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MEASURES OF NONCOMPACTNESS AND SOME APPLICATIONS

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Abstract. The aim of this paper is to present a summary of the lectures I gave at the *Fourth Jordanian Mathematical Conference, Irbid, Jordan, 24–26 August, 1998*, and at the conference *Topology and Analysis, Kraljevo–Mataruška Banja, Yugoslavia, 4–7 June, 1998*.

0. Introduction, notation and preliminaries

The first *measure of noncompactness*, the function α, was defined and studied by *Kuratowski* [42] in 1930. It is surprising that later in 1955 *Darbo* [13] was the first who continued to use the function α. Darbo proved that if T is a continuous self-mapping of a nonempty, bounded, closed and convex subset C of a Banach space X such that

$$(0.0.1) \qquad \alpha(T(Q)) \leq k\alpha(Q) \quad \text{for all } Q \subset C,$$

($k \in (0,1)$ is a constant) then T has at least one fixed point in the set C. Darbo's fixed point theorem is a very important generalization of Schauder's fixed point theorem and it includes the existence part of Banach's fixed point theorem.

Other measures were introduced by *Goldenstein, Gohberg and Markus* (the *ball measures of noncompactness, Hausdorff measure of noncompactness*) [21] in 1957 (later studied by *Goldenstein and Markus* [22] in 1968), *Istrăţesku* [34] in 1972 and others. Apparently Goldenstein, Gohberg and Markus were unawere of the work of Kuratowski and Darbo. It is surprising that Darbo's theorem was almost never noticed and applied, not till in the seventies mathematicians working in operator theory, functional analysis

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and differential equations begun to apply Darbo's theorem and develop the theory connected with measures of noncompactness.

The use of these measures is discussed for example in the monographs [2, 6, 7, 28, 29, 32, 35, 43, 70, , 82, 83], Ph. D. Thesis [1, 4, 53, 57, 64, 86], expository papers [47, 78, 88]. We refer the reader to these works with references given there.

Now let us recall some definitions and results which are probably well known.

If M and S are subsets of a metric space (X,d) and $\epsilon > 0$, then the set S is called *ϵ-net* of M if for any $x \in M$ there exists $s \in S$, such that $d(x,s) < \epsilon$. If the set S is finite, then the ϵ-net S of M is called *finite ϵ-net.* The set M is said to be *totally bounded* if it has a finite ϵ-net for every $\epsilon > 0$. It is well known, that a subset M of a metric space X is compact if every sequence (x_n) in M has a convergent subsequence, and in this case the limit of that subsequence is in M. The set M is said to be *relatively compact* if the closure $\overline{M}$ of M is a compact set. If the set M is relatively compact, then M is totally bounded. If the metric space (X,d) is complete, then the set M is relatively compact if and only if it is totally bounded. It is easy to prove that a subset M of a metric space X is relatively compact if and only if every sequence (x_n) in M has a convergent subsequence; in that case the limit of that subsequence need not be in M.

If $x \in X$ and $r > 0$, then the open ball with centre at x and radius r is denoted by $B(x,r)$, $B(x,r) = \{y \in X : d(x,y) < r\}$. If X is a normed spece, then we denote by B_X the closed unit ball in X and by S_X the unit sphere in X.

Let $\mathcal{M}_X$ (or simply $\mathcal{M}$) be the set of all nonempty and bounded subsets of a metric space (X,d), and let $\mathcal{M}^c{}_X$ (or simply $\mathcal{M}^c$) be the subfamily of $\mathcal{M}_X$ consisting of all closed sets. Further, let $\mathcal{N}_X$ (or simply $\mathcal{N}$) be the set of all nonempty and relatively compact subsets of (X,d). Let $d_H : \mathcal{M}_X \times \mathcal{M}_X \mapsto \mathbb{R}$ be the function defined by

$$(0.0.2) \qquad d_H(S,Q) = \max\{\sup_{x\in S} d(x,Q), \sup_{y\in Q} d(y,S)\} \quad (S,Q \in \mathcal{M}_X).$$

The function d_H is called *Hausdorff distance*, and $d_H(S,Q)$ $(S,\ Q \in \mathcal{M}_X)$ is the *Hausdorff distance of sets S and Q.*

Let us remark that if $\varnothing \neq F \subset X$, $r > 0$ and

$$(0.0.3) \qquad B(F,r) = \bigcup_{x\in F} B(x,r) = \{y \in X : d(y,F) < r\}$$

is the *open ball with centre in* F *and radius* r, then (0.0.1) is equivalent to
(0.0.4)
$d_H(S,Q) = \inf\{\epsilon > 0 : S \subset B(Q,\epsilon) \quad \text{and} \quad Q \subset B(S,\epsilon)\}, \quad (S,Q \in \mathcal{M}_X).$

It is well known that $(\mathcal{M}_X, d_H)$ is a pseudometric space and that $(\mathcal{M}^c_X, d_H)$ is a metric space.

Let X and Y be infinite-dimensional complex Banach spaces and denote the set of bounded linear operators from X into Y by $B(X,Y)$. Set $B(X) = B(X,X)$. For T in $B(X,Y)$ throughout this paper $N(T)$ and $R(T)$ will denote, respectively, the null space and the range space of T. A linear operator A from X to Y is called *compact* (or *completely continuous*) if the domain of A, $D(A) = X$ and for every sequence $\{x_n\} \subset X$ such that $\|x_n\| \leq C$, the sequence $\{Ax_n\}$ has a subsequence which converges in Y. A compact operator is bounded. Operator A in $B(X,Y)$ is of *finite rank* if $\dim R(A) < \infty$. An operator of finite rank is clearly compact. Let $F(X,Y)$, $K(X,Y)$ denote the set of all finite rank and compact operators from X to Y, respectively. Set $F(X) = F(X,X)$ and $K(X) = K(X,X)$.

Let X be a vector space over the field $\mathbb{F}$, $(\mathbb{F} = \mathbb{R}, \mathbb{C})$. A subset E of X is said to be *convex* if

$$\lambda x + (1-\lambda)y \in E \quad \text{for all } x,y \in E \text{ and for all } \lambda \in (0,1). \tag{0.0.5}$$

Clearly the intersection of any family of convex sets is a convex set. If F is a subset of X, then the intersection of all convex sets that contain F is called *convex cover* or *convex hull* of F and denoted by $\mathrm{co}(F)$. Let Q be a nonempty and bounded subset of a normed space X. Then the *convex closure* of Q, is denoted by $\mathrm{Conv}(Q)$, and $\mathrm{Conv}(Q)$ is the smallest convex and closed subset of X that contains Q. It is easy to prove that $\mathrm{Conv}(Q) = \overline{\mathrm{co}(Q)}$.

1. The Kuratowski measure of noncompactness

The notation of measure of noncompactness ($\alpha-$ measure or set-measure), introduced by Kuratowski [42], and the associated notion of an $\alpha-$ contraction, have proved useful in several areas of functional analysis, operator theory, differential equations, ... (see for example, [2, 6, 7]). We start with some results from Kuratowski [42, 43].

Definition 1.1. Let (X,d) be a metric space and Q a bounded subset of X. Then the *Kuratowski measure of noncompactness (the set- measure of noncompactness, α–measure)* of Q, denoted by $\alpha(Q)$, is the infimum of the

set of all numbers $\epsilon > 0$ such that Q can be covered by a finite number of sets with diameters $< \epsilon$, that is
(1.1.1)

$$\alpha(Q) = \inf\left\{\epsilon > 0 : Q \subset \bigcup_{i=1}^{n} S_i,\ S_i \subset X,\ \mathrm{diam}(S_i) < \epsilon\,(i = 1, \ldots, n;\ n \in \mathbb{N})\right\}.$$

The function α is called *Kuratowski's measure of noncompactness.* Clearly

$$\alpha(Q) \leq \mathrm{diam}(Q) \quad \text{for each bounded subset } Q \text{ of } X.$$

As an consequence of Definition 1.1, we obtain.

Lemma 1.2. *Let Q, Q_1 and Q_2 be bounded subsets of a complete metric space (X, d). Then*

(1.2.1) $\alpha(Q) = 0$ *if and only if* $\overline{Q}$ *is compact,*

(1.2.2) $\alpha(Q) = \alpha(\overline{Q})$,

(2.1.3) $Q_1 \subset Q_2$ *implies* $\alpha(Q_1) \leq \alpha(Q_2)$,

(2.1.4) $\alpha(Q_1 \cup Q_2) = \max\{\alpha(Q_1), \alpha(Q_2)\}$,

(2.1.5) $\alpha(Q_1 \cap Q_2) \leq \min\{\alpha(Q_1), \alpha(Q_2)\}$.

(2.1.6) $|\alpha(Q_1) - \alpha(Q_2)| \leq 2d_H(Q_1, Q_2)$.

The next theorem is a generalization of the well–known Cantor intersection theorem.

