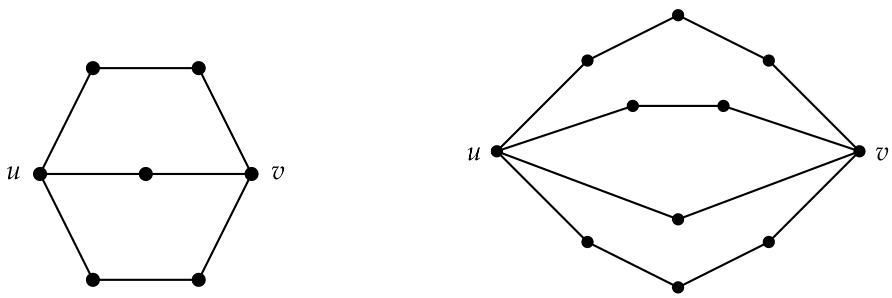


F_1 F_2 F_3 F_4 F_t

Figure 1: The friendship graphs F_1, F_2, F_3, F_4 and F_t

2. The number of perfect matchings of $C\theta C$ graphs

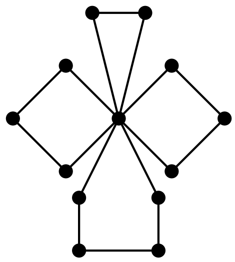
In this section first we introduce some new families of graphs that appear in the proof of the main result of the paper. Then we find the number of perfect matchings of them. Let $k \geq 1$ and $q_1, \ldots, q_k$ be some positive integers. The *generalized theta graph* $\Theta_{q_1,\ldots,q_k}$ is the graph that is formed by taking a pair of vertices u and v (called the end vertices of $\Theta_{q_1,\ldots,q_k}$) and joining them by k internally disjoint paths of lengths $q_1, \ldots, q_k$. Hence u and v have degree k. The generalized theta graph with three paths is called *theta graph*. In other words, by theta graph we mean Θ_{q_1,q_2,q_3}, for some positive integers q_1, q_2 and q_3. We note that $\Theta_{q_1} \cong P_{q_1+1}$ and $\Theta_{q_1,q_2} \cong C_{q_1+q_2}$, if $q_1 + q_2 \geq 3$. In Figure 2 the theta graph $\Theta_{2,3,3}$ and the generalized theta graph $\Theta_{2,3,4,4}$ have been shown. Let $q_1, \ldots, q_k \geq 3$ be some positive integers. By $C_{q_1,\ldots,q_k}$ we mean the graph that obtained by joining the cycles $C_{q_1}, \ldots, C_{q_k}$ with a common vertex (called the central vertex of $C_{q_1,\ldots,q_k}$). Note that the central vertex of $C_{q_1}, \ldots, C_{q_k}$ has degree $2k$. If $q_1 = \cdots = q_k = 3$, then $C_{3,\ldots,3}$ is called the friendship graph F_k. In other words F_k obtained by joining k copies of the cycle graph C_3 with a common vertex. In Figure 3 the graph $C_{3,4,4,5}$ has been shown.



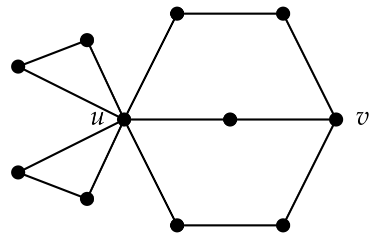
$\Theta_{2,3,3}$ $\Theta_{2,3,4,4}$

Figure 2: The theta graph $\Theta_{2,3,3}$ and the generalized theta graph $\Theta_{2,3,4,4}$.

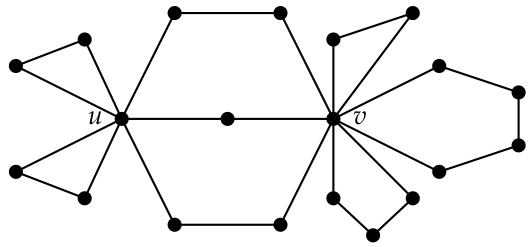
Let $k, r, s \geq 1$ be some integers. Suppose that $q_1, \ldots, q_k \geq 1$, $m_1, \ldots, m_r \geq 3$ and $n_1, \ldots, n_s \geq 3$ are some integers. Consider the generalized theta graph $\Theta_{q_1,\ldots,q_k}$ such that u and v are the end vertices of $\Theta_{q_1,\ldots,q_k}$. Let u' and v' be the central vertices of $C_{m_1,\ldots,m_r}$ and $C_{n_1,\ldots,n_s}$, respectively. By $C\Theta(m_1, \ldots, m_r; q_1, \ldots, q_k)$ we mean the graph that obtained by joining $C_{m_1,\ldots,m_r}$ and $\Theta_{q_1,\ldots,q_k}$ at the vertices u and u'. By $C\Theta C(m_1, \ldots, m_r; q_1, \ldots, q_k; n_1, \ldots, n_s)$ we mean the graph that constructed by joining $C\Theta(m_1, \ldots, m_r; q_1, \ldots, q_k)$ and $C_{n_1,\ldots,n_s}$ at the vertices v and v'.



