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FINITE DIFFERENCE SCHEMES ON NONUNIFORM MESHES FOR A PARABOLIC PROBLEM WITH WEAK SOLUTION

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Abstract. We investigate the convergence of finite difference schemes for two dimensional heat conduction equation on nonuniform rectangular meshes. For schemes with averaged right hand sides convergence rate estimates almost consistent with the smoothness of the solution in discrete L_2 norm are obtained.

1. Introduction

Nonuniform meshes are often used for approximation of problems with generalized solutions. In this case, the order of local error is usually reduced.

In many papers it is shown that the accuracy of the method can be increased using approximation of the considered differential equation in some non–mesh points (see e.g. [1], [2]). In [9] for one dimensional heat conduction equation on a nonuniform in space variable rectangular mesh finite difference schemes (FDSs) of the second order accuracy on x are constructed. The convergence of these schemes in discrete C–norm is proved under some restrictions on the step sizes of the mesh. In [10] similar results for the Poisson equation are obtained. FDSs for two dimensional heat conduction equation are constructed in [8]. Problems with generalized solutions are considered in [3] and [4].

In present paper we investigate the convergence of FDSs for two dimensional heat conduction equation in discrete L_2–norm assuming that the generalized solution of the considered initial boundary value problem (IBVP) belongs to the corresponding Sobolev space. Obtained convergence rate estimates are "almost" consistent with the smoothness of the solution of IBVP

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(see [5]). The Bramble–Hilbert lemma, which usually involve unnecessary restrictions on the mesh step sizes is not applied in the proofs. Contrary to FDSs considered in [8–10], where the right hand sides of equations are taken in some intermediate non–mesh points, we replace the right hand sides with some averaged values. This is necessary because in the problems with generalized solutions, the right hand sides of equations may be discontinuous functions.

2. Preliminaries and Notation

Let us consider the first initial–boundary value problem for the heat conduction equation in the domain $Q = \Omega \times (0,\, T)$, $\Omega = (0,\, l_1) \times (0,\, l_2)$

$$\begin{gathered}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + f(x,\, t)\,, \qquad x = (x_1,\, x_2) \in \Omega, \quad t \in (0,\, T)\,; \\ u(x,\, 0) = 0\,, \quad x \in \Omega\,; \qquad u(x,\, t) = 0\,, \quad x \in \Gamma = \partial\Omega\,, 0 \quad t \in (0,\, T)\,. \end{gathered} \tag{1}$$

In the domain $\bar{Q}$ we define the mesh $\bar{Q}_{h\tau} = \hat{\bar{\omega}}_h \times \bar{\omega}_\tau$, where $\hat{\bar{\omega}}_h = \hat{\bar{\omega}}_1 \times \hat{\bar{\omega}}_2$, $\hat{\bar{\omega}}_\alpha = \big\{x_\alpha = x_\alpha^{(i_\alpha)} = x_\alpha^{(i_\alpha - 1)} + h_\alpha^{(i_\alpha)}\,,\ i_\alpha = 1,\, 2,\, \ldots,\, n,\ x_\alpha^{(0)} = 0\,,\ x_\alpha^{(n)} = l_\alpha\big\}$ $\Big(\sum_{i_\alpha=1}^{n} h_\alpha^{(i_\alpha)} = l_\alpha\Big)$ is nonuniform mesh on $[0,\, l_\alpha]$ $(\alpha = 1,\, 2)$ and $\bar{\omega}_\tau$ is uniform mesh on $[0,\, T]$ with the step size $\tau = T/m$. We assume that

$$\frac{1}{c_1} \le \frac{h_\alpha^{(i_\alpha+1)}}{h_\alpha^{(i_\alpha)}} \le c_1\,,$$

where c_1 is a positive constant. Denote

$$\hat{\omega}_\alpha = \hat{\bar{\omega}}_\alpha \cap (0,\, l_\alpha)\,, \qquad \hat{\omega}_\alpha^+ = \hat{\bar{\omega}}_\alpha \cap (0,\, l_\alpha]\,, \qquad \omega_\tau^+ = \bar{\omega}_\tau \cap (0,\, T]\,,$$
$$Q_{h\tau} = \hat{\omega}_h \times \omega_\tau^+\,.$$