Theorem (Kuratowski, 1930, [42]) 1.3. *Let (X, d) be a complete metric space. If (F_n) is a decreasing sequence of nonempty, closed and bounded subsets of X and $\lim_n \alpha(F_n) = 0$, then the intersection $F_\infty = \cap_{n=1}^{\infty} F_n$ is a nonempty and compact subset of X.*

If X is a normed space, then the function α has some addtitional properties connected with the vector (linear) structures of a normed space ([13]).

Theorem (Darbo, 1955, [13]) 1.4. *Let Q, Q_1 and Q_2 be bounded subsets of a normed space X. Then*

(1.4.1) $\alpha(Q_1 + Q_2) \leq \alpha(Q_1) + \alpha(Q_2)$,

(.1.4.2) $\alpha(Q + x) = \alpha(Q)$ *for each* $x \in X$,

(1.4.3) $\alpha(\lambda Q) = |\lambda|\alpha(Q)$ *for each* $\lambda \in \mathbb{F}$,

(1.4.4) $\alpha(Q) = \alpha(co(Q))$.

Definition (Darbo) 1.5. Let (X, d) be a metric space. A mapping $T: X \mapsto X$ is said to be *α–contraction (set–contraction)* if T is bounded and continuous, and there exists k, $0 < k < 1$, such that for all bounded subset Q of X :

$$\alpha(T(Q)) \leq k\alpha(Q). \tag{1.5.1}$$

Theorem (Darbo, 1955, [13]) 1.6. *Let X be a complex Banach space, C a nonempty bounded closed and convex subset of X and suppose $T : C \mapsto C$ is α– contraction. Then T has a fixed point and*

$$\alpha(\{x \in C : T(x) = x\}) = 0. \tag{1.6.1}$$

Let us remark that G. Darbo [13] proved (1.4.4) and than applied it in the proof of his famous fixed point theorem. His fixed point theorem is a very important generalization of the Schauder fixed point theorem and it includes the existence part of Banach's fixed point theorem.

Let us mention that first Sadovskii [77] and then several other authors investigated the so called *α–condensing (set–condensing, densifying)* mappings.

Definition (Sadovskii) 1.7. Let (X, d) be a metric space. A mapping $T : X \mapsto X$ is said to be *α–condensing* if T is bounded and continuous, and for all bounded subset Q of X for which $\alpha(Q) > 0$:

$$\alpha(T(Q)) < \alpha(Q). \tag{1.7.1}$$

Obviously every α–contraction is a α–condensing map, but Nussbaum [59] has shown that there exist α–condensing maps which are not α–contractions.

Theorem (Sadovskii, 1967, [77]) 1.8. *Let X be a complex Banach space, C a nonempty bounded closed and convex subset of X and suppose $T : C \mapsto C$ is α– condensing. Then T has a fixed point and*

$$\alpha(\{x \in C : T(x) = x\}) = 0. \tag{1.8.1}$$

Hence, Sadovskii's theorem is a generalization of Darbo's theorem, but it is rather difficult to check whether it holds in a concrete example.

Remark 1.9. Let X be a complex Banach space, and let $T : X \mapsto X$ be a linear α– condensing map. R. Leggett [45] has proved that there exists an equivalent norm on X such that, with respect to this new norm, T is α–contraction. He also has shown that, in general, this result does not hold for a nonlinear maps.

Let X be an infinite-dimensional normed space and B_X be the closed unit ball in X. Then, clearly $\alpha(B_X) \leq 2$, but Furi and Vignoli [20] and Nussbaum [59] have shown more precisely:

Theorem (Furi and Vignoli, 1970, [20]; Nussbaum, 1971, [59]) 1.10. *Let X be an infinite-dimensional normed space. Then*

$$\alpha(B_X) = 2. \tag{1.10.1}$$

2. The Hausdorff measure of noncompactness

Usually it is complicated to find the exact value of $\alpha(Q)$. Another measure of noncompactness, which is more applicable in many cases, were introduced and studied by *Goldenstein, Gohberg and Markus* (the *ball measures of noncompactness, Hausdorff measure of noncompactness*) [21] in 1957 (later studied by *Goldenstein and Markus* [22] in 1968), is given in the next definition.

Definition 2.1. Let (X, d) be a metric space and Q a bounded subset of X. Then the *Hausdorff measure of noncompactness (the ball- measure of noncompactness, χ–measure)* of the set Q, denoted by $\chi(Q)$ is defined to be the infimum of the set of all reals $\epsilon > 0$ such that Q can be covered by a finite number of balls of radii $< \epsilon$, that is
(2.1.1)

$$\chi(Q) = \inf\left\{\epsilon > 0 : Q \subset \bigcup_{i=1}^{n} B(x_i, r_i),\ x_i \in X,\ r_i < \epsilon\,(i = 1, \dots, n)\, n \in \mathbb{N}\right\}.$$

The function χ is called *Hausdorff measure of noncompactness*.

Let us remark that in the definition of the Hausdorff measure of noncompactness of the set Q it is not supposed that centers of the balls which cover Q belong to Q. Hence, (2.1.1) can be equivalently formulated as follows:

$$\chi(Q) = \inf\{\epsilon > 0 : Q \quad \text{has a finite } \epsilon\text{–net in } X\}. \tag{2.1.2}$$

The measures α and χ are different although thay have a good deal in common. Let us remark that the above mentioned properties of α (Section 1) are also valid for χ. As in the case of α, corresponding to χ we have *χ–contraction (ball-contraction)* and *ball-condensing* maps. In generall it is not known the precise relationship between maps defined in terms of α and χ. There are maps which are set-condensing but is unknown whether they are also ball condensing and vica versa (see e.g. T. Dominguez Benavides [14], T. Dominguez Benavides and J. M. Ayerbe [15], Petryshin [60]).

The next theorem shows that the functions α and χ are in some sense equivalent.

Theorem 2.2. *Let (X,d) be a metric space and Q be a bounded subset of X. Then*

$$\chi(Q) \leq \alpha(Q) \leq 2\chi(Q). \tag{2.2.1}$$

Let us remark that the inequalities (2.2.1) are best possible in general, as example show. These measures are closely related to geometrical properties of the space and it is possible to improve on the inequality $\chi(Q) \leq \alpha(Q)$ in certain spaces (see e.g. T. Dominguez Benavides and J. M. Ayerbe [15], J. R. L. Webb and Weiyu Zhao [87]). For example (see [2], [7]) in Hilbert space

$$\sqrt{2}\chi(Q) \leq \alpha(Q) \leq 2\chi(Q), \tag{2.2.2}$$

and in l^p, $1 \leq p < \infty$, space

$$\sqrt[p]{2}\chi(Q) \leq \alpha(Q) \leq 2\chi(Q), \tag{2.2.3}$$

Theorem 2.3. *Let X be an infinite-dimensional normed space and B_X be the closed unit ball of X. Then*

$$\chi(B_X) = 1. \tag{2.3.1}$$

Obviously $\chi(B_X) \leq 1$, and let us remark that Theorem 2.3 follows from Theorems 1.10 and 2.2.

The next theorem shows that the Hausdorff measure of noncompactness is connected with the *Hausdorff distance.*

Theorem 2.4. *Let (X,d) be a metric space, $Q, Q_1, Q_2 \in \mathcal{M}_X$, and $\mathcal{N}_X^c$ be the set of all nonempty and compact subsets of (X,d). Then*

$$|\chi(Q_1) - \chi(Q_2)| \leq d_H(Q_1, Q_2), \tag{2.4.1}$$

$$\chi(Q) = d_H(Q, \mathcal{N}_X^c). \tag{2.4.2}$$

Corollary 2.5. *Let $\mathcal{N}_X^c$ be the set of all nonempty and compact subsets of a complete metric space (X,d). Then $\mathcal{N}_X^c$ is a closed subset of $(\mathcal{M}_X^c, d_H)$.*

Definition 2.6. Let (X,d) be a metric space and Q a bounded subset of X. Then the *inner Hausdorff measure of noncompactness* of the set Q, denoted by $\chi_i(Q)$ is defined to be the infimum of the set of all reals $\epsilon > 0$ such that

Q can be covered by a finite number of balls of radii $< \epsilon$ and centers in Q, that is

(2.6.1)

$$\chi(Q) = \inf\left\{\epsilon > 0 : Q \subset \bigcup_{i=1}^{n} B(x_i, r_i),\ x_i \in Q,\ r_i < \epsilon\,(i = 1, \ldots, n)\, n \in \mathbb{N}\right\}.$$

The function χ_i is called *inner Hausdorff measure of noncompactness.*

Hence, (2.6.1) can be equivalently formulated as follows:

(2.6.2) $$\chi_i(Q) = \inf\{\epsilon > 0 : Q \quad \text{has a finite } \epsilon\text{–net in } Q\}.$$