$C_{3,4,4,5}$



$C\Theta(3,3;2,3,3)$



$C\Theta C(3,3;2,3,3;3,4,5)$

Figure 3: The graphs $C_{3,4,4,5}$, $C\Theta(3,3;2,3,3)$ and $C\Theta C(3,3;2,2,3;3,4,5)$.

Similarly one can define $\Theta C(q_1,\ldots,q_k;n_1,\ldots,n_s)$. In fact $\Theta C(q_1,\ldots,q_k;n_1,\ldots,n_s) \cong C\Theta(n_1,\ldots,n_s;q_1,\ldots,q_k)$. We say G is a $C\Theta C$ graph if $G \cong C_{m_1,\ldots,m_r}$ or $G \cong \Theta_{q_1,\ldots,q_k}$ or $G \cong C\Theta(m_1,\ldots,m_r;q_1,\ldots,q_k)$ or $G \cong C\Theta C(m_1,\ldots,m_r;q_1,\ldots,q_k;n_1,\ldots,n_s)$ for some positive integers $q_1,\ldots,q_k \geq 1$, $m_1,\ldots,m_r \geq 3$ and $n_1,\ldots,n_s \geq 3$. In Figure 3 the graphs $C\Theta(3,3;2,3,3)$ and $C\Theta C(3,3;2,3,3;3,4,5)$ have been shown.

Now we find the number of perfect matchings of $C\Theta C$ graphs. Let M be a matching of G. Then the two ends of each edge of M are said to be matched under M, and each vertex incident with an edge of M is said to be covered by M. We recall that $pm(G)$ is the number of perfect matchings of G.

Theorem 1. *Let k, r and s be some positive integers. Let $m_1,\ldots,m_r \geq 3$, $q_1,\ldots,q_k \geq 1$ and $n_1,\ldots,n_s \geq 3$ be some integers. Suppose that λ is the number of even numbers of $m_1,\ldots,m_r$, and ξ is the number of even numbers of $q_1,\ldots,q_k$ and γ is the number of even numbers of $n_1,\ldots,n_s$. Let $G = C\Theta C(m_1,\ldots,m_r;q_1,\ldots,q_k;n_1,\ldots,n_s)$. Then the following hold:*

1. *$pm(G) = 2$ if and only if $(\lambda,\xi,\gamma) \in \{(0,2,0),(0,1,1),(1,1,0)\}$ or $k = 2$ and $(\lambda,\xi,\gamma) = (0,0,0)$.*
2. *$pm(G) = 4$ if and only if $(\lambda,\xi,\gamma) = (1,0,1)$ or $k = 4$ and $(\lambda,\xi,\gamma) = (0,0,0)$.*
3. *Assume that $k \notin \{2,4\}$. Then $pm(G) = k$ if and only if $(\lambda,\xi,\gamma) = (0,0,0)$.*
4. *$pm(G) = 0$ if and only if $(\lambda,\xi,\gamma) \notin \{(0,0,0),(1,1,0),(1,0,1),(0,1,1),(0,2,0)\}$.*

Proof. Let $H = \Theta_{q_1,\ldots,q_k}$. Assume that u and v are the end vertices of H. Let u' and v' be the central vertices of $C_{m_1,\ldots,m_r}$ and $C_{n_1,\ldots,n_s}$, respectively. Thus G is obtained by joining $C_{m_1,\ldots,m_r}$ to H by identifying u and u' and joining $C_{n_1,\ldots,n_s}$ to H by identifying v and v'. Suppose that M is a perfect matching of G. We consider the following cases:

A. Assume that u is covered by M with one of the paths of H between u and v with odd length. Thus assume that P is a path of H with odd length between u and v, and $e = ua$ is an edge of P such that $e \in M$ and $a \in V(H)$. Since all vertices of G except u and v have degree two, the vertex v is also covered by M with an edge of P. This shows that all paths of H between u and v have odd length. More precisely all numbers $q_1,\ldots,q_k$ are odd. In addition, one can see that all the cycles of $C_{m_1,\ldots,m_r}$ and $C_{n_1,\ldots,n_s}$ have odd length. In other words all numbers $m_1, \ldots, m_r$ and $n_1, \ldots, n_s$ are odd. On the other hand if all numbers $m_1, \ldots, m_r$, $q_1, \ldots, q_k$ and $n_1, \ldots, n_s$ are odd, then the number of perfect matchings of G is k. In other words if $(\lambda, \xi, \gamma) = (0, 0, 0)$, then $pm(G) = k$.

B. Assume that u is covered by M with one of the paths of H between u and v with even length. Thus suppose that P' is a path of H with even length between u and v, and $e' = ua'$ is an edge of P' such that $e' \in M$ and $a' \in V(H)$. Thus $\xi \geq 1$. If $\xi \geq 2$, then there exists another path of even length between u and v in H, say P''. It is easy to see that v is covered by M with an edge of P''. It is not hard to see that there is no path of even length except P' and P'' between u and v (in H). In other words $\xi = 2$. In addition one can see that all cycles of $C_{m_1,\ldots,m_r}$ and $C_{n_1,\ldots,n_s}$ have odd length. Therefore $(\lambda, \gamma) = (0, 0)$. On the other hand if $(\lambda, \xi, \gamma) = (0, 2, 0)$, then $pm(G) = 2$ (since the two even paths of $\Theta_{q_1,\ldots,q_k}$ construct an even cycle, and any even cycle has just two perfect matchings, by considering these perfect matchings, the other edges of perfect matchings of G are uniquely determined).

Now suppose that P' is the unique path of even length between u and v (in H), that is $\xi = 1$. Thus v is covered by M with one of the cycles of $C_{n_1,\ldots,n_s}$, say C_{n_i}, where $1 \leq i \leq s$. Thus n_i is even. One can see that the other cycles of $C_{n_1,\ldots,n_s}$ have odd length. This means that all numbers $n_1, \ldots, n_s$ except n_i are odd. Thus $\gamma = 1$. One can see that all cycles of $C_{m_1,\ldots,m_r}$ have odd length. Thus $m_1, \ldots, m_r$ are odd. So $\lambda = 0$. On the other hand if $(\lambda, \xi, \gamma) = (0, 1, 1)$, then $pm(G) = 2$ (since any even cycle has only two perfect matchings, by considering the perfect matchings of the even cycle of $C_{n_1,\ldots,n_s}$ and the even path of $\Theta_{q_1,\ldots,q_k}$ the other edges of perfect matchings of G are uniquely determined).

C. Assume that u is covered by M with one of the cycles of $C_{m_1,\ldots,m_r}$, say C_{m_h}, where $1 \leq h \leq r$. Thus m_h is even. In addition, the length of the other cycles of $C_{m_1,\ldots,m_r}$ is odd. Hence $\lambda = 1$. Suppose that v is covered by M with one of the cycles of $C_{n_1,\ldots,n_s}$, say C_{n_i} (the case that v is covered by M with one of the paths of H between u and v can be considered similar to the parts A and B). Thus n_i is also even and the other cycles of $C_{n_1,\ldots,n_s}$ have odd length. Thus $\gamma = 1$. One can see that all paths of H between u and v have odd length. In other words $q_1, \ldots, q_k$ are odd and so $\xi = 0$. On the other hand if $(\lambda, \xi, \gamma) = (1, 0, 1)$, then $pm(G) = 4$ (by considering the perfect matchings of the even cycles of $C_{m_1,\ldots,m_r}$ and $C_{n_1,\ldots,n_s}$ the other edges of perfect matchings of G are uniquely determined).