In the sequel we shall use the notation

$$\begin{gathered} x = (x_1^{(i_1)},\, x_2^{(i_2)})\,, \qquad x^{\pm 1} = (x_1^{(i_1 \pm 1)},\, x_2^{(i_2)})\,, \qquad x^{\pm 2} = (x_1^{(i_1)},\, x_2^{(i_2 \pm 1)})\,, \\ t = t_j = j\,\tau\,, \qquad h_\alpha = h_\alpha^{(i_\alpha)}\,, \qquad h_\alpha^{\pm} = h_\alpha^{(i_\alpha \pm 1)}\,, \qquad \hbar_\alpha = (h_\alpha + h_\alpha^+)/2\,, \\ v = v(x,\, t)\,, \qquad v^{\pm\alpha} = v(x^{\pm\alpha},\, t)\,, \qquad \hat{v} = v(x,\, t+\tau)\,, \qquad \check{v} = v(x,\, t-\tau)\,. \end{gathered}$$

We introduce finite differences in a standard way (see [7])

$$v_{x_\alpha} = \big(v^{+\alpha} - v\big)/h_\alpha^+ , \qquad v_{\bar{x}_\alpha} = \big(v - v^{-\alpha}\big)/h_\alpha , \qquad v_{\hat{x}_\alpha} = \big(v^{+\alpha} - v\big)/\hbar_\alpha ,$$
$$v_t = (\hat{v} - v)/\tau , \qquad v_{\bar{t}} = (v - \check{v})/\tau ,$$

Let us define the discrete inner products

$$(u,\, v)_* = \sum_{x \in \hat{\omega}_h} u\, v\, \hbar_1\, \hbar_2 , \qquad (u,\, v] = \sum_{x \in \hat{\omega}_1^+ \times \hat{\omega}_2^+} u\, v\, h_1\, h_2$$

and the norms

$$\|v\|^2_{L_2(\hat{\omega}_h)} = \|v\|_*^2 = (v,\, v)_* , \qquad \|v]|^2_{L_2(\hat{\omega}_h)} = \|v]|^2 = (v,\, v] ,$$
$$\|v\|^2_{L_2(Q_{h\tau})} = \tau \sum_{t \in \omega_\tau^+} \|v(\,\cdot\,,\, t)\|_*^2 , \qquad \|v]|^2_{L_2(Q_{h\tau})} = \tau \sum_{t \in \omega_\tau^+} \|v(\,\cdot\,,\, t)]|^2 .$$

In the following, by C and C_i we shall denote positive generic constants independent of mesh step sizes.

3. Divergent Schemes

Let us define difference operators

$$\Lambda_\alpha v = -v_{\bar{x}_\alpha \hat{x}_\alpha} , \qquad B_\alpha v = -\Big(\frac{h_\alpha^2}{6}\, v_{\bar{x}_\alpha}\Big)_{\hat{x}_\alpha} ,$$
$$\Lambda = \Lambda_1 + \Lambda_2 , \qquad \Lambda' = \Lambda_1 + \Lambda_2 - \Lambda_1\, B_2 - \Lambda_2\, B_1 .$$

(As usual, we set $\Lambda_\alpha v = B_\alpha v = 0$ for $x \in \gamma$).

We approximate the IBVP (1) by implicit FDS

$$(2) \qquad (I - B_1)\,(I - B_2)\, v_t + \Lambda' \hat{v} = T_1^2 T_2^2 T_t f , \qquad v = 0 \ \text{ for } \ t = 0 .$$

The error $z = u - v$ satisfies the following conditions

$$(3) \ \ (I - B_1)\,(I - B_2)\, z_t + \Lambda' \hat{z} = \Lambda_1\, \varphi_1 + \Lambda_2\, \varphi_2 + \psi_t , \qquad z = 0 \ \text{ for } \ t = 0 ,$$

where

$$\varphi_\alpha = \hat{u} - B_{3-\alpha}\, \hat{u} - T_{3-\alpha}^2 T_t\, u \qquad \text{and} \qquad \psi = (I - B_1)\,(I - B_2)\, u - T_1^2 T_2^2\, u .$$

Equation (3) can be rewritten as

$$(4) \qquad z_t + \Lambda' \hat{z} = \Lambda_1 \varphi_1 + \Lambda_2 \varphi_2 + \psi'_t\,, \qquad z = 0 \;\text{ for }\; t = 0\,,$$

where $\psi' = \psi + (B_1 + B_2 - B_1 B_2)\, z$.