If Q, Q_1 and Q_2 be bounded subsets of the metric space (X, d), then

(2.6.3) $$\chi_i(Q) = 0 \quad \text{if and only if} \quad Q \quad \text{is totally bounded},$$

(2.6.4) $$\chi_i(Q) = \chi_i(\overline{Q}),$$

but in general

(2.6.5) $$Q_1 \subset Q_2 \quad \not\Rightarrow \quad \chi_i(Q_1) \le \chi_i(Q_2),$$

and

(2.6.6) $$\chi_i(Q_1 \cup Q_2) \ne \max\{\chi_i(Q_1), \chi_i(Q_2)\}.$$

Let Q, Q_1 and Q_2 be bounded subset of the normed space X. Then

(2.6.7) $$\chi_i(Q_1 + Q_2) \le \chi_i(Q_1) + \chi_i(Q_2),$$

(2.6.8) $$\chi_i(Q + x) = \chi_i(Q) \quad \text{for each } x \in X,$$

(2.6.9) $$\chi_i(\lambda Q) = |\lambda|\chi_i(Q) \quad \text{for each } \lambda \in \mathbb{F},$$

but in general

(2.6.10) $$\chi_i(Q) \ne \chi(\mathrm{co}(Q)).$$

In the fixed point theory in normed space (or more generaly in locally convex spaces) the relation $\alpha(Q) = \alpha(co(Q))$ is of great importance. Let us remark that O. Hadžić [30], among other things, studied the inner Hausdorff measure of noncompactness in paranormed spaces. She proved under some additional conditions the inequality $\chi_i(co(Q)) \le \varphi[\chi_i(Q)]$, where $\varphi : [0,\infty) \mapsto [0,\infty)$, and then got some fixed point theorems for multivalued mappings in general topological vector spaces.

Now we shall point out the well–known result of Goldenštein, Gohberg and Markus [21, Theorem 1] (see also [2, 1.8.1] or [7, Theorem 6.1.1]) concerning the Hausdorff measure of noncompactness in Banach spaces with Schauder basis. Let X be a Banach space with a Schauder basis $\{e_1, e_2, \dots\}$. Then each element $x \in X$ has a unique representation

$$x = \sum_{i=1}^{\infty} \phi_i(x) e_i$$

where the functions ϕ_i are the basis functionals. Let $P_n : X \mapsto X$ be the projector onto the linear span of $\{e_1, e_2, \dots, e_n\}$, that is

$$P_n(x) = \sum_{i=1}^{n} \phi_i(x) e_i.$$

Then, in view of the Banach-Steinhaus theorem, all oparators P_n and $I - P_n$ are equibounded.

Theorem (Goldenštein, Gohberg and Markus, 1957, [21]) 2.7. *Let X be a Banach space with a Schauder basis $\{e_1, e_2, \dots\}$, Q be a bounded subset of X, and $P_n : X \mapsto X$ the projector onto the linear span of $\{e_1, e_2, \dots, e_n\}$. Then*

$$\text{(2.7.1)} \qquad \frac{1}{a} \inf_n \sup_{x\in Q} \|(I - P_n)x\| \le \chi(Q) \le \inf_n \sup_{x \in Q} \|(I - P_n)x\|.$$

where $a = \limsup_{n\to\infty} \|I - P_n\|$.

Let us remark that Banaś and Goebel [7, Theorem 6.1.1] proved that the function

$$\text{(2.7.2)} \qquad \mu(Q) = \limsup_{n\to\infty} \left(\sup_{x\in Q} \|(I - P_n)x\| \right),$$

is a regular measure of noncompactness (see Section 4) in X and, moreover the following inequality holds for any $Q \in \mathcal{M}$:

$$\text{(2.7.3)} \qquad \frac{1}{a}\mu(Q) \le \chi(Q) \le \inf_n \sup_{x\in Q} \|(I - P_n)x\| \le \mu(Q).$$

Juan Arias de Reyna and Tomás Dominguez Benavides [3] studied the function

$$\nu(Q) = \liminf_{n\to\infty} \left(\sup_{x\in Q} \|(I - P_n)x\| \right), \tag{2.7.4}$$

They proved, among other things, that the following inequality holds for any $Q \in \mathcal{M}$:

$$\frac{1}{a}\,\nu(Q) \leq \chi(Q) \leq \nu(Q). \tag{2.7.5}$$

Let us mention that concerning the number a in Theorem 2.7 if $X = c_0$, then $a = 1$, but if $X = c$, then $a = 2$ (see e.g. [7, pp.22]).

The next theorem shows how to compute the Hausdorff measure of noncompactness in the spaces ℓ_p, $1 \leq p < \infty$ and c_0.

Theorem ([21]), 2.8. *Let Q be a bounded subset of the normed space X, where X is ℓ_p for $1 \leq p < \infty$ or c_0. If $P_n : X \to X$ is the operator defined by $P_n(x_1, x_2, \dots) = (x_1, x_2, \dots, x_n, 0, 0, \dots)$ for $(x_1, x_2, \dots) \in X$, then*

$$\chi(Q) = \lim_{n\to\infty} \sup_{x\in Q} \|(I - P_n)x\|. \tag{2.8.1}$$

Concerning the space ℓ_∞, to the best of our knowledge is the following theorem.

Theorem (T. Dominguez Benavides, 1988, [14]) 2.9. *Let ℓ_∞ be the real normed space of bounded sequences with* sup-*norm and Q be a bounded subset of ℓ_∞. Then*

$$\alpha(Q) = 2\chi(Q). \tag{2.9.1}$$

We set as on open problem to prove (or disprove) the equality (2.9.1) in the complex normed space ℓ_∞.

3. The Istrăţesku's measure of noncompactness

Let (X, d) be a complete metric space and Q a bounded subset of X. Let us recall that set Q is called *ϵ–discrete* if

$$d(x, y) \geq \epsilon, \qquad \forall x, y \in Q, \quad x \neq y. \tag{3.0.1}$$

Obviously, the set Q is relatively compact if and only if every ϵ–discrete set is finite for all $\epsilon > 0$.

Definition (Istrăţescu, 1972, [34]) 3.1. Let (X, d) be a complete metric space and Q a bounded subset of X. Then the *Istrăţescu measure of noncompactness (β–measure, I –measure)* of Q, is denoted by $\beta(Q)$, and defined by

$$\beta(Q) = \inf\{\epsilon > 0 : Q \quad \text{has no infinite } \epsilon\text{–discrete subsets}\}. \tag{3.1.1}$$

The function β is called *Istrăţesku's measure of noncompactness.*

Let us remark [12] that β can be defined also by

$$\beta(Q) = \sup\{\epsilon > 0 : Q \quad \text{contains an infinite } \epsilon\text{–discrete set}\}, \tag{3.1.2}$$

and the above mentioned properties of α (Section 1) are also valid for β (see e.g. [2], [7], [12]).

Theorem (Daneš, 1972, [12]) 3.2. *Let (X, d) be a metric space and Q be a bounded subset of X. Then*

$$\chi(Q) \le \chi_i(Q) \le \beta(Q) \le \alpha(Q) \le 2\chi(Q). \tag{3.2.1}$$

Hence, in particular,

$$\frac{1}{2}\alpha(Q) \le \beta(Q) \le \alpha(Q) \quad \text{and} \quad \chi(Q) \le \beta(Q) \le 2\chi(Q). \tag{3.2.2}$$

From the best of our knowladge the following problem is still open.

Problem (Daneš, 1972, [12]) 3.3. *Compute $\beta(B_X)$ where B_X is the closed unit ball in a normed space X of infinite dimension.*

From our opinion Daneš's problem is connected with the packing constant of X, $\Lambda(X)$, defined by the formula

(3.2.3)

$$\Lambda(X) = \sup\{r > 0 : \exists (x_n)_{n=1}^{\infty} \quad \text{in } X \text{ such that } \|x_n\| \le 1 - r, \quad \text{and } \|x_m - x_n\| \ge 2r \text{ if } m \ne n\}.$$

Kottman [41] has proved that for any infinite dimensional Banach space X, we have

$$\Lambda(X) = \frac{D(X)}{2 + D(X)}, \tag{3.2.4}$$

where

$$D(X) = \sup\{\inf_{m\neq n} \|x_m - x_n\| : (x_n)_{n=1}^{\infty} \quad \text{contained in } S_X\}, \tag{3.2.5}$$

and S_X is the unit sphere in X. Let us remark that $D(X) = \beta(S_X) = \beta(B_X)$.

