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THE VECUA-BERNOULLI AND VECUA-RICCATI EQUATIONS

Miloje Rajović and Dragan Dimitrovski

Abstract. In accordance to the classical theory of differential equations we define the Vecua-Bernoulli and Vecua-Riccati equations in the space of functions of two complex variables z and $\overline{z}$ (where the conjugate of unknown function w is a novelty) and give possibilities for solving of these equations using basic I. N. Vecua's equations with analytic coefficients $A(z,\overline{z})$, $B(z,\overline{z})$ which are defined by their convergent areolar series in a domain D_{xOy}.

1. The Vecua-Bernoulli equation

Definition 1.1. Let $u = u(z,\overline{z})$ be a unknown differentiable function of two conditionally independent variables z and $\overline{z}$ with respect to operations $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \overline{z}}$. Suppose there are the inverse operations $\int d\overline{z}$ and $\int dz$. Then a nonlinear equation

$$\frac{\partial u}{\partial \overline{z}} = u[A + B(\frac{\overline{u}}{u})^{\alpha}], \tag{1}$$

where $A(z,\overline{z})$, $B(z,\overline{z})$ are given analytic coefficients of $z, \overline{z}$, α a given complex constant, is called the *Vecua-Bernoulli equation.*

There are (at least) two reasons for such a definition:

1^0. The equation (1) has a typical structure of the Bernoulli equation – on the lefthand side one has a derivative and on the righthand side one has the sum of a linear and a non-linear member:

$$\frac{\partial u}{\partial \overline{z}} = Au + Bue^{-2i\alpha Arg u} \tag{1'.}$$

2^0. The second reason is contained in the following assertion.

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Theorem 1.1. *The equation (1) can be reduced to the Vecua equation with analytic coefficients, hence it can be solved by symmetric iterations.*

Proof. The substitution

$$u^\alpha = w \tag{2}$$

(from where we have also $\overline{u}^\alpha = \overline{w}$), after differentiation gives

$$\alpha u^{\alpha-1}\frac{\partial u}{\partial \overline{z}} = \frac{\partial w}{\partial \overline{z}}$$

and consequently (1) becomes

$$\frac{1}{\alpha u^{\alpha-1}}\frac{\partial w}{\partial \overline{z}} = Au + B\frac{\overline{u}^\alpha}{u^{\alpha-1}},$$

from where we have

$$\frac{\partial w}{\partial \overline{z}} = \alpha A w + \alpha B \overline{w}, \tag{3}$$

which is the Vecua equation that can be solved by iterations. □

If in (3) we take only first integral

$$w = \alpha \int A w d\overline{z} + \alpha \int B \overline{w} d\overline{z} + \Phi(z)$$

and if we are going only up to the first iteration

$$\begin{aligned} w_1 = \alpha \int A[\alpha \int A w d\overline{z} + \alpha \int B\overline{w} d\overline{z} + \Phi(z)]d\overline{z} \\ + \alpha \int B[\alpha \int \overline{A}\overline{w} dz + \overline{\alpha} \int \overline{B} w dz + \overline{\Phi(z)}]d\overline{z} + \Phi(z), \end{aligned}$$

we obtain the first iteration symmetric with respect to coefficients:

$$\begin{aligned} w_1 = \Phi(z) + \alpha\Phi(z)\int A d\overline{z} + \alpha \int B\overline{\Phi(z)}d\overline{z} \\ + \alpha^2 \int A(\int A w d\overline{z} + \alpha^2 \int A(\int B\overline{w} d\overline{z})d\overline{z} \\ + \alpha\overline{\alpha}\int B(\int \overline{A}\overline{w} dz)d\overline{z} + \alpha\overline{\alpha}\int B(\int \overline{B} w dz)d\overline{z}. \end{aligned} \tag{4}$$

From (3) and (4) it follows that symmetry will be seen also in iterations of higher order. In [5] and [6] detail estimates for remainder were given; it consists of the last two lines of (4). Therefore, the remainder is of the order

$$4|\alpha|^2 Max_D(|A|,|B|) Max_D|w|\frac{|z|^2}{2!}, \tag{5}$$

where D is the domain of analicity of coefficients $A(z,\overline{z})$, $B(z,\overline{z})$ and in a small neighbourhood of $O(0,0)$: $|z| \leq 1$, because the other factors are bounded, already the first iteration gives a small remainder. Obviously, continuing by iterations we will get the estimate of the type $N\frac{|z|^n}{n!}$, and so $N\frac{|z|^n}{n!} \to 0$ as $n \to \infty$.