Now we can enumerate the perfect matchings of G. For every perfect matching of G, say M, one of the following holds:

1. The vertices u and v are covered by M with some paths of $\Theta_{q_1,\ldots,q_k}$ between u and v. If u is covered by a path of odd length, say P, then by Case A, v is also covered by P (it means that v is covered by a path of odd length). Thus $m_1, \ldots, m_r$, $q_1, \ldots, q_k$, and $n_1, \ldots, n_s$ are odd. In other words $(\lambda, \xi, \gamma) = (0, 0, 0)$. Therefore by Case A, G has exactly k perfect matchings. Now assume that u is covered by a path of even length, say P. Thus by Case A, v is covered by another path of even length, say P'. By Case B, $P \neq P'$. Also by Case B we obtain that $(\lambda, \xi, \gamma) = (0, 2, 0)$. Thus $pm(G) = 2$.
2. u is covered by M with a path of $\Theta_{q_1,\ldots,q_k}$ between u and v, say P, and v is covered by M with a cycle of $C_{n_1,\ldots,n_s}$, say C_{n_i}. By Case A, the length of P is even. On the other hand n_i is even. Thus by Case B, the path P is the unique path of $\Theta_{q_1,\ldots,q_k}$ with even length. Therefore by Case B, $m_1, \ldots, m_r$ are odd and only one of the numbers $q_1, \ldots, q_k$ is even. In addition only one of the numbers $n_1, \ldots, n_s$ is even. Thus $(\lambda, \xi, \gamma) = (0, 1, 1)$. This shows that $pm(G) = 2$.
3. u is covered by M with a cycle of $C_{m_1,\ldots,m_r}$, say C_{m_j}, and v is covered by M with a path of $\Theta_{q_1,\ldots,q_k}$ between u and v, say P. Similar to the previous part, we obtain that $n_1, \ldots, n_s$ are odd and exactly one of the numbers $m_1, \ldots, m_r$ is even. In addition exactly one of the numbers $q_1, \ldots, q_k$ is even. Hence $(\lambda, \xi, \gamma) = (1, 1, 0)$. Thus $pm(G) = 2$.
4. u is covered by M with a cycle of $C_{m_1,\ldots,m_r}$ and v is covered by M with a cycle of $C_{n_1,\ldots,n_s}$. Thus by Case C, $q_1, \ldots, q_k$ are odd, that is $\xi = 0$. In addition $(\lambda, \gamma) = (1, 1)$. By Case C, $pm(G) = 4$.

The proof is complete. □

Similar to Theorem 1 one can obtain the number of perfect matchings of other $C\Theta C$ graphs.

Theorem 2. *Let k and r be some positive integers. Let $m_1, \ldots, m_r \geq 3$ and $q_1, \ldots, q_k \geq 1$ be some integers. Assume that λ is the number of even numbers of $m_1, \ldots, m_r$ and ξ is the number of even numbers of $q_1, \ldots, q_k$. Let $G = C\Theta(m_1, \ldots, m_r; q_1, \ldots, q_k)$. Then the following hold:*

1. *$pm(G) = 2$ if and only if $(\lambda, \xi) \in \Big\{(0, 2), (1, 1)\Big\}$ or $k = 2$ and $(\lambda, \xi) = (0, 0)$.*
2. *$pm(G) = k$ if and only if $(\lambda, \xi) = (0, 0)$ or $k = 2$ and $(\lambda, \xi) \in \Big\{(0, 2), (1, 1)\Big\}$.*
3. *$pm(G) = 0$ if and only if $(\lambda, \xi) \notin \Big\{(0, 0), (1, 1), (0, 2)\Big\}$.*

Theorem 3. *Let $r \geq 1$ and $m_1, \ldots, m_r \geq 3$ be some integers. Suppose that λ is the number of even numbers of $m_1, \ldots, m_r$. Let $G = C_{m_1,\ldots,m_r}$. Then the following hold:*

1. *$pm(G) = 2$ if and only if $\lambda = 1$.*
2. *$pm(G) = 0$ if and only if $\lambda \neq 1$.*

Theorem 4. *Let k and $q_1, \ldots, q_k$ be some positive integers. Assume that ξ is the number of even numbers of $q_1, \ldots, q_k$. Let $G = \Theta_{q_1,\ldots,q_k}$. Then the following hold:*

1. *$pm(G) = 2$ if and only if $\xi = 2$ or $k = 2$ and $\xi = 0$.*
2. *$pm(G) = k$ if and only if $\xi = 0$ or $k = 2$ and $\xi = 2$.*
3. *$pm(G) = 0$ if and only if $\xi \neq 0, 2$.*

3. Edge cover polynomial of friendship graphs

In this section we study the edge cover polynomial of friendship graphs. We show that the friendship graphs are uniquely determined by their edge cover polynomials. Also we show that the edge cover polynomials of friendship graphs are unimodal. We need the following results. For more details we refer to Lemma 2 and Corollaries 1 and 2 of [5]. For a graph G, by $a_k(G)$ we mean the number of vertices of G with degree k.