4. The axiomatic approach of Banaś and Goebel

The properties of the Kuratowski, Hausdorff and Istrăţesku measures of noncompactness α, χ and χ suggest an axiomatic approach for an abstract definition of a measure of noncompactness. There have appeared a lot of papers containing axiomatic approach to the notion of a measure of noncompactness (see e.g. [2], [7], [30], [78]). Following Banaś and Goebl approach [7] : *the set of axioms should satisfy two requirements; first, it should have netural realizations and second, it should provide useful tools for applications* we recall:

Definition (Banaś and Goebl, 1980, [7]) 4.1. Let X be a Banach space. A nonempty family $\mathcal{P} \subset \mathcal{N}_X$ is called the *kernel (of a measure of noncompactness)* if it satisfies the following conditions:

1. $E \in \mathcal{P}$ implies $\overline{E} \in \mathcal{P}$,
2. $E \in \mathcal{P}$ and $\varnothing \neq F \subset E$ imply $F \in \mathcal{P}$,
3. $E, F \in \mathcal{P}$ and $\lambda \in (0,1)$ imply $\lambda E + (1-\lambda)F \in \mathcal{P}$,
4. $E \in \mathcal{P}$ implies $\mathrm{Conv}(E) \in \mathcal{P}$
5. $\mathcal{P}^c$ is closed in $(\mathcal{M}^c, d_H)$.

Definition 4.2. The function $\mu : \mathcal{M}_X \to [0, +\infty)$ is called a *measure of noncompactness with kernel* $\mathcal{P}$ (ker $\mu = \mathcal{P}$) if it satisfies the following conditions:

1. $\mu(E) = 0$ if and only if $E \in \mathcal{P}$,
2. $\mu(E) = \mu(\overline{E})$,
3. $F \subset E$ implies $\mu(F) \leq \mu(E)$,
4. $\mu(\mathrm{Conv}(E)) = \mu(E)$,
5. $\mu(\lambda E + (1-\lambda)F) \leq \lambda\mu(E) + (1-\lambda)\mu(F)$ for all $\lambda \in (0,1)$,
6. if $F_n \in \mathcal{M}_X$, $n = 1, 2, \dots$, is a decreasing sequence of closed subsets of X and $\lim_{n\to\infty} \mu(F_n) = 0$, then $F_\infty = \cap_{n=1}^{\infty} F_n \neq \varnothing$.

The measure of noncompactness with kernel $\mathcal{P} = \mathcal{N}$ is called *full*, or *complete measure*. Obviously, the measures of noncompactness α, β and χ are full. Simple examples of measures of noncompactness with with kernel $\mathcal{P} \neq \mathcal{N}$ are the *norm of bounded nonempty subsets Q of X*, defined by

(4.2.1) $$\|Q\| = \sup_{x \in Q} \|x\|,$$

and the *diameter* of Q, the well–known function

(4.2.2) $$\operatorname{diam}(Q) = \sup_{x,y \in Q} \|x - y\|.$$

The kernel of the measure of noncompactness (4.2.1) consists of one set $\{0\}$, and the kernel of the measure of noncompactness (4.2.2) is the family of all singletons $\{x\}$ where $x \in X$.

Measures of noncompactness may be defined in an equivalent way without introducing the concept of kernel first [7].

Definition (Banaś and Goebl, 1980, [7]) 4.3. The function $\mu : \mathcal{M}_X \to [0, +\infty)$ is called a *measure of noncompactness* if it satisfies the conditions 2.– 6. of Definition 4.2 and

1′. the family $\mathcal{P} = \{Q \in \mathcal{M} : \mu(Q) = 0\}$ is nonempty, $\mathcal{P} \subset \mathcal{N}$ and $\mathcal{P}^c$ is closed in $\mathcal{M}^c$.

Let Q, Q_1 and Q_2 be bounded subsets of X. Then

$$\max\{\mu(Q_1), \mu(Q_2)\} \leq \mu(Q_1 \cup Q_2);$$

if

(4.3.1) $$\max\{\mu(Q_1), \mu(Q_2)\} = \mu(Q_1 \cup Q_2),$$

than it is said that measure μ has *maximum property*; if

(4.3.2) $$\alpha(\lambda Q) = |\lambda| \alpha(Q) \text{ for each } \lambda \in \mathbb{F},$$

than measure μ is said to be *homogeneous*, and if

(4.3.3) $$\alpha(Q_1 + Q_2) \leq \alpha(Q_1) + \alpha(Q_2),$$

than measure μ is called *subadditive*.

The measure μ is said to be *sublinear* if it is homogeneous and subadditive. The measure μ is called *regular* if it is full, sublinear and has maximum property.

Theorem (Banaś and Goebl, 1980, [7]) 4.4. *If μ is a regular measure, then*

$$\mu(Q) \leq \mu(B_X) \cdot \chi(Q), \quad Q \in \mathcal{M}, \tag{4.4.1}$$

It is an interesting question [7] whether measures μ and χ are equivalent, that is does there exist a constant $c > 0$ such that

$$\mu(Q) \geq c\chi(Q), \quad Q \in \mathcal{M}? \tag{4.4.2}$$

Theorem (Banaś and Goebl, 1980, [7]) 4.5.

(i) If μ is a sublinear measure, then μ is Lipschitzian with respect to the Hausdorff distance, that is

$$|\mu(Q_1) - \mu(Q_2)| \leq \mu(B_X) \cdot d_H(Q_1, Q_2), \quad Q_1, Q_2 \in \mathcal{M}, \tag{4.5.1}$$

(ii) Each measure of noncompactnessf μ is locally Lipschitzian (hence continuous) with respect to the Hausdorff distance.

Let us point out that, among other things, a Darbo type theorem is true for abstract measure of noncompactness and corresponding set-contractions [7].

5. Operators

In the above sections we *measured* the noncompactness of a bounded subset of a metric space. Now we *measure* the noncompactness of an operator.

Definition ([2]) 5.1 Let μ_1 and μ_2 be measures of noncompactness on the Banach spaces X and Y, respectively. An operator $A : X \to Y$ is said to be (μ_1, μ_2)*-bounded* if

$$A(Q) \in \mathcal{M}_Y \quad \text{for each} \quad Q \in \mathcal{M}_X \tag{5.1.1}$$

and there exists a real k with $0 \leq k < \infty$ such that

$$\mu_2(AQ) \leq k\mu_1(Q) \quad \text{for each } Q \in \mathcal{M}_X. \tag{5.1.2}$$

If an operator A is (μ_1, μ_2)-*bounded* then the number $\|A\|_{\mu_1,\mu_2}$ is defined by

$$\|A\|_{\mu_1,\mu_2} = \inf\{k \geq 0 : \mu_2(AQ) \leq k\mu_1(Q) \quad \text{for each } Q \in \mathcal{M}_X\} \tag{5.1.4}$$

and called (μ_1, μ_2)*-operator norm of* A, or (μ_1, μ_2)*-measure of noncompactness of* A, or simply *measures of noncompactness of* A.

If $\mu_1 = \mu_2 = \mu$ then we write $\|A\|_\mu$ instead of $\|A\|_{\mu,\mu}$.

Let us mention that if $A \in B(X, Y)$, then

$$\|A\|_\chi = \chi(AS_X) = \chi(AB_X). \tag{5.1.5}$$

If X is a Banach space, we write X^* for the dual space of X. Further, if $T \in B(X, Y)$ we write $T^* \in B(Y^*, X^*)$ for the adjoint of T. The following sharp estimates have been proved by Goldenstein and Markus [22].

Theorem (Goldenstein and Markus, 1968, [22]) 5.2. *Let* $T \in B(X,Y)$. *Then*

$$\frac{1}{2} \cdot \|T\|_{\chi} \leq \|T^*\|_{\chi} \leq 2 \cdot \|T\|_{\chi}. \tag{5.2.1}$$

Nussbaum [59] proved that

$$\|T^*\|_{\alpha} \leq \|T\|_{\chi} \quad \text{and} \quad \|T\|_{\alpha} \leq \|T^*\|_{\chi}. \tag{5.2.2}$$

Clearly, bouth (5.2.1) and (5.2.2) generalize a classical theorem doe to Schauder (see e.g. [31], [32], [70] or [79]).