To derive the a priori estimate in discrete L_2–norm, let us set $z = z^{(1)} + z^{(2)}$ where $z^{(1)}$ and $z^{(2)}$ are the solutions of the following FDSs

$$(5) \qquad z_t^{(1)} + \Lambda' \hat{z}^{(1)} = \Lambda_1 \varphi_1 + \Lambda_2 \varphi_2\,, \qquad z^{(1)} = 0 \;\text{ for }\; t = 0\,,$$

and

$$(6) \qquad z_t^{(2)} + \Lambda' \hat{z}^{(2)} = \psi'_t\,, \qquad z^{(2)} = 0 \;\text{ for }\; t = 0\,,$$

Let ζ be a mesh function satisfying $\Lambda \zeta = z$. Multiplying (5) by $\hat{\zeta}$ and summing over the mesh $\hat{\omega}_h$, we immediately obtain

$$(7) \qquad \begin{aligned} &\frac{\tau}{2}\, \|\zeta_t\|_\Lambda^2 + \frac{1}{2\tau} \left(\|\hat{\zeta}\|_\Lambda^2 - \|\zeta\|_\Lambda^2\right) + (\hat{z}^{(1)},\, \Lambda' \hat{\zeta})_* \\ &= (\varphi_1,\, \Lambda_1 \hat{\zeta})_* + (\varphi_2,\, \Lambda_2 \hat{\zeta})_* \le \left(\|\varphi_1\|_*^2 + \|\varphi_2\|_*^2\right)^{1/2} \|\hat{z}^{(1)}\|_*\,, \end{aligned}$$

where $\|\zeta\|_\Lambda^2 = (\Lambda\, \zeta,\, \zeta)_*$. Using inequality [4]

$$(8) \qquad (\hat{z}^{(1)},\, \Lambda' \hat{\zeta})_* \ge \frac{(\sqrt{2} - 1)^2}{3}\, \|\hat{z}^{(1)}\|_*^2$$

and ε–inequality from (7) we obtain

$$(9) \qquad \begin{aligned} &\tau^2\, \|\zeta_t\|_\Lambda^2 + \|\hat{\zeta}\|_\Lambda^2 - \|\zeta\|_\Lambda^2 + 2\,\tau\, \frac{(\sqrt{2} - 1)^2}{3}\, \|\hat{z}^{(1)}\|_*^2 \\ &\le \varepsilon\, \tau\, \|\hat{z}^{(1)}\|_*^2 + \frac{\tau}{\varepsilon} \left(\|\varphi_1\|_*^2 + \|\varphi_2\|_*^2\right). \end{aligned}$$

From (9), setting $\varepsilon = (\sqrt{2} - 1)^2/3$ and summing over the mesh ω_τ^+, we obtain the estimate

$$(10) \qquad \|z^{(1)}\|_{L_2(Q_{h\tau})} \le 3\, (\sqrt{2} + 1)^2 \left(\|\varphi_1\|_{L_2(Q_{h\tau})}^2 + \|\varphi_2\|_{L_2(Q_{h\tau})}^2\right)^{1/2}.$$

Let us now estimate $z^{(2)}$. We multiply (6) in a scalar way by $\hat{\eta}$, where

$$-\eta_t = z^{(2)} \quad \text{for} \quad t \in \bar{\omega}_\tau\,, \qquad \eta = 0 \quad \text{for} \quad t = T + \tau\,.$$

Applying partial summing on t, we obtain

$$\tau\sum_{t=\tau}^{T}\|z^{(2)}\|_*^2+\frac{1}{2}\|\eta(\tau)\|_{\Lambda'}^2+\frac{\tau^2}{2}\sum_{t=\tau}^{T}\|\eta_t\|_{\Lambda'}^2=\tau\sum_{t=\tau}^{T}(\psi',z^{(2)})_*$$

wherefrom follows

$$\|z^{(2)}\|_{L_2(Q_{h\tau})}\le\|\psi'\|_{L_2(Q_{h\tau})}\,. \tag{11}$$

From (10) and (11) we obtain the desired a priori estimate for FDS (4)

$$\|z\|_{L_2(Q_{h\tau})}\le 3\,(\sqrt{2}+1)^2\left(\|\varphi_1\|_{L_2(Q_{h\tau})}+\|\varphi_2\|_{L_2(Q_{h\tau})}\right)+\|\psi'\|_{L_2(Q_{h\tau})}\,. \tag{12}$$