Remark 1. [5] Let G be a graph of order n with no isolated vertex. Then $n \leq 2\rho(G)$.

Lemma 1. [5] *Let G and H be two graphs with no isolated vertex. Let $E(G, x) = E(H, x)$. Then the following hold:*

i) The number of edges of G and H are the same.
ii) The minimum degree of G and H are the same.
iii) If $\delta(G) \geq 2$, then for $k = 1, \ldots, 2\delta(G) - 2$, $a_k(G) = a_k(H)$.

Now we prove the main result of the paper.

Theorem 5. *Let $t \geq 1$ be an integer and F_t be the friendship graph. Let G be a simple graph such that $E(G, x) = E(F_t, x)$. Then $G \cong F_t$.*

Proof. Since $\delta(F_t) = 2 \neq 0$, $E(F_t, x) \neq 0$. Thus $E(G, x) \neq 0$. This shows that $\delta(G) \neq 0$. Therefore G and F_t have no isolated vertex. By the first and the second parts of Lemma 1 we obtain that $\delta(G) = 2$ and the number of edges of G is $3t$. Using the third part of Lemma 1 we obtain that $a_2(G) = a_2(F_t)$.

If $t = 1$, then $F_1 = C_3$. Thus $\delta(G) = 2$, $a_2(G) = 3$ and the number of edges of G is three. So $G \cong C_3$.

Now assume that $t \geq 2$. Thus $a_2(G) = a_2(F_t) = 2t$. Since $\delta(G) = 2$ we have

$$2(3t) = \sum_{v \in V(G)} deg(v) = 2(2t) + \sum_{v \in V(G),\ deg(v) \geq 3} deg(v).$$

This shows that

$$\textstyle\sum_{v \in V(G),\ deg(v) \geq 3} deg(v) = 2t. \tag{1}$$

Let n be the order of G. The last equality shows that $n \geq a_2(G) + 1$. Thus $n \geq 2t + 1$. On the other hand $E(G, x) = E(F_t, x)$ implies that $\rho(G) = \rho(F_t)$. By Remark 1 one has $\rho(F_t) \geq \frac{2t+1}{2}$. By this fact it is easy to see that $\rho(F_t) = t + 1$. Hence by Remark 1, $t + 1 = \rho(F_t) = \rho(G) \geq \frac{n}{2}$. Thus $n \leq 2t + 2$. Since $n \geq 2t + 1$, $n = 2t + 1$ or $n = 2t + 2$. Therefore we consider the following cases:

(i) Assume that $n = 2t + 1$. Since $a_2(G) = 2t$, by Equation (1) we obtain that G has only one vertex of degree $2t$. This shows that $G \cong F_t$.

(ii) Suppose that $n = 2t + 2$. The equality $\rho(G) = \rho(F_t) = t + 1$ implies that $\rho(G) = \frac{|V(G)|}{2}$. Thus M is an edge covering of G with cardinality $t+1$ if and only if M is a perfect matching of G. Thus $pm(G) = e(G, t+1)$. On the other hand $E(G, x) = E(F_t, x)$ implies that $e(G, t + 1) = e(F_t, t + 1) = 3t$. Thus $pm(G) = 3t$. Since $a_2(G) = 2t$ and the order of G is $2t + 2$, by Equation (1) we obtain that G has exactly two vertices of degree greater than two, say u and v. Thus $deg(u), deg(v) \geq 3$ and $deg(u) + deg(v) = 2t$. Since the maximum degree of the vertices of $G \setminus \{u, v\}$ is at most two, every connected component of $G \setminus \{u, v\}$ is a path or a cycle or a single vertex. This shows that G is isomorphic with one of the following graphs:

1) $G \cong C_{m_1,\ldots,m_r} + C_{n_1,\ldots,n_s} + C_{t_1} + \cdots + C_{t_h}$, where $r \geq 2$, $s \geq 2$, $h \geq 0$ and $m_1, \ldots, m_r \geq 3$, $n_1, \ldots, n_s \geq 3$ and $t_1, \ldots, t_h \geq 3$ are some integers. If $h = 0$, then $G \cong C_{m_1,\ldots,m_r} + C_{n_1,\ldots,n_s}$. Thus by Theorem 3, $pm(G) = pm(C_{m_1,\ldots,m_r})\, pm(C_{n_1,\ldots,n_s}) \leq 4$. On the other hand $pm(G) = 3t$. Thus $3t \leq 4$, a contradiction (since $t \geq 2$). Thus $h \geq 1$. Since $pm(G) = 3t \neq 0$ all numbers $t_1, \ldots, t_h$ are even. Thus $pm(C_{t_i}) = 2$, for $i = 1, \ldots, h$. Hence $pm(G) = 2^h pm(C_{m_1,\ldots,m_r}) pm(C_{n_1,\ldots,n_s})$. By Theorem 3, $pm(C_{m_1,\ldots,m_r}), pm(C_{n_1,\ldots,n_s}) \in \{0, 2\}$. Since $pm(G) \neq 0$, the latter equality shows that $pm(C_{m_1,\ldots,m_r}) = pm(C_{n_1,\ldots,n_s}) = 2$. Therefore
$$3t = pm(G) = 2^h\ pm(C_{m_1,\ldots,m_r})\ pm(C_{n_1,\ldots,n_s}) = 2^{h+2}.$$
This is a contradiction, since 2^{h+2} is not divided by 3.

2) $G \cong C\Theta C(m_1, \ldots, m_r; q_1, \ldots, q_k; n_1, \ldots, n_s) + C_{t_1} + \cdots + C_{t_h}$, where $r \geq 1$, $k \geq 1$, $s \geq 1$, $h \geq 0$ and $m_1, \ldots, m_r \geq 3$, $n_1, \ldots, n_s \geq 3$, $q_1, \ldots, q_k \geq 1$ and $t_1, \ldots, t_h \geq 3$ are some integers. If $h = 0$, then $G \cong C\Theta C(m_1, \ldots, m_r; q_1, \ldots, q_k; n_1, \ldots, n_s)$. Hence by Theorem 1, $pm(G) \in \{0, 2, 4, k\}$. On the other hand $pm(G) = 3t$. Thus $pm(G) = k$. Therefore $k = 3t \geq 6$. On the other hand G has two vertices, say a and b, such that $deg(a), deg(b) \geq k$ (the end vertices of $\Theta_{q_1,\ldots,q_k}$). Therefore $deg(a), deg(b) \geq 3$ and $deg(a) + deg(b) \geq 2k = 6t$, a contradiction (by Equation (1)). Thus $h \geq 1$. Since $pm(G) = 3t \neq 0$, all numbers $t_1, \ldots, t_h$ are even. Thus $pm(C_{t_i}) = 2$, for $i = 1, \ldots, h$. That is $t_1, \ldots, t_h$ are even. Hence $pm(G) = 2^h\, pm(C\Theta C(m_1, \ldots, m_r; q_1, \ldots, q_k; n_1, \ldots, n_s))$. By Theorem 1, $pm(C\Theta C(m_1, \ldots, m_r; q_1, \ldots, q_k; n_1, \ldots, n_s)) \in \{0, 2, 4, k\}$. Since $pm(G) = 3t \neq 0$, the latter equality shows that $pm(C\Theta C(m_1, \ldots, m_r; q_1, \ldots, q_k; n_1, \ldots, n_s)) = k$. Thus by the third part of Theorem 1 all numbers $m_1, \ldots, m_r, q_1, \ldots, q_k, n_1, \ldots, n_s$ are odd. In addition we get
$$3t = pm(G) = 2^h\, pm(C\Theta C(m_1, \ldots, m_r; q_1, \ldots, q_k; n_1, \ldots, n_s)) = 2^h k. \tag{2}$$
On the other hand the number of edges of $C\Theta C(m_1, \ldots, m_r; q_1, \ldots, q_k; n_1, \ldots, n_s) + C_{t_1} + \cdots + C_{t_h}$ is $\sum_{i=1}^r m_i + \sum_{i=1}^k q_i + \sum_{i=1}^s n_i + \sum_{i=1}^h t_i$. Since G has $3t$ edges, we obtain that
$$\sum_{i=1}^r m_i + \sum_{i=1}^k q_i + \sum_{i=1}^s n_i + \sum_{i=1}^h t_i = 3t. \tag{3}$$
Since G has two vertices with degrees $2r+k$ and $2s+k$, by Equation (1) we obtain that $2r+k+2s+k = 2t$. Therefore
$$r + s + k = t. \tag{4}$$
This shows that $k < t$, since $r, s \geq 1$. Hence by Equation (2) we obtain $h \geq 2$. Since G is a simple graph, there is at most one $i \in \{1, \ldots, k\}$ such that $q_i = 1$. First assume that $q_j = 1$ for some j, $1 \leq j \leq k$. Since $q_1, \ldots, q_k$ are odd, the others are at least 3. Thus $\sum_{i=1}^k q_i \geq 3k - 2$. Therefore $\sum_{i=1}^r m_i + \sum_{i=1}^k q_i + \sum_{i=1}^s n_i + \sum_{i=1}^h t_i \geq 3r + 3k - 2 + 3s + 4h$. Thus by Equation (4) we obtain $\sum_{i=1}^r m_i + \sum_{i=1}^k q_i + \sum_{i=1}^s n_i + \sum_{i=1}^h t_i \geq 3t + 4h - 2$. Since $h \geq 2$, thus $\sum_{i=1}^r m_i + \sum_{i=1}^k q_i + \sum_{i=1}^s n_i + \sum_{i=1}^h t_i > 3t$, a contradiction (see Equation (3)). Now assume that all numbers $q_1, \ldots, q_k$ are at least 3. Similarly we obtain $\sum_{i=1}^r m_i + \sum_{i=1}^k q_i + \sum_{i=1}^s n_i + \sum_{i=1}^h t_i > 3t$, a contradiction.