Another measure of noncompactness on $B(X,Y)$ which is more widely used is $\|\cdot\|_K$ the *quotient norm* on the Banach space $B(X,Y)/K(X,Y)$, that is

$$\|A\|_K = \inf_{k \in K(X,Y)} \|A - K\|, \quad A \in B(X,Y). \tag{5.2.3}$$

It is easy to prove

Lemma 5.3. *Let* X, Y *and* Z *be Banach spaces,* $A \in B(X,Y)$, $B \in B(Y,Z)$ *and* $\|\cdot\|_K$ *the quotient norm on the Banach space* $B(X,Y)/K(X,Y)$. *Then* $\|\cdot\|_{\chi}$ *is a seminorm on* $B(X,Y)$ *and*

$$\|A\|_{\chi} = 0 \quad \textit{if and only if} \quad A \in K(X,Y), \tag{5.3.1}$$

$$\|A\|_{\chi} \leq \|A\|, \tag{5.3.2}$$

$$\|A + K\|_{\chi} = \|A\|_{\chi}, \quad \textit{for each } K \in K(X,Y). \tag{5.3.3}$$

$$\|BA\|_{\chi} \leq \|B\|_{\chi} \|A\|_{\chi}. \tag{5.3.4}$$

$$\|A\|_{\chi} \leq \|A\|_K. \tag{5.3.5}$$

Recall also that another measures of noncompactness, $\|A\|_m$, is defined in [44, p. 7] to be the greatest lower bound of those numbers η for which there exists a subspace M of X with finite codimension and such that

$$\|Ax\| \leq \eta\|x\|, \quad x \in M. \tag{5.3.6}$$

Let us mention that $\|\cdot\|_{\chi}$ and $\|\cdot\|_m$ are equivalent seminorms on $B(X,Y)$ ([44, Theorem 3.1]),

$$\frac{1}{2} \cdot \|T\|_{\chi} \leq \|T\|_m \leq 2 \cdot \|T\|_{\chi}, \tag{5.3.7.}$$

and (see [25] or [75])

$$\|T\|_m = \|T^*\|_{\chi}, \quad T \in B(X,Y). \tag{5.3.8.}$$

If Y is a Hilbert space then we have

Theorem ([7]) 5.4. *Let X be an arbitrary Banach space and Y be a Hilbert space. For $A \in B(X,Y)$ we have*

$$\|A\|_K = \|A\|_\chi \tag{5.4.1}$$

Motivated by this result Ylinen [91] has studied the measures of noncompactness for elements of C^*–algebras. Let $\mathcal{A}$ be a C^*–algebra. An element $u \in \mathcal{A}$ is called *compact* if the mapping $x \mapsto uxu$, $x \in \mathcal{A}$, is compact operator on $\mathcal{A}$. We denote by $C(\mathcal{A})$ the set of the compact elements of $\mathcal{A}$. Let us mention that $C(\mathcal{A})$ is a closed two-sided ideal of $\mathcal{A}$ [9], and in the C^*–algebra $B(X)$, where X is a Hilbert space, the compact elements are the same as the compact operators on X.

Definition (Ylinen, 1981, [91]) 5.5. Let $\mathcal{A}$ be a C^*–algebra. If $u \in \mathcal{A}$, we denote $k(u) = \inf\{\|u - x\| : x \in C(\mathcal{A})\}$ and call $k(u)$ the *(quotient) measure of noncompactness* of u.

Lemma (Ylinen, 1981, [91]) 5.6. *Let $\mathcal{A}$ be a C^*-algebra and $\mathcal{I}$ a closed two-sided ideal of $\mathcal{A}$. Denote $q(x) = \inf\{\|x - y\| : y \in \mathcal{I}\}$ for $x \in \mathcal{A}$. Let $p : \mathcal{A} \mapsto \mathbb{R}$ be a seminorm such that $p(x) \leq q(x)$ and $p(xy) \leq p(x)p(y)$ for all $x, y \in \mathcal{A}$, and $\{x \in \mathcal{A} : p(x) = 0\} = \mathcal{I}$. Then $p = q$.*

As a corollary we get

Theorem (Ylinen, 1981, [91]) 5.7. *If x is a complex Hilbert space and $T \in B(X)$, then*

$$\|T\|_\chi = \|T\|_m = \|T\|_K. \tag{5.7.1}$$

Let us recall that $A \in B(\ell_1, \ell_1)$ if and only if there exists a scalar matrix $(a_{nk})_{n,k=1}^\infty$ such that

$$\sup_k \sum_{n=1}^\infty |a_{nk}| < \infty,$$

and

$$Ax = y,\ y = (y_n),\ y_n = \sum_{k=1}^\infty a_{nk}x_k\ (n = 1, 2, \dots) \quad \text{for each} \quad x = (x_k) \in \ell_1.$$

In that case

$$\|A\| = \sup_k \sum_{n=1}^\infty |a_{nk}| < \infty,$$

and the operator A uniquely determines the matrix $(a_{nk})_{n,k=1}^\infty$. The operator A is said to be *given (defined) by the matrix* $(a_{nk})_{n,k=1}^\infty$.

The next result gives a technique for the evaluation of the Hausdorff measure of noncompactness of an operator on the space ℓ_1.

Theorem (Goldenštein, Gohberg and Markus, 1957, [21]) 5.8. *Let $A : \ell_1 \to \ell_1$ be a bounded linear operator given by the matrix $(a_{nk})_{n,k=1}^{\infty}$. Then*

$$\|A\|_{\chi} = \lim_{m\to\infty} \sup_k \sum_{n=m}^{\infty} |a_{nk}|. \tag{5.8.1}$$

Corollary 5.9. *Let $A : \ell_1 \to \ell_1$ be a linear and bounded operator given by the matrix $(a_{nk})_{nk=1}^{\infty}$. Then A is compact if and only if*

$$\lim_{m\to\infty} \sup_k \sum_{n=m}^{\infty} |a_{nk}| = 0. \tag{5.9.1}$$

As an application of Theorem 2.7 and Kellog [38, Theorem 1] we shall mention the next our result.

Let $l^{p,q}$, $0 < p,\ q \le \infty$, be the mixed norm sequence space, and $T_\lambda : l^{r,s} \mapsto l^{u,v}$ the operator defined by the multiplier $T_\lambda(a) = \{\lambda_n a_n\}$, $\lambda = \{\lambda_n\} \in l^{\infty}$, $a = \{a_n\} \in l^{r,s}$. ([38]). We have investigated the Hausdorff measure of noncompactness of the operator T_λ in the cases when $0 \le r, u, s, v \le \infty$, and prove necessary and sufficient conditions for T_λ to be compact.

Theorem (Jovanović and Rakočević, 1994, [36]; 1997, [37]) 5.10. *Let $0 < r, u, s, v \le \infty$, and define p and q by*

$$1/p = 1/u - 1/r \quad \text{if} \quad r > u, \qquad p = \infty \quad \text{if} \quad r \le u,$$

$$1/q = 1/v - 1/s \quad \text{if} \quad s > v, \qquad q = \infty \quad \text{if} \quad s \le v.$$

Then $(l^{r,s}, l^{u,v}) = l^{p,q}$, and the operator $T_\lambda : l^{r,s} \mapsto l^{u,v}$, defined by the multiplier $T_\lambda(a) = \{\lambda_n a_n\}$, $\lambda = \{\lambda_n\} \in l^{p,q}$, $a = \{a_n\} \in l^{r,s}$, is well defined.

Now we have:

$$\|T_\lambda\|_\chi = 0, \ (v < s),$$

$$\|T_\lambda\|_\chi = \limsup_{n\to\infty} |\lambda_n|^v, \ (s \le v < 1, \ r \le u),$$

$$\|T_\lambda\|_\chi = \limsup_{m\to\infty} \Big(\sum_{n\in I(m)} |\lambda_n|^p \Big)^{v/p}, \ (s \le v < 1, \ r > u),$$

$$\|T_\lambda\|_\chi = \limsup_{n\to\infty} |\lambda_n|, \ (1 \le v < \infty, \ s \le v, \ r \le u, \ 1 \le u),$$

$$\|T_\lambda\|_\chi = \limsup_{m\to\infty} \Big(\sum_{n\in I(m)} |\lambda_n|^p \Big)^{1/p}, \ (1 \le v < \infty, \ s \le v, \ r > u, \ 1 \le u),$$

$$\|T_\lambda\|_\chi = \limsup_{n\to\infty} |\lambda_n|^u, \ (1 \le v < \infty, \ s \le v, \ r \le u, \ u < 1),$$

$$\|T_\lambda\|_\chi = \limsup_{m\to\infty} \Big(\sum_{n\in I(m)} |\lambda_n|^p \Big)^{u/p}, \ (1 \le v < \infty, \ s \le v, \ r > u, \ u < 1),$$

$$\frac{1}{2} \cdot \limsup_{n\to\infty} |\lambda_n| \le \|T_\lambda\|_\chi \le \limsup_{n\to\infty} |\lambda_n|, \ (v = \infty, \ r \le u, \ 1 \le u),$$

$$\frac{1}{2} \cdot \limsup_{m\to\infty} \Big(\sum_{n\in I(m)} |\lambda_n|^p \Big)^{1/p} \le \|T_\lambda\|_\chi \le$$

$$\le \limsup_{m\to\infty} \Big(\sum_{n\in I(m)} |\lambda_n|^p \Big)^{1/p}, \ (v = \infty, \ r > u, \ 1 \le u).$$

$$\frac{1}{2} \cdot \limsup_{n\to\infty} |\lambda_n|^u \le \|T_\lambda\|_\chi \le \limsup_{n\to\infty} |\lambda_n|^u, \ (v = \infty, \ r \le u, \ u < 1),$$

$$\frac{1}{2} \cdot \limsup_{m\to\infty} \Big(\sum_{n\in I(m)} |\lambda_n|^p \Big)^{u/p} \le \|T_\lambda\|_\chi \le$$

$$\le \limsup_{m\to\infty} \Big(\sum_{n\in I(m)} |\lambda_n|^p \Big)^{u/p}, \quad (v = \infty, \ r > u, \ u < 1).$$