Further

$$\|\psi'\|_{L_2(Q_{h\tau})}\le\|\psi\|_{L_2(Q_{h\tau})}+\|(B_1+B_2-B_1\,B_2)\,z\|_{L_2(Q_{h\tau})}\,. \tag{13}$$

Operators B_1 and B_2 satisfy the relations

$$B_1\,B_2=B_2\,B_1\,,\qquad B_\alpha=B_\alpha^*\,,\qquad 0\le B_\alpha\le\frac{2}{3}\,I\,,\quad \alpha=1,\,2\,,$$
$$B_1+B_2-B_1\,B_2=(B_1+B_2-B_1\,B_2)^*\ge 0\,,$$

where I is the unit operator. Using partial summation we obtain

$$\begin{aligned}
&\big((B_1+B_2-B_1\,B_2)\,z,\,z\big)_*=\\
&=\frac{1}{6}\sum(z-z^{-1})^2\,h_1\,\hbar_2+\frac{1}{6}\sum(z-z^{-2})^2\,h_2\,\hbar_1\\
&-\frac{1}{36}\sum(z-z^{-1}-z^{-2}+z^{-1,-2})^2\,h_1\,h_2\\
&=\frac{4}{6}\sum z^2\,\hbar_1\,\hbar_2-\frac{1}{36}\sum(z+z^{-1}+z^{-2}+z^{-1,-2})^2\,h_1\,h_2\\
&-\frac{1}{9}\sum z\,z^{-1}\,h_1\,\hbar_2-\frac{1}{9}\sum z\,z^{-2}\,h_2\,\hbar_1\\
&\le\frac{2}{3}\sum z^2\,\hbar_1\,\hbar_2+\frac{1}{18}\sum\left[z^2+(z^{-1})^2\right]h_1\,\hbar_2\\
&+\frac{1}{18}\sum\left[z^2+(z^{-2})^2\right]h_2\,\hbar_1=\frac{8}{9}\,\|z\|_*^2
\end{aligned}$$

wherefrom follows $B_1+B_2-B_1\,B_2\le\frac{8}{9}\,I$ and

$$\|(B_1+B_2-B_1\,B_2)\,z\|_{L_2(Q_{h\tau})}\le\frac{8}{9}\,\|z\|_{L_2(Q_{h\tau})}\,. \tag{14}$$

From (12), (13) and (14) one obtains the following a priori estimate for FDS (3)

$$(15)\quad \|z\|_{L_2(Q_{h\tau})} \le 27\,(\sqrt{2}+1)^2\,\big(\|\varphi_1\|_{L_2(Q_{h\tau})}+\|\varphi_2\|_{L_2(Q_{h\tau})}\big)+9\,\|\psi\|_{L_2(Q_{h\tau})}\,.$$

In such a manner, to obtain the convergence rate estimate of FDS (2), we need to estimate the right hand side terms in (15). Let us set

$$\begin{aligned}
\varphi_\alpha &= \varphi_{\alpha\,1}+\varphi_{\alpha\,2}+\varphi_{\alpha\,3}+\varphi_{\alpha\,4}\\
&= T^2_{3-\alpha}(\hat{u}-T_t u)+T_\alpha(\hat{u}-B_{3-\alpha}\hat{u}-T_{3-\alpha}\hat{u})\\
&\quad + T_t\big[(u-B_{3-\alpha}u-T_{3-\alpha}u)-T_\alpha(\hat{u}-B_{3-\alpha}\hat{u}-T_{3-\alpha}\hat{u})\big]\\
&\quad + \Big\{\big[(\hat{u}-B_{3-\alpha}\hat{u}-T_{3-\alpha}\hat{u})-T_\alpha(\hat{u}-B_{3-\alpha}\hat{u}-T_{3-\alpha}\hat{u})\big]\\
&\quad - T_t\big[(u-B_{3-\alpha}u-T_{3-\alpha}u)-T_\alpha(\hat{u}-B_{3-\alpha}\hat{u}-T_{3-\alpha}\hat{u})\big]\Big\}
\end{aligned}$$