If we suppose that the analytic function $\Phi(z)$ is finite (without singularities) in D, we obtain that the general solution of the equation (3) is given by the integral equation (4), where iterations can be symmetrically extended, as it was already shown in the paper [5]. So the general solution of the equation (1) is:

$$\begin{aligned} u = \Big\{ & \Phi(z) + \alpha\Phi(z)\int A d\overline{z} + \alpha\int B\Phi d\overline{z} \\ & + \alpha^2\int A\Big(\int A w d\overline{z}\Big)d\overline{z} + \alpha^2\int A\Big(\int B\overline{w} d\overline{z}\Big)d\overline{z} \\ & + |\alpha|^2\int B\Big(\int \overline{A}\overline{w} dz\Big)d\overline{z} + |\alpha|^2\int B\Big(\int \overline{B} w dz\Big)d\overline{z}\Big\}^{1/\alpha}, \end{aligned} \tag{6}$$

where one can replace w by w_1 from (4) in order to obtain a necessary degree of accuracy which can be obtained from (5).

Multiformity of the solution (6) is obvious since α is a complex number.

2. A more general Vecua-Bernoulli equation

It is the equation (with another member)

$$\frac{\partial u}{\partial \overline{z}} = Au + Bu\left(\frac{\overline{u}}{u}\right)^{\alpha} + \frac{F}{u^{\alpha-1}} \tag{7}$$

which can be solved by the same substitution (2). Then (7) becomes

$$\frac{\partial w}{\partial \overline{z}} = \alpha(Aw + B\overline{w} + F), \tag{8}$$

and this is a non-homogeneous Vecua equation, which was solved in [5].

Separating real and imaginary parts of left and right sides of the equations (1) and (7) we obtain different systems of real quasilinear equations. Therefore, solving (6) and (8) we also solve systems of quasilinear partial differential equations with six arbitrary real functions.

Example 2.1. For $\alpha = 1$ the equation (1) is Vecua equation with analytic coefficients ("pseudolinear"):

$$\frac{\partial u}{\partial \overline{z}} = Au + B\overline{u}.$$

Example 2.2. If $\alpha = 2$ we have the equation

$$\frac{\partial u}{\partial \overline{z}} = Au + B\overline{u},$$

which is obviously the Riccati equation; it can be solved by the substitution $u^2 = w$. One obtains

$$\frac{\partial w}{\partial \overline{z}} = 2(Au^2 + B\overline{u}^2),$$

and this is the standard Vecua equation with analytic coefficients.

Example 2.3. Let us consider the equation

$$\frac{\partial u}{\partial \overline{z}} = Au + Bu\frac{u}{\overline{u}}, \tag{9}$$

which can be written in the form

$$\frac{\partial u}{\partial \overline{z}} = Au + Bu\frac{1}{\frac{\overline{u}}{u}},$$

or

$$\frac{\partial u}{\partial \overline{z}} = Au + Bu\left(\frac{\overline{u}}{u}\right)^{-1}.$$

The substitution $u = w^{-1}$ gives the equation

$$\frac{\partial w}{\partial \overline{z}} = -Aw - B\overline{w},$$

which is again one standard Vecua equation. According to (4) we have

$$u = \frac{1}{w(z, \overline{z}} = \frac{1}{\Phi(z) - \Phi \int Ad\overline{z} - \int B\overline{\Phi}d\overline{z} - \ldots}$$

3. The Vecua-Riccati equation

After previous considerations it is natural to ask: does there exist some Vecua-Riccati equation? It shoud be pointed out that in difference of the usual Riccati equation, here the assertions concerning (known) particular integrals cannot be used, because here we have partial differential equations for which particular integrals do not have such a role as in the case of usual differential equations. Thus looking for forms of Vecua-Riccati equations we can start only from teh following three properties:

1. invariance of equations with respect to inversions;
2. nore general, invariance of the form of equations with respect to bilinear transformations;
3. anharmonic ratio

So, we start from the standard Vecua equation with analytic coefficients

$$\frac{\partial w}{\partial \overline{z}} = Aw + B\overline{w} + F$$

and put

$$w = f + \frac{1}{u}, \tag{10}$$

where u is the new unknown function and f can be determined. If we substitute the derivative

$$\frac{\partial w}{\partial \overline{z}} = f'_z - \frac{1}{u^2}\frac{\partial u}{\partial \overline{z}}$$

in (9) we obtain

$$-\frac{\partial u}{\partial \overline{z}} = Au + (Af + B\overline{f} + F - f'_z)u^2 + Bu\frac{u}{\overline{u}} \tag{11}$$

which is typical structure of the Riccati equation.

The following assertion is obvious.

Theorem 3.1. *If $f(z, \overline{z})$ is a solution of the Vecua equation*

$$\frac{\partial f}{\partial \overline{z}} = Af + B\overline{f} + F$$

then the equation (11) is one Vecua-Bernoulli equation from Example 2.3 and can be solved by iterations.