Now as a corollary of the above theorem we have

Corollary 5.11. *Let* $1 \le r, u \le \infty, 0 < s, v \le \infty$, *and define* p *and* q *by*

$$1/p = 1/u - 1/r \quad \textit{if} \quad r > u, \qquad p = \infty \quad \textit{if} \quad r \le u,$$

$$1/q = 1/v - 1/s \quad if \quad s > v, \qquad q = \infty \quad if \quad s \leq v.$$

Then, for $\lambda \in (l^{r,s}, l^{u,v}) = l^{p,q}$, *we have:*

T_λ *is a compact, if* $v < s$,

T_λ *is a compact* $\iff \limsup_{n\to\infty} |\lambda_n| = 0$, *if* $s \leq v$ *and* $r \leq u$,

T_λ *is a compact* $\iff \limsup_{m\to\infty} \Big(\sum_{n\in I(m)} |\lambda_n|^p \Big)^{1/p} = 0$, *if* $s \leq v$ *and* $r > u$.

Remark 5.12. Recently Malkowsky and Rakočević have studied measures of noncompactness in connection with the theory of summability, and we shall give a brief review of these results.

In [49] we investigate linear operators between certain sequence spaces X and Y. Among other things, if X is any p–normed space and $Y = w_0^1, w^1, w_\infty^1, c_0(\mu), c(\mu)$, or $c_\infty(\mu)$ we find necessary and sufficient conditions for A to map X into Y. Then the Hausdorff measure of noncompactness is applied to give necessary and sufficient conditions for A to be a compact operator.

In [50] if X is any BK space and Y is a sequence space of bounded or convergent m^{th}– order differences, then we find necessary and sufficient conditions for infinite matrices A to map X into Y. Further the Hausdorff measure of noncompactness is applied to give necessary and sufficient conditions for A to be a compact operator.

In [51] this paper we investigate linear operators between arbitrary BK spaces X and spaces Y of sequences that are Nq summable or bounded. We give necessary and sufficient conditions for infinite matrices A to map X into Y. Further the Hausdorff measure of noncompactness is applied to give necessary and sufficient conditions for A to be a compact operator.

In [52] we give necessary and sufficient conditions for infinite matrices to map a sequence space X into a sequence space Y where $X = l_1$ and $Y = w_\infty^p, w^p, w_o^p$ $(1 \leq p < \infty)$, or $X = w_0, w, w_\infty$ and $Y = l_p$ $(1 \leq p \leq \infty)$, or $X = w_0, w, w_\infty$ and $Y = w_o^p, w^p, w_i^p$ $(1 \leq p < \infty)$. Furthermore the Hausdorff measure of noncompactness is applied to give necessary and sufficient conditions for a linear operator between these spaces to be compact.

6. Equivalence problem

This section is concerned with an equivalence problem for measures of noncompactness (see e.g. [5], [22], [44], [58]).

Let us recall that a Banach space X has the *λ–compact approximation property* (briefly, λ–CAP) if for each compact subset $D \subset X$ and for each $\epsilon > 0$ there exists a compact operator $K \in K(X)$ such that

$$\sup_{x \in D} \|Kx - x\| \leq \epsilon \quad \text{and} \quad \|K - I\| \leq \lambda; \tag{6.0.1}$$

X has the *bounded compact approximation property* (briefly, BCAP) if it has the λ–CAP for some $\lambda \geq 1$ (see [5], [44]).

Most Banach spaces have the BCAP, but there exists a Banach space without the BCAP ([5], [44], [86]).

Let us recall that if X and Y are Banach spaces, $A \in B(X, Y)$, then

$$\|A\|_\chi \leq \|A\|_K. \tag{6.0.2}$$

Seminorms $\|\cdot\|_\chi$ and $\|\cdot\|_K$ induce norms on the quotient space $B(X,Y)/K(X,Y)$. It is known that this space is complete with respect to the norm induced by $\|\cdot\|_K$. If $B(X,Y)/K(X,Y)$ is complete with respect to the norm induced by $\|\cdot\|_\chi$ then it followes from the closed graph theorem that these norms are equivalent. Since $\|A\|_\chi \leq \|A\|_K$, χ defines a complete norm in $B(X,Y)/K(X,Y)$ if and only if $\|A\|_K \leq c\|A\|_\chi$ for some $c > 0$ depending only of X and Y.

Theorem (Lebow and Schechter, 1971, [44]) 6.1. *Suppose that the Banach space Y has the λ–CAP, then*

$$\|A\|_K \leq \lambda \, \|A\|_\chi, \qquad A \in B(X,Y). \tag{6.0.3}$$

For the converse we have

Theorem (Astala and Tylli, 1987, [5]) 6.2. *Let Y be a Banach space. If*

$$\|A\|_K < \|A\|_\chi \tag{6.0.4}$$

holds for every operator $A \in B(X,Y) \setminus K(X,Y)$ and every Banach space X, then Y has the λ–CAP.

Further we have

Theorem (Astala and Tylli, 1987, [5]) 6.3. *A Banach space Y has the bounded compact approximation property if and only if the quotient $(B(X,Y)/K(X,Y); \chi)$ is complete for every Banach space X.*

Problem (Astala and Tylli, 1987, [5]) 6.4. *It is an open problem whether in Theorem 6.3 it sufficies to assume the completness of* χ *only in the Calkin algebra* $C(X) = B(X)/K(X)$*?*

Let us recall that there is an example of a Banach space X such that $C(X)$ is not complete with respect to χ [5, Example 2.6].

7. Fredholm and semi-Fredholm operators

Although the theory of Fredholm and semi-Fredholm operators is usually approached with Banach algebra techiques (e.g. [9], [11], [16], [32], [79]) there has been some interest in *measures of noncompactness* (e.g [2, 10, 17, 18, 19, 21, 22, 26, 44, 53, 58, 78, 85, 86, 88, 90, 92, 93]).

Let X, Y denote infinite-dimensional complex Banach spaces. An operator $T \in B(X,Y)$ is said to be in $\Phi_+(X,Y)$ (*upper semi-Fredholm operator,* Φ_+*- operator*) if $R(T)$ is closed in Y and the dimension $nul(T)$ of the null space $N(T)$ of T is finite. It is said to be in $\Phi_-(X,Y)$ (*lower semi-Fredholm operator,* Φ_-*- operator*) if $R(T)$ is closed in Y and the codimension $def(T)$ of $R(T)$ in Y is finite. Operators in $\Phi_+(X,Y) \cap \Phi_-(X,Y)$ are called *semi-Fredholm* operators from X to Y. For such operators the *index* is defined as $ind(T) = nul(T) - def(T)$. We set $\Phi(X,Y) = \Phi_+(X,Y) \cup \Phi_-(X,Y)$. The operators in $\Phi(X,Y)$ are called *Fredholm operators* (Φ*- oparators*) from X to Y. They are natural extensions of operators of the form $I - K$, K compact.

We set $B(X) = B(X,X)$, $\Phi_+(X) = \Phi_+(X,X)$, $\Phi_-(X) = \Phi_-(X,X)$ and $\Phi(X) = \Phi(X,X)$. It is well known that $\Phi_+(X)$ and $\Phi_-(X)$ are open semigroups in $B(X)$ (see [11], [32], [79]).

Theorem (Goldenštein, Gohberg and Markus, 1957, [21]) 7.1. *If* $A \in B(X,Y)$ *and* $\|A^n\|_\chi < 1$ *for some* $n \geq 1$*, then* $I - A \in \Phi(X)$ *and* $ind(I - A) = 0$.

The fact that $K(X)$ is a closed two-sided ideal in $B(X)$ enables us to define the *Calkin algebra* over X as the quotient algebra $C(X) = B(X)/K(X)$. $C(X)$ is itself a Banach algebra in the quotient algebra norm

$$\|A\|_K \equiv \|T + K(X)\| = \inf_{K \in K(X)} \|T + K\|.$$

Set

$$\|A\|_F = \inf_{K \in F(X)} \|T + K\|.$$

We shall use π to denote the natural homomorphism of $B(X)$ onto $C(X)$; $\pi(T) = T + K(X)$, $T \in B(X)$. Let $r_e(T) = \lim \|\pi(T^n)\|^{1/n}$ be the *essential* *spectral radius* of T. An operator $T \in B(X)$ is *Riesz* operator if and only if $r_e(T) = 0$ ([11], [32]), i.e., if and only if $\pi(T)$ is quasinilpotent in $C(X)$. Let $R(X)$ denote the set of Riesz operators in $B(X)$.