Using analogous estimates for one–dimensional equation [3] we immediately obtain

$$(16)\qquad \|\varphi_{\alpha\,1}\|_{L_2(Q_{h\tau})} \le C\,(h^2_{\alpha,\,max}+\tau)\,\|u\|_{W_2^{2,\,1}(Q)}\,,$$

and

$$(17)\qquad \|\varphi_{\alpha\,2}\|_{L_2(Q_{h\tau})} \le C\,(h^2_{3-\alpha,\,max}+\tau)\,\|u\|_{W_2^{2,\,1}(Q)}\,.$$

In a similar manner one obtains

$$(18)\qquad \|\varphi_{\alpha\,3}\|_{L_2(Q_{h\tau})} \le C\,(h^2_{1,\,max}+h^2_{2,\,max})\,\|u\|_{W_2^{2,\,1}(Q)}\,,$$

and

$$(19)\qquad \|\varphi_{\alpha\,4}\|_{L_2(Q_{h\tau})} \le C\,h^\sigma_{1,\,max}\,h^\sigma_{2,\,max}\,\tau^\sigma\,|u|_{(\sigma,\,\sigma,\,\sigma),\,Q}\,,$$

where

$$\begin{aligned}
|u|_{(\sigma_1,\,\sigma_2,\,\varrho),\,Q} = \Big\{ \int_0^{l_1}\!\int_0^{l_1}\!\int_0^{l_2}\!\int_0^{l_2}\!\int_0^{T}\!\int_0^{T} \big| u(x_1',x_2',t')-u(x_1'',x_2',t')\\
-u(x_1',x_2',t'')+u(x_1'',x_2',t'')-u(x_1',x_2'',t')+u(x_1'',x_2'',t')+u(x_1',x_2'',t'')\\
-u(x_1'',x_2'',t'')\big|^2\,\frac{dt'\,dt''\,dx_2'\,dx_2''\,dx_1'\,dx_1''}{|x_1'-x_1''|^{1+2\sigma_1}\,|x_2'-x_2''|^{1+2\sigma_2}\,|t'-t''|^{1+2\varrho}}\Big\}^{1/2},
\end{aligned}$$

and $1/2 < \sigma < 1$. From interpolation theory of function spaces [6] follows

$$|u|_{(\sigma,\sigma,\sigma),Q} \leq C\,\|u\|_{W_2^{4\sigma,\,2\sigma}(Q)}\,.$$

In such a way, setting $\sigma = (1+\varepsilon)/2$, from (19) follows

$$(20) \qquad \|\varphi_{\alpha 4}\|_{L_2(Q_{h\tau})} \leq C\,(h_{1,max}^{2+2\varepsilon} + h_{2,max}^{2+2\varepsilon} + \tau^{1+\varepsilon})\,\|u\|_{W_2^{2+2\varepsilon,\,1+\varepsilon}(Q)}\,.$$

Finally, from (16), (17), (18) and (20) one obtains

$$(21) \quad \|\varphi_\alpha\|_{L_2(Q_{h\tau})} \leq C\,(h_{1,max}^2 + h_{2,max}^2 + \tau)\,\|u\|_{W_2^{2+2\varepsilon,\,1+\varepsilon}(Q)}\,, \qquad \varepsilon > 0\,.$$

Similarly,

$$\psi = T_1^2\,\psi_1 + (I - B_2)\,\psi_2\,,$$

where

$$\psi_\alpha = u - B_{3-\alpha}u - T_{3-\alpha}^2 u = \check{\varphi}_\alpha - \check{\varphi}_{\alpha 1}\,.$$

From previous estimates immediately follows

$$(22) \quad \|\psi\|_{L_2(Q_{h\tau})} \leq C\,(h_{1,max}^2 + h_{2,max}^2 + \tau)\,\|u\|_{W_2^{2+2\varepsilon,\,1+\varepsilon}(Q)}\,, \qquad \varepsilon > 0\,.$$

From (15), (21) and (22) we obtain the desired convergence rate estimate for FDS (2)

$$(23) \quad \|u-v\|_{L_2(Q_{h\tau})} \leq C\,(h_{1,max}^2 + h_{2,max}^2 + \tau)\,\|u\|_{W_2^{2+2\varepsilon,\,1+\varepsilon}(Q)}\,, \qquad \varepsilon > 0\,.$$

In such a manner, the following assertion is proved.

