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THE SEQUENCE OF EXPONENTS OF KARAMATA'S CONVERGENCE

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Abstract. In this paper we give the concept of the sequence of exponents of Karamata's convergence as analogon of the sequence of exponents of convergence introduced by Pringsheim and considered by Borel, Pólya - Szegö, Petrović, Adamović and Tasković.

1. Introduction

Let $f : [a, +\infty) \to (0, +\infty)$ $(a > 0)$ be a measurable function, and (c_n) be a sequence of positive numbers. We shall consider the next asymptotic properties of the function $f(x)$ at $+\infty$:

$$\lim_{x \to +\infty} f(x) = c \quad (c \in (0, +\infty)); \tag{1}$$

$$\lim_{x \to \infty} \frac{f(\lambda x)}{f(x)} = r_f(\lambda) < +\infty \quad (\lambda > 0); \tag{2}$$

$r_f(\lambda)$ $(\lambda > 0)$ is the *index function* of the function $f(x)$ $(x \geq a)$ and by the **Characterization Theorem** (see e.g. [2]) it holds $r_f(\lambda) = \lambda^\rho$ for every $\lambda > 0$ and for some $\rho \in R$. The number ρ is called the *index* of the regular variability. Asumptotic property (1) characterizes the set K of all functions converging to $c > 0$ as $x \to +\infty$. So (2) defines the class of *regularly varying* functions, denoted RV. The class SRV, which is a subclass of RV for $\rho = 0$, is called the class of *slowly regularly varying* functions. It is known that the next relations hold: $K \subseteq$ SRV $\subseteq$ RV and $K \neq$ SRV $\neq$ RV.

The mentioned classes of regularly varying functions have a great importance in qualitative investigations of divergente asymptotic processes (see e.

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g. [2]). Together with above functional classes, we also consider the following conditions related to a numerical sequence (c_n):

$$\lim_{n\to+\infty} c_n = c \quad (c \in (0,+\infty)); \tag{1'}$$

$$\lim_{n\to\infty} \frac{c_{[\lambda x]}}{c_n} = r_c(\lambda) < +\infty \quad (\lambda > 0). \tag{2'}$$

The case when (a_n) is a sequence of positive numbers tending to zero is considered in [1], [3], [4] and [5], and the notion of the sequence of exponents of convergence is introduced.

Definition 1. If (a_n) is a sequence of positive numbers tending to 0, then a sequence of positive numbers (λ_n) is a sequence of exponents of convergence for the sequence (a_n) if for every $\varepsilon > 0$ the series $\sum_{n=1}^{+\infty} a_n^{\lambda_n(1+\varepsilon)}$ converges, and the series $\sum_{n=1}^{+\infty} a_n^{\lambda_n(1-\varepsilon)}$ diverges.

Many informations about this subject can be found in [6].

Now we shall define the sequence of exponents of convergence in the Karamata sense. **Definition 2.** If (a_n) is a sequence of positive numbers, then a sequence of real numbers (λ_n) is the sequence of exponents of the slow variability for the sequence (a_n), if for every $\varepsilon \geq 0$ the sequence (S_n), $S_n = \sum_{k=1}^{n} a_k^{\lambda_k(1+\varepsilon)}$ $(n \in \mathbf{N})$ is SRV, and for every $\mu \in (-\infty, 0)$ the sequence (S'_n), $S'_n = \sum_{k=1}^{n} a_k^{\lambda_k(1+\mu)}$ $(n \in \mathbf{N})$, is not SRV.

2. Main result

Proposition 1. *Let (a_n) be a sequence of positive numbers such that $a_n \neq 1$ $(n \geq n_0)$. Then (a_n) has at least one sequence of exponents of slow variability.*

Proof. Define $\lambda_n = 1$ $(n < n_o)$ and $\lambda_n = -\frac{\ln n}{\ln a_n}$ $(n \geq n_o)$. Then for any $\varepsilon > 0$, the sequence (S_n), $S_n = \sum_{k=1}^{n} a_k^{\lambda_k(1+\varepsilon)}$ $(n \in \mathbf{N})$, converges in the sense of (1'), so it is SRV. If $\varepsilon = 0$ then

$$S_n = \sum_{k=1}^{n} a_k^{\lambda_k} \sim \int_1^{n+1} \frac{dx}{[x]} \sim \int_1^{n+1} \frac{dx}{x} = \ln(n+1)$$

as $n \to \infty$. Hence (S_n) is SRV. Since the sequence (S'_n), $S'_n = \sum_{k=1}^{n} a_k^{\lambda_k(1+\mu)}$ $(n \in \mathbf{N})$, for any $\mu < 0$ and $\alpha = 1 + \mu$ satisfies

$$S'_n = \sum_{k=1}^{n} a_k^{\lambda_k(1+\mu)} \sim \int_1^{n+1} \frac{dx}{[x]^\alpha} \sim \int_1^{n+1} \frac{dx}{x^\alpha} = \frac{1}{1-\alpha}\left((n+1)^{1-\alpha} - 1\right)$$

as $n \to +\infty$, we conclude that (S'_n) is not SRV. □

Example. Let $a_n = n^{-1/2}$ $(n \in \mathbf{N})$. Since $\frac{1}{[x]^{1/2}} \sim \frac{1}{x^{1/2}}$ as $x \to +\infty$, we have that $\sum_{k=1}^{n} a_k = \int_1^{n+1} \frac{1}{[x]^{1/2}} dx \sim \int_1^{n+1} x^{-1/2} dx$ as $n \to +\infty$. Hence $c_n = \sum_{k=1}^{n} a_k \sim 2((n+1)^{1/2} - 1)$ as $n \to +\infty$, so (a_n) is $\mathrm{RV}_{1/2}$, thus it is not SRV, and the sequence of exponents of the slow variability for the sequence (a_n) is (λ_n), $\lambda_1 = 1$ and $\lambda_n = 2$ $(n \geq 2)$. □