Theorem (Goldenštein, Gohberg and Markus [85]) 7.2. *If $A \in B(X)$ then*

$$(7.2.1) \qquad \begin{aligned} r_e(T) &= \lim \|T^n\|_K^{1/n} = \lim \|T^n\|_F^{1/n} = \lim \|T^n\|_\chi^{1/n} \\ &= \inf\{\epsilon \geq 0 : T - \lambda I \in \Phi(X) \quad for \quad |\lambda| > \epsilon\}. \end{aligned}$$

Let $\mathcal{A}$ be a unital C^*– algebra and Inv $(\mathcal{A})$ the set of invertible elements of $\mathcal{A}$. If $\mathcal{I}$ is a closed two-sided ideal in $\mathcal{A}$, let $x + \mathcal{I}$ denote the coset in the quotient algebra $\mathcal{A}/\mathcal{I}$ containing x. For $x \in \mathcal{A}$ denote by $r(x)$ $(r(x+\mathcal{I}))$ the spectral radius of the element x $(x + \mathcal{I})$. Let us recall

Theorem (Rakočević, 1991, [67]) 7.3. *Let $\mathcal{A}$ be a unital C^*– algebra and $\mathcal{I}$ a closed two-sided ideal in $\mathcal{A}$. Then*

$$(7.3.1) \qquad r(x + \mathcal{I}) = \inf_{s \in Inv(\mathcal{A})} \|s^{-1}xs + \mathcal{I}\|.$$

Now as a corollary of Lemma 5.6, we get the main result of Mau-Hsiang Shin [81, Theorem 1]

Corollary ([67], [81, Theorem 1]) 7.4. *Let X be a Hilbert space. Then*

$$(7.4.1) \qquad r_e(T) = \inf_{S \in Inv(B(X))} \|S^{-1}TS\|_\alpha.$$

Theorem (Lebow and Schechter, 1971, [44]) 7.5. *An operator $A \in B(X, Y)$ is in $\Phi_+(X)$ if and only if there is a constant c such that*

$$(7.5.1) \qquad \chi(Q) \leq c\chi[A(Q)]$$

holds for all bounded subsets Q of X.

Theorem (Rakočević, 1983, [64], [65]) 7.6. *An operator $A \in B(X, Y)$ is in $\Phi_+(X)$ if and only if for each bounded set $Q \subset X$, there exists an integer $n = n(Q)$ such that*

$$(7.6.1) \qquad \chi(Q) \leq c\chi[A^n(Q)],$$

where c is a constant independent of Q.

For $T \in B(X, Y)$ let

$$(7.6.2) \qquad m(T) = \inf\{\|Tx\| : \|x\| = 1\}$$

be the *minimum modulus (injection modulus)* of T, and

$$q(T) = \sup\{\epsilon \geq 0 : \epsilon B_Y \subset T(B_X)\} \tag{7.6.3}$$

be the *surjection modulus* of T. Operator T is *bounded below* if $m(T) > 0$. Let us recall that for $T \in B(X)$ the following asymptotic formulae of m and q were obtained by Makai and Zemánek [48] and Müller [55]

$$\lim_{n\to\infty} m(T^n)^{1/n} = \inf\{|\lambda| : m(A - \lambda I) = 0\}, \tag{7.6.4}$$

$$\lim_{n\to\infty} q(T^n)^{1/n} = \inf\{|\lambda| : q(A - \lambda I) = 0\}. \tag{7.6.5}$$

We consider the following families of closed subspaces of X:

$$\begin{aligned}
\mathcal{S}(X) &= \{M \subset X : M \text{ is an infinite dimensional subspace of } X\},\\
\mathcal{S}^*(X) &= \{M \subset X : M \text{ is a finite codimensional subspace of } X\},\\
\mathcal{Q}(X) &= \{M \subset X : X/M \text{ is an infinite dimensional space}\},\\
\mathcal{Q}_*(X) &= \{M \subset X : M \text{ is a finite dimensional subspace of } X\}.
\end{aligned}$$

Further, let J_M stands for the inclusion $M \mapsto X$ and Q_M stands for the quotient map $X \mapsto X/M$.

Several operational quontities (with a different notation) have appeared in the literature characterizing upper and lower semi-Fredholm operators (see e.q [17, 19, 25, 26, 44, 53, 54, 62, 63, 64, 80, 85, 86, 92, 93, 97]).

For $T \in B(X)$ set

$$\begin{aligned}
d_+(T) &= \operatorname{dist}\big(T, B(X) \setminus \Phi_+(X)\big),\\
d_-(T) &= \operatorname{dist}\big(T, B(X) \setminus \Phi_-(X)\big).
\end{aligned}$$

Hence, $d_+(T) > 0$ if and only if $T \in \Phi_+(X)$, and $d_-(T) > 0$ if and only if $T \in \Phi_-(X)$. The *semi-Fredholm radii* of the operator T are

$$\begin{aligned}
r_+(T) &= \sup\{\epsilon \geq 0 : T - \lambda I \in \Phi_+(X) \quad \text{for} \quad |\lambda| < \epsilon\},\\
r_-(T) &= \sup\{\epsilon \geq 0 : T - \lambda I \in \Phi_-(X) \quad \text{for} \quad |\lambda| < \epsilon\}.
\end{aligned}$$

Let us remark that $r_+(T) \geq d_+(T)$ and $r_-(T) \geq d_-(T)$.

For $T \in B(X)$, concerrning Φ_+–operators, set

$$m_\chi(T) = \inf\left\{\frac{\chi(TQ)}{\chi(Q)} : Q \subset X \quad \text{bounded}, \quad \chi(Q) > 0\right\}.$$

$$\chi_{cb}(T) = \inf\left\{\frac{\chi(TQ)}{\chi(Q)} : Q \subset X \quad \text{bounded countable}, \quad \chi(Q) > 0\right\}.$$

$$i_\chi(T) = \inf\{\chi(TJ_M) : M \in \mathcal{S}(X)\}.$$

$$i_m(T) = \inf\{m(TJ_M) : M \in \mathcal{S}(X)\}.$$

$$s_m^*(T) = \sup\{m(TJ_M) : M \in \mathcal{S}^*(X)\}.$$

$$s_m(T) = \sup\{m(TJ_M) : M \in \mathcal{S}(X)\}.$$

$$is_m(T) = \inf\{s_m(TJ_M) : M \in \mathcal{S}(X)\}.$$

Further, for $T \in B(X)$, concerrning Φ_-–operators, set

$$s_*q'(T) = \sup\{q((Q_UT) : U \in \mathcal{Q}_*(Y)\}$$

$$sq'(T) = \sup\{q((Q_UT) : U \in \mathcal{Q}(Y)\}$$

$$isq'(T) = \inf\{sq'((Q_UT) : U \in \mathcal{Q}(Y)\}$$

$$i\chi'(T) = \inf\{\chi(Q_UT) : U \in \mathcal{Q}(Y)\}.$$

Let us remark ([26], [54], [85]) that

$$i\chi'(T) = \inf\{\|Q_UT\| : U \in \mathcal{Q}(Y)\}.$$

Let us mention that in the series of papers (see e.q [17, 19, 25, 26, 44, 53, 54, 62, 63, 64, 80, 85, 86, 92, 93, 97]) quantities d_+, d_-, r_+, r_-, m_χ, χ_{cb}, i_χ, i_m, s_m^*, is_m, s_*q', $isq', i\chi'$, and several other operational quantities have been studied characterizing upper and lower semi-Fredholm operators. M. González and A. Martinón [26] have proved that these quantities can be divided into three classes, in such a way that two of them are equivalent if they belong to the same class, and are comparable and not equivalent if they belong to different classes.

Theorem 7.7. *Let $T \in B(X)$. Then*

(7.7.1) $$T \in \Phi_+(X) \text{ if and only if } a(T) > 0,$$

for $a = m_\chi, \chi_{cb}, i_\chi, i_m, s_m^*, is_m$.

(7.7.2) $$T \in \Phi_-(X) \text{ if and only if } b(T) > 0,$$

for $b = s_*q', isq', i\chi'$.

For the semi-Fredholm radii we get

Theorem (Fainshtein, 1985, [17]; Tylli, 1985, [85]; Zemánek, 1984, [92], [93]; González and Martinón, 1995, [26]) 7.8. *For every $T \in B(X)$ we have*

$$\begin{aligned} r_+(T) &= \lim_{n\to\infty} d_+(T^n)^{1/n} = \sup_n d_+(T^n)^{1/n} \\ &= \lim_{n\to\infty} a(T^n)^{1/n}, \end{aligned} \tag{7.8.1}$$

for $a = m_\chi, \chi_{cb}, i_\chi, i_m, s_m^*, is_m$,
and

$$\begin{aligned} r_-(T) &= \lim_{n\to\infty} d_-(T^n)^{1/n} = \sup_n d_-(T^n)^{1/n} \\ &= \lim_{n\to\infty} b(T^n)^{1/n}, \end{aligned} \tag{7.8.2}$$

for $b = s_*q', isq', i\chi'$.