Theorem 1. *FDS (2) converges in the discrete L_2 norm and the inequality (23) holds.*

Notice, that the convergence rate estimate (23) is "almost consistent" with the smoothness of the solution of IBVP (1) (see [5]).

FDS (2) is not efficient, because on each time level one must solve the corresponding elliptic problem. Let us therefore consider the factorized scheme (24)

$$(I - B_1 + \tau\,\Lambda_1)\,(I - B_2 + \tau\,\Lambda_2)\,v_t + \Lambda' v = T_1^2 T_2^2 T_t f\,, \qquad v = 0 \text{ for } t = 0\,.$$

FDS (24) can be rewritten in the form

$$\left[(I - B_1)\,(I - B_2) + \tau^2\,\Lambda_1\,\Lambda_2)\right] v_t + \Lambda'\hat{v} = T_1^2 T_2^2 T_t f\,, \qquad v = 0 \text{ for } t = 0\,.$$

The error $z = u - v$ satisfies the following conditions

(25)
$$\big[(I - B_1)\,(I - B_2) + \tau^2\,\Lambda_1\,\Lambda_2)\big]\,z_t + \Lambda'\hat{z} = \Lambda_1\varphi_1 + \Lambda_2\varphi_1 + \Lambda_1\Lambda_2\chi + \psi_t\,,$$
$$z = 0 \quad \text{for} \quad t = 0\,,$$

where φ_1, φ_2 and ψ are already defined and $\chi = \tau^2\,u_t$.

Analogously as in the previous case one obtains the a priori estimate

(26)
$$\|z\|_{L_2(Q_{h\tau})} \leq 27\,(\sqrt{2}+1)^2\,\big(\|\varphi_1\|_{L_2(Q_{h\tau})} + \|\varphi_2\|_{L_2(Q_{h\tau})} + \|\chi_{\bar{x}_1\bar{x}_2}]|_{L_2(Q_{h\tau})}\big) + 9\,\|\psi\|_{L_2(Q_{h\tau})}\,,$$

and the estimate

$$(27)\quad \|\chi_{\bar{x}_1\bar{x}_2}]|_{L_2(Q_{h\tau})} \leq C\,\frac{\tau^{1+\varrho}}{h_{1,\,min}^{1-\sigma}\,h_{2,\,min}^{1-\sigma}}\,|u|_{(\sigma,\,\sigma,\,\varrho),\,Q}\,, \qquad 1/2 < \sigma,\ \varrho \leq 1\,.$$

From interpolation theory of function spaces [6] follows

$$(28)\qquad |u|_{(\sigma,\,\sigma,\,\varrho),\,Q} \leq C\,\|u\|_{W_2^{2\,(\sigma+\varrho),\,\sigma+\varrho}(Q)}\,.$$

In such a way, from (26), (21), (22), (27) and (28) we obtain the following convergence rate estimate for FDS (24)

(29)
$$\|u - v\|_{L_2(Q_{h\tau})} \leq C\,\big[(h_{1,\,max}^2 + h_{2,\,max}^2 + \tau)\,\|u\|_{W_2^{2+2\varepsilon,\,1+\varepsilon}(Q)} + \frac{\tau^{1+\varrho}}{h_{1,\,min}^{1-\sigma}\,h_{2,\,min}^{1-\sigma}}\,\|u\|_{W_2^{2\,(\sigma+\varrho),\,\sigma+\varrho}(Q)}\big]\,, \qquad \varepsilon > 0\,, \quad 1/2 < \sigma,\ \varrho \leq 1\,.$$

Unlike (2), the FDS (24) converges conditionally. For example, setting $\sigma = \varrho = (1+\varepsilon)/2$ from (29) follows the estimate in the form (23), under condition

$$\tau \asymp (h_{1,\,min}\,h_{2,\,min})^{\frac{1-\varepsilon}{1+\varepsilon}}\,.$$