3) $G \cong C\Theta(m_1, \ldots, m_r; q_1, \ldots, q_k) + C_{t_1} + \cdots + C_{t_h}$, where $r \geq 1$, $k \geq 3$, $h \geq 0$ and $m_1, \ldots, m_r \geq 3$, $q_1, \ldots, q_k \geq 1$ and $t_1, \ldots, t_h \geq 3$ are some integers. Similar to the previous part one can see that this case does not happen.

4) $G \cong \Theta_{q_1,\ldots,q_k} + C_{t_1} + \cdots + C_{t_h}$, where $k \geq 3$, $h \geq 0$ and $q_1,\ldots,q_k \geq 1$ and $t_1,\ldots,t_h \geq 3$ are some integers. Similar to the second part one can see that this case does not happen.

Therefore the case (**ii**) does not happen. In other words $n = 2t + 1$. Hence by case (**i**), $G \cong F_t$. The proof is complete.
□

In sequel we study the unimodality of edge cover polynomial of friendship graphs. First we compute $E(F_t, x)$ for every t. There are some recursive formulas for computing the edge cover polynomials [5, 16]. Let G and H be two disjoint graphs. Let $u \in V(G)$ and $v \in V(H)$. By $G \cdot uv \cdot H$ we denote the graph that obtained by identifying the vertices u and v. For example if u is a vertex of the cycle C_n and v is a vertex of C_m. Then $C_n \cdot uv \cdot C_m \cong C_{m,n}$. In fact $C_{m_1,\ldots,m_k}$ is obtained from the cycles $C_{m_1},\ldots,C_{m_k}$ by this operation, see Figure 3.

Lemma 2. [16] *Let G and H be two disjoint graphs. Let $u \in V(G)$ and $v \in V(H)$. Then*

$$E(G \cdot uv \cdot H, x) = E(G, x)E(H, x) + E(G \setminus u, x)E(H, x) + E(G, x)E(H \setminus v, x).$$

Theorem 6. [5] *Let G be a graph with connected components $G_1,\ldots,G_k$. Then $E(G, x) = \prod_{i=1}^{k} E(G_i, x)$.*

Lemma 3. *Let t be a positive integer. Then*

(i) $E(F_t, x) = x^t\Big((x^2 + 3x + 1)^t - 1\Big)$.

(ii) Every non-zero root of $E(F_t, x)$ is simple. In other words the multiplicity of every non-zero root of $E(F_t, x)$ is one.

(iii) If $z \notin \{0, -3\}$ is a root of $E(F_t, x)$, then $z = -\frac{3}{2} \pm \frac{1}{2}\sqrt{5 + 4\exp^{\frac{2k\pi i}{t}}}$, where $k \in \{1,\ldots,t-1\}$.