Theorem 7.7 and Theorem 7.8 show that that the operational quantities which characterize the upper (lower) semi-Fredholm operators have the same asymptotic behaviour. To give an application of Theorem 7.8 we need some notations.

Recall that $\operatorname{asc}(T)$ ($\operatorname{des}(T)$), the *ascent (descent)* of $T \in B(X)$, is the smallest non-negative integer n such that $N(T^n) = N(T^{n+1})$ ($R(T^n) = R(T^{n+1})$). If no such n exists, than $\operatorname{asc}(T) = \infty$ ($\operatorname{des}(T) = \infty$). An operator T is called *upper semi-Browder* if $T \in \Phi_+(X)$ and $\operatorname{asc}(T) < \infty$; T is called *lower semi-Browder* if $T \in \Phi_-(X)$ and $\operatorname{des}(T) < \infty$ [32, Definition 7.9.1]. Let $\mathcal{B}_+(X)\,(\mathcal{B}_-(X))$ denote the set of upper (lower) semi-Browder operators. An operator in a Banach space is called semi-Browder if it is upper semi-Browder or lower semi-Browder. Semi-Browder operators were studied by many authors; see e.g. [27], [32], [40], [66], [71], [96]. The name was introduced in [32]. An operator T is *Browder* if it is both upper semi-Browder and lower semi-Browder [32, Definition 7.7.1]. Let $\mathcal{B}(X)$ denote the set of Browder operators, i.e., $\mathcal{B}(X) = \mathcal{B}_+(X) \cap \mathcal{B}_-(X)$. Let us recall that $\mathcal{B}_+(X)$ and $\mathcal{B}_-(X)$ are open subsets in $B(X)$, but not stable under finite-rank perturbations [1, pp. 13-14].

As an application of Theorem 7.8 we have

Theorem (Rakočević, 1997, [72]) 7.9. *Suppose that $T,\ S \in B(X)$ and $TS = ST$. Then*

$$T \in \mathcal{B}_+(X) \quad \text{and} \quad r_e(S) < r_+(T) \quad \Longrightarrow \quad T + S \in \mathcal{B}_+(X). \tag{7.9.1}$$

$$(7.9.2) \qquad T \in \mathcal{B}_-(X) \quad and \quad r_e(S) < r_-(T) \quad \Longrightarrow \quad T + S \in \mathcal{B}_-(X).$$

Let us remark that the commutativity condition in Theorem 7.9 is essential, even for finite dimensional perturbation S [11, pp. 13–14]. Theorem 7.9 generalize the well known Grabiner's theorem [27, Theorem 2] and our result [71, Theorem 1] on the perturbations of semi-Fredholm operators with finite ascent or descent (see Corollary 7.11 and Corollary 7.12 below).

Corollary 7.10. *Suppose that $T \in B(X)$, $S \in R(X)$ and $TS = ST$. Then*

$$(7.10.1) \qquad T \in \mathcal{B}_+(X) \quad \Longrightarrow \quad T + S \in \mathcal{B}_+(X).$$

$$(7.10.2) \qquad T \in \mathcal{B}_-(X) \quad \Longrightarrow \quad T + S \in \mathcal{B}_-(X).$$

Corollary 7.11. *Suppose that $T \in B(X)$, $S \in K(X)$ and $TS = ST$. Then*

$$(7.11.1) \qquad T \in \mathcal{B}_+(X) \quad \Longrightarrow \quad T + S \in \mathcal{B}_+(X).$$

$$(7.11.2) \qquad T \in \mathcal{B}_-(X) \quad \Longrightarrow \quad T + S \in \mathcal{B}_-(X).$$

Recall that the *perturbation classes* associated with $\Phi_+(X)$ and $\Phi_-(X)$ are denoted, respectively, by $P(\Phi_+(X))$ and $P(\Phi_-(X))$, i.e.,

$$P(\Phi_+(X)) = \{T \in B(X) : T + S \in \Phi_+(X) \quad \text{for all} \quad S \in \Phi_+(X)\}$$

and

$$P(\Phi_-(X)) = \{T \in B(X) : T + S \in \Phi_-(X) \quad \text{for all} \quad S \in \Phi_-(X)\}.$$

Corollary 7.12. *Suppose that T, $K \in B(X)$ and $TK = KT$. Then*

$$(7.12.1) \qquad T \in \mathcal{B}_+(X) \quad and \quad K \in P(\Phi_+(X)) \quad \Longrightarrow \quad T + K \in \mathcal{B}_+(X).$$

$$(7.12.2) \qquad T \in \mathcal{B}_-(X) \quad and \quad K \in P(\Phi_-(X)) \quad \Longrightarrow \quad T + K \in \mathcal{B}_-(X).$$

For further applications of Theorem 7.9 to spectra based on semi-Browder operators see e.g. ([40], [71], [72]).

Let us explain some arguments concerning the proof of Theorem 7.8. Recall that B. N. Sadovskii [78] and later (independently) Bouoni, Harte and Wickstead [10] (see also [23]) introduced the next usefull functorial construction. For a Banach space X we set

$$l^\infty(X) = \{x = (x_n) : x_n \in X, \quad \sup_n \|x_n\| < \infty\}.$$

Clearly, $l^\infty(X)$ is a Banach space equipped with the supremum norm. Let $m(X)$ denotes the closed subspace

$$\{x = (x_n) \in l^\infty(X) : \{x_n : n \in \mathbb{N}\} \quad \text{relatively compact}\}$$

of $l^\infty(X)$ and set

$$P(X) = l^\infty(X)/m(X).$$

If Y is another Banach space, and $T \in B(X,Y)$ then T defines an operator $P(T) : P(X) \mapsto P(Y)$ by

$$P((x_n) + m(X)) = (Tx_n) + m(Y), \quad (x_n) \in l^\infty(X).$$

The correspodence $X \mapsto P(X)$, $T \mapsto P(T)$ defines a functor on the category of Banach spaces and bounded linear operators and $\|P(T)\| \leq \|T\|, T \in B(X,Y)$. P is called the *Sadovskii functor*. Further, $P(T) = O$ if and only if $T \in K(X,Y)$. Hence, the mapping

$$\pi(T) = T + K(X,Y) \mapsto P(T) : B(X,Y)/K(X,Y) \mapsto B(P(X), P(Y))$$

is one-one and norm-decreasing. Buoni, Harte and Wickstead [10] asked whether this mapping is also bounded below. From Theorem 6.3 it followes that this is not always the case [5].

The connection of the P-functor with the Hausdorff measure of noncompactness is given by the following facts

Theorem (Harte and Wickstead, 1977, [33]) 7.13. *If $(x_n) \in l^\infty(X)$ then*

$$\|(x_n) \in m(X)\| = \chi(\{x_n : n \in \mathbb{N}\}).$$

Theorem (Harte and Wickstead, 1977, [33]) 7.13. *If $T \in B(X,Y)$ there is inequality*

$$\|P(T)\| \leq \sup\{\chi(TQ) : \chi(Q) \leq 1\} \leq 2\|P(T)\|,$$

and equality

$$\sup\{\chi(TQ) : \chi(Q) \leq 1\} = \chi(TB_X).$$

Theorem (Buoni, Harte and Wickstead, 1977, [10]; Sadovskii, 1972, [78]) 7.14. *If $T \in B(X,Y)$ then the following are equivalent:*

$$P(T) : P(X) \mapsto P(Y) \quad \textit{is one-one}$$
$$T : X \mapsto Y \quad \textit{is upper semi-Fredholm}$$
$$P(T) : P(X) \mapsto P(Y) \quad \textit{is bounded below.}$$

Theorem (Förster and Liebetrau, 1983, [19]; Fainstein, 1980, [18]) 7.15. *If $T \in B(X,Y)$ then the following are equivalent:*

$$T : X \mapsto Y \quad \textit{is lower semi-Fredholm}$$
$$P(T) : P(X) \mapsto P(Y) \quad \textit{is surjective.}$$

Theorem (Buoni, Harte and Wickstead, 1977, [10]; Sadovskii 1972, [78]) 7.16. *If $T \in B(X,Y)$ then the following are equivalent:*

$$P(T) : P(X) \mapsto P(Y) \quad \textit{is invertible}$$
$$T : X \mapsto Y \quad \textit{is Fredholm.}$$

Finally, let us mention that the proofs of Theorem 7.8 are established by some connections (comparisons) between the minimum modulus and the surjection modulus of $P(T)$ with the asymptotic formulas (7.6.4) and (7.6.5). For further related results and generalizations see e.g., ([39, 46, 61, 68, 69, 74, 75, 94, 95]).

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