Setting $\sigma = 1$, $\varrho = (1+\varepsilon)/2$ from (29) one obtains

$$\|u - v\|_{L_2(Q_{h\tau})} \leq C\,\big[(h_{1,\,max}^2 + h_{2,\,max}^2 + \tau)\,\|u\|_{W_2^{2+2\varepsilon,\,1+\varepsilon}(Q)} + \tau^{(3+\varepsilon)/2}\,\|u\|_{W_2^{3+\varepsilon,\,(3+\varepsilon)/2}(Q)}\big]\,, \qquad \varepsilon > 0\,.$$

In this case no restrictions on the mesh step sizes, but increased smoothness of the solution of IBVP (1) is required. In such a manner, the following assertion is valid.

Theorem 2. *FDS (24) is conditionally convergent in the discrete L_2 norm and the inequality (29) holds.*

4. Monotonous Schemes

Suppose that the mesh $\hat{\omega}_h$ gets progressively denser towards the vertix (1, 1), i.e. that

$$h_\alpha \ge h_\alpha^+ \ge q\, h_\alpha\,, \qquad \alpha = 1,\, 2, \qquad 0 < q \le 1\,,$$

and define difference operators

$$\tilde{B}_\alpha v = -\frac{h_\alpha^+ - h_\alpha}{3}\, v_{\bar{x}_\alpha}\,, \qquad \Lambda'' = \Lambda_1 + \Lambda_2 - \Lambda_1\, \tilde{B}_2 - \Lambda_2\, \tilde{B}_1\,.$$

Let us approximate now the IBVP (1) with the following factorized implicite monotonous FDS

$$(I - \tilde{B}_1)\,(I - \tilde{B}_2)\, v_t + \Lambda'' \hat{v} = T_1^2 T_2^2 T_t f\,, \qquad v = 0 \ \text{ for } \ t = 0\,. \tag{30}$$

Analogous nonfactorized FDS is proposed in [8].

The error $z = u - v$ satisfies the following conditions

$$(I - \tilde{B}_1)\,(I - \tilde{B}_2)\, z_t + \Lambda'' \hat{z} = \Lambda_1\, \tilde{\varphi}_1 + \Lambda_2\, \tilde{\varphi}_2 + \tilde{\psi}_t\,, \qquad z = 0 \ \text{ for } \ t = 0\,,$$

where

$$\tilde{\varphi}_\alpha = \hat{u} - \tilde{B}_{3-\alpha}\, \hat{u} - T_{3-\alpha}^2 T_t\, u \qquad \text{and} \qquad \tilde{\psi} = (I - \tilde{B}_1)\,(I - \tilde{B}_2)\, u - T_1^2 T_2^2\, u\,.$$

Relations analogous to (8), (10), (11) and (12) are satisfied. Further

$$\|\tilde{B}_\alpha v\|_* \le \frac{2}{3}\,(1 - q)\, \|v\|_*\,,$$

wherefrom follows

$$\|(\tilde{B}_1 + \tilde{B}_2 - \tilde{B}_1\, \tilde{B}_2)\, z\|_{L_2(Q_{h\tau})} \le \left[\frac{4}{3}\,(1 - q) + \frac{4}{9}\,(1 - q)^2\right] \|z\|_{L_2(Q_{h\tau})}\,.$$

Inequality

$$\frac{4}{3}\,(1 - q) + \frac{4}{9}\,(1 - q)^2 < 1$$

is satisfied for $(5 - 3\sqrt{2})/2 = 0.37867965644 < q < (5 + 3\sqrt{2})/2 = 4.62132034356$. Consequently

$$\|(\tilde{B}_1 + \tilde{B}_2 - \tilde{B}_1\, \tilde{B}_2)\, z\|_{L_2(Q_{h\tau})} < \|z\|_{L_2(Q_{h\tau})} \qquad \text{for} \qquad 0.379 \le q \le 1\,.$$

In such a manner, analogously as in the previous case, one obtains the a priori estimate

$$\|z\|_{L_2(Q_{h\tau})} \le C_1\, \big(\|\tilde{\varphi}_1\|_{L_2(Q_{h\tau})} + \|\tilde{\varphi}_2\|_{L_2(Q_{h\tau})}\big) + C_2\, \|\tilde{\psi}\|_{L_2(Q_{h\tau})}\,.$$

Estimates analogous to (21) and (22) hold. In such a manner, the following assertion is valid.