Definition 3.1. The non-linear equation

$$\frac{\partial u}{\partial \overline{z}} = Au + Bu^2 + Cu\frac{u}{\overline{u}} \tag{12}$$

is called the *incomplete Vecua-Riccati equation.*

Theorem 3.2. *By substitution* $u = \dfrac{1}{w-f}$ *the equation (12) can be reduced to the non-homogeneous Vecua equation.*

Proof. After elementary calculations we get

$$\frac{\partial w}{\partial \overline{z}} = -Aw - C\overline{w} + (Af + C\overline{f} + f'_z, \tag{13}$$

which is a non-homogeneous Vecua equation (which can be solved by iterations).

Special choice of functions $f(z, \overline{z})$ can only simplify coefficients A, B, C or give the equation (13) to be homogeneous with respect to w. So, the incomplete Vecua-Riccati equation can always be reduced to simple convergent iterations.

Definition 3.2. The equation

$$\frac{\partial u}{\partial \overline{z}} = Au + Bu^2 + Cu\frac{u}{\overline{u}} + F \tag{14}$$

is called the *complete Vecua-Riccati equation*

There are justifications for such a definition. Namely, using inversion $u = w^{-1}$ we obtain

$$-\frac{\partial w}{\partial \overline{z}} = Aw + B + C\overline{w} + F\overline{w}^2 \tag{15}$$

which is the classical form of the Riccati equation.

Special cases of this equation are:

1^0. For $C = 0$ we have the usual areolar Riccati equation;

2^0. For $F = 0$ it is the Vecua equation;

3^0. For $B = C = 0$ we have the Bernoulli areolar equation;

4^0. For $A = B = F = 0$, we have the conjugate Vecua equation for exponential functions.

All these equation we can solve.

The problem of generalized symmetric iteration in (15) remains open.

4. The Sarkisyan equation

If we continue with analogies we can add two new members in (14), so that we have:

Definition 4.1. The equation (derived from (14))

$$\frac{\partial u}{\partial \overline{z}} = Au + Bu^2 + Cu\frac{u}{\overline{u}} + F + D\frac{u}{\overline{u}} + E\frac{u^2}{\overline{u}^2}, \tag{16}$$

which, by inversion, can be transformed in well-known Sakisyan equation

$$-\frac{\partial w}{\partial \overline{z}} = Aw + B + C\overline{w} + Fw^2 + Dw\overline{w} + E\overline{w}^2, \tag{17}$$

is called the Riccati equation of II type with conjugate of the function u.

The equation (17) is investigated from the functional analysis point of view.

The problem for symmetric analytic iterations for equations (16) and (17) remains open.

5. Some consequences

Applying the operator

$$\frac{\partial u}{\partial \overline{z}} = \frac{1}{2}[(u_x - v_y) + i(u_y + v_x)]$$

to $u(z,\overline{z}) = u(x,y) + iv(x,y)$; $z = x + iy$, the Vecua-Bernoulli equation

$$\frac{\partial u}{\partial \overline{z}} = u[A + B\left(\frac{\overline{u}}{u}\right)^{\alpha}],$$

separating real and imaginary parts, gives a quasilinear system

$$u_x - v_y = f(u, v, a_i(x,y), b_i(x,y)), \ : u_y + v + x = g(u, v, a_i(x,y), b_i(x,y)).$$

By eliminating v we obtain the equation

$$\Delta u = F(u, u_x, u_y, a_i, b_i, \alpha),$$

which can be solved by iterations.

6. Open problems

1. Can some of the previous equations be reduced either to a linear equation of second order

$$\frac{\partial^2 w}{\partial \overline{z}^2} = A\frac{\partial w}{\partial \overline{z}} + Bw + C\overline{w} + F,$$

or to the equation

$$\frac{\partial^2 w}{\partial \overline{z}^2} = C\overline{w}?$$

2. What is behaviour of (16) and (17) with respect to the bilinear transformation

$$u = \frac{A_1 w + B_1 \overline{w} + C_1}{A_2 w + B_2 \overline{w} + C_2}?$$

3. Study problems of anharmonic ratio for these equations.

4. Consider homogeneous equations

(a) $\dfrac{\partial w}{\partial \overline{z}} = f\left(\dfrac{\overline{w}}{w}\right)$,

(b) $\dfrac{\partial w}{\partial \overline{z}} = f\left(\dfrac{A_1 w + B_1 \overline{w} + C_1}{A_2 w + B_2 \overline{w} + C_2}\right)$

and find conditions under which they can be reduced to the previous types of equations.

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Department of Mathematics, Faculty of Mechanical Engineering,
36000 Kraljevo, Yugoslavia

Mathematical Institute, Faculty of Natural Sciences and Mathematics,
P.O. Box 162, 91000 Skopje, Republic of Macedonia