Proof. (i) Since $E(F_1, x) = x^3 + 3x^2$, for $t = 1$ there is nothing to prove. Now let $t \geq 2$. Let u be a vertex of F_1 and v be the vertex of F_{t-1} with degree $2t-2$. Thus $F_t \cong F_1 \cdot uv \cdot F_{t-1}$. Since $E(P_2, x) = x$, by Lemma 2 and Theorem 6 we obtain that

$$E(F_t, x) = E(F_1, x)E(F_{t-1}, x) + xE(F_{t-1}, x) + E(F_1, x)x^{t-1}.$$

By the fact that $E(F_1, x) = x^3 + 3x^2$ we find that $E(F_t, x) = (x^3 + 3x^2 + x)E(F_{t-1}, x) + E(F_1, x)x^{t-1}$. Using this recursive formula (applying $t-1$ times) we obtain that

$$E(F_t, x) = E(F_1, x)\textstyle\sum_{j=0}^{t-1} x^j g^{t-1-j}, \qquad (5)$$

where $g = x^3 + 3x^2 + x$. On the other hand

$$\sum_{j=0}^{t-1} x^j g^{t-1-j} = x^{t-1}\sum_{j=0}^{t-1}(x^2 + 3x + 1)^{t-1-j} = x^{t-1}\ \frac{(x^2 + 3x + 1)^t - 1}{x^2 + 3x}.$$

Thus by Equation 5, $E(F_t, x) = x^t\Big((x^2 + 3x + 1)^t - 1\Big)$.

(ii) First note that for every graph G with no isolated vertex, the multiplicity of zero as a root of $E(G, x)$ is $\rho(G)$. Since $\rho(F_t) = t + 1$, the multiplicity of zero as a root of $E(F_t, x)$ is $t + 1$. Now assume that z is a non-zero root of $E(F_t, x)$. That is $E(F_t, z) = 0$. Thus by the first part, z is a root of $h(x) = (x^2 + 3x + 1)^t - 1$. Obviously $z \neq -\frac{3}{2}$. This shows that $h'(z) = t(2z + 3)(z^2 + 3z + 1)^{t-1} \neq 0$, where $h'(x)$ is the derivative of $h(x)$ with respect to x. Thus all non-zero roots of $E(F_t, x)$ are simple.

(iii) First we note that by the first part, 0 and -3 are two roots of $E(F_t, x)$. Now suppose that $z \notin \{0, -3\}$ and $E(F_t, z) = 0$ (so $t \geq 2$). Thus by the first part, $(z^2 + 3z + 1)^t = 1$. Therefore $z^2 + 3z + 1 = \exp^{\frac{2k\pi i}{t}}$ for some $k \in \{0, 1,\ldots,t-1\}$. Since $z \notin \{0, -3\}$, $z^2 + 3z + 1 = \exp^{\frac{2k\pi i}{t}}$ for some $k \in \{1,\ldots,t-1\}$. On the other hand $z^2 + 3z + 1 = \exp^{\frac{2k\pi i}{t}}$ implies that $(z + \frac{3}{2})^2 = \frac{5}{4} + \exp^{\frac{2k\pi i}{t}}$. Thus $z = -\frac{3}{2} \pm \frac{1}{2}\sqrt{5 + 4\exp^{\frac{2k\pi i}{t}}}$, where $k \in \{1,\ldots,t-1\}$.
□

We use the following result to show that the edge cover polynomial of friendship graphs are unimodal.

Theorem 7. [29] *Let $f(x)$ and $g(x)$ be some polynomials with positive coefficients. Then*

(i) If both $f(x)$ and $g(x)$ are log-concave, then so is their product $f(x)g(x)$.

(ii) If $f(x)$ is log-concave and $g(x)$ is unimodal, then their product $f(x)g(x)$ is unimodal.

(iii) If both $f(x)$ and $g(x)$ are symmetric and unimodal, then so is their product $f(x)g(x)$.

Theorem 8. *For every positive integer t, $E(F_t, x)$ is log-concave and unimodal.*

Proof. Since $1 + 3x + x^2$ is log-concave, by the first part of Theorem 7, $(1 + 3x + x^2)^t$ is log-concave. Thus $(1+3x+x^2)^t - 1$ is log-concave. Therefore $x^t\left((1+3x+x^2)^t - 1\right)$ is also log-concave. Hence by the first part of Lemma 3, $E(F_t, x)$ is log-concave. Similarly by the second part of Theorem 7 one can easily see that $E(F_t, x)$ is unimodal. We note that by the third part of Theorem 7, $(1 + 3x + x^2)^t$ is symmetric. □

By Lemma 3, every non-zero root of $E(F_t, x)$ is simple. For example $E(F_1, x) = x^2(x + 3)$ and $E(F_2, x) = x^3(x + 1)(x + 2)(x + 3)$. It is also one of the interesting properties of friendship graphs. We are interested to find all graphs with this property. We finish the paper by the following problem:

Problem. Characterize all graphs G such that all non-zero roots of $E(G, x)$ are simple.

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