Theorem 3. *For* $0.379 \le q \le 1$ *FDS (30) converges in the discrete* L_2 *norm and the inequality (23) holds.*

Finally, let us consider the factorized scheme

(31)

$$(I - \tilde{B}_1 + \tau \Lambda_1)\,(I - \tilde{B}_2 + \tau \Lambda_2)\, v_t + \Lambda'' v = T_1^2 T_2^2 T_t f\,, \qquad v = 0 \ \text{ for } \ t = 0\,.$$

FDS (31) can be rewritten as

$$\left[(I - \tilde{B}_1)\,(I - \tilde{B}_2) + \tau^2 \Lambda_1 \Lambda_2\right] v_t + \Lambda'' \hat{v} = T_1^2 T_2^2 T_t f\,, \qquad v = 0 \ \text{ for } \ t = 0\,.$$

The error $z = u - v$ satisfies the conditions

$$\left[(I - \tilde{B}_1)\,(I - \tilde{B}_2) + \tau^2 \Lambda_1 \Lambda_2\right] z_t + \Lambda'' \hat{z} = \Lambda_1 \tilde{\varphi}_1 + \Lambda_2 \tilde{\varphi}_2 + \Lambda_1 \Lambda_2 \chi + \tilde{\psi}_t\,,$$
$$z = 0 \ \text{ for } \ t = 0\,.$$

The a priori estimate

$$\|z\|_{L_2(Q_{h\tau})} \le C_1 \left(\|\tilde{\varphi}_1\|_{L_2(Q_{h\tau})} + \|\tilde{\varphi}_2\|_{L_2(Q_{h\tau})} + \|\chi_{\bar{x}_1 \bar{x}_2}]|_{L_2(Q_{h\tau})}\right) + C_2\, \|\tilde{\psi}\|_{L_2(Q_{h\tau})}\,,$$

holds, wherefrom follows a convergence rate estimate in the form (29). In such a manner, the following assertion is valid.

Theorem 4. *For* $0.379 \le q \le 1$ *FDS (31) is conditionally convergent in the discrete* L_2 *norm and the inequality (29) holds.*

References

[1] Berkovskiĭ B.M., Polevikov V.K., *Numerical experiments in convection*, Minsk, 1988.

[2] Gushchin V.A., Shchennikov V.V., *On a monotonous difference scheme of second order of precision*, Zh. Vychisl. Mat. i Mat. Fiz. **14** (1974), 789–792.

[3] Jovanović B.S., *Finite difference schemes on nonuniform meshes for boundary value problems with generalized solutions*, In: A.A. Samarskiĭ (ed.), Proc. of Second Int. Conf. "Finite Difference Methods: Theory and Application", vol. 2, Minsk 1998,, pp. 42–49.

[4] Jovanović B.S., Matus P.P., *Convergence rate estimates for difference schemes on nonuniform meshes for elliptic problems with generalized solutions*, Zh. Vychisl. Mat. Mat. Fiz. (to appear).

[5] Lazarov R.D., Makarov V.L., Samarskiĭ A.A., *Application of exact difference schemes for construction and investigation of difference schemes for generalized solutions*, Mat. Sbornik **117** (1982), 469–480.

[6] Lions J.L., Magenes E., *Problèmes aux limites non homogènes et applications*, Dunod, Paris, 1968.

[7] Samarskiĭ A.A., *Theory of difference schemes*, Nauka, Moscow, 1983.

[8] Samarskiĭ A.A., Mazhukin V.I., Matus P.P., *Difference schemes on nonuniform meshes for twodimensional parabolic equation*, Differentsial'nye Uravneniya **34** (1998).

[9] Samarskiĭ A.A., Vabishchevich P.N., Matus P.P., *Difference schemes of high order of precision on irregular grids*, Differentsial'nye Uravneniya **32** (1996), 265–274.

[10] Samarskiĭ A.A., Vabishchevich P.N., Matus P.P., *Difference schemes of second order of precision on irregular grids*, Zh. Vychisl. Mat. i Mat. Fiz. **38** (1998), 413–424.

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