The sequence (λ_n) from the previous proposition is called the *standard sequence of exponents of the slow variability* for the sequence (a_n).

Remark. Many basic properties of the sequence of exponents of the slow variability can be derived analogously to the properties of sequences of exponents of convergence (see [1] and [6]).

The sequence of exponents of the slow variability obviously generates an operator $F(a_n^{\bullet}) = (S_n)$ in the class of positive sequences (a_n) such that $a_n \neq 1$ $(n \geq n_o)$, where the sequence $S_n = \sum_{k=1}^{n} a_k^{\lambda_k(1+\varepsilon)}$ $(n \in \mathbf{N})$, ($\varepsilon \geq 0$ is fixed) is SRV.

Proposition 2. *If (a_n) is a sequence of positive numbers and (b_n), $b_n = a_n^{\lambda_n}$ $(n \in \mathbf{N})$ is RV_{-1}, then (λ_n) is the sequence of exponents of slow variability for the sequence (a_n).*

Proof. The function $f(x) = b[x]$ $(x \geq 1)$ is regularly varying of index -1. Then $\int_1^x f(t)dt$ is a slow varying function, which witnesses that the sequence $(\sum_{k=1}^{n} b_k)$ $(n \in \mathbf{N})$ is SRV.

If $\mu < 0$ we have that

$$\int_1^x (f(t))^{1+\mu} dt = \int_1^x \frac{l(t)}{t^{1+\mu}} dt = \int_1^x t^{-1} \frac{l(t)}{t^{\mu}} dt, \quad (l(t),\ (t \geq 1),\ \text{is SRV}),$$

is $\mathrm{RV}_{-\mu}$. Hence the sequence $\left(\sum_{k=1}^{n} b_k^{1+\mu}\right)$ $(n \in \mathbf{N})$ is $\mathrm{RV}_{-\mu}$, so it is not SRV. On the other hand, for any $\varepsilon > 0$ we have that

$$\int_1^x (f(t))^{1+\varepsilon} dt = \int_1^x \frac{l(t)}{t^{1+\varepsilon}} dt = \int_1^x t^{-1-\varepsilon/2} \frac{l(t)}{t^{\varepsilon/2}} dt = \int_1^x h(t) dt\,.$$

Since $h(t) = o\left(t^{-1-\varepsilon/2}\right)$ as $t \to +\infty$, we find that the last integral converges. Hence the sequence $\left(\sum_{k=1}^{n} b_k^{1+\varepsilon}\right)$ $(n \in \mathbf{N})$ converges, hence it is SRV. □

Proposition 3. Let (a_n) be a sequence of positive numbers, and let the sequence (b_n), $b_n = a_n^{\lambda_n}$ $(n \in \mathbf{N})$ be RV. Then (λ_n) is a sequence of exponents of slow variability for the sequence (a_n) only if the sequence $(b_n) \in \mathrm{RV}_{-1}$.

Proof. Since (b_n) is RV_{-1}, by Proposition 2 the sequence (λ_n) is a sequence of exponents of slow variability for the sequence (a_n). If (b_n) is RV_ρ for some $\rho > -1$, then $f(x) = b_{[x]}$ $(x \geq 1)$ is a regularly varying function of index ρ, hence $\int_1^x f(t)dt$ is a regularly varying function whose index is $\rho + 1$. Hence it is not slow varying, which gives that sequence $(\sum_{k=1}^n b_k)$ $(n \in \mathbf{N})$ is also not slow varying, and consequently the sequence (λ_n) is not a sequence of exponents of slow variability.

If (b_n) is RV_ρ $(\rho < -1)$, then $f(x) = b_{[x]}$ $(x \geq 1)$ is a regularly varying function with the same index, and $p = -1 - \rho > 0$. Then for $\mu = \frac{p}{2\rho} < 0$ we have that

$$\int_1^x (f(t))^{1+\mu} dt = \int_1^x t^{\rho(1+\mu)} l(t) = \int_1^x t^{\rho + \frac{p}{2}} l(t)dt, \quad (l(t),\ (t \geq 1),\ \text{is SRV}).$$

Since $\rho + \frac{p}{2} < -1$, analogously to Proposition 2, we find that the last integral converges. Hence if $\mu = \frac{p}{2\rho} < 0$, then the sequence $\left(\sum_{k=1}^n b_k^{1+\mu}\right)$ $(n \in \mathbf{N})$ is SRV.

Consequently, the sequence (λ_n) is not a sequence of exponents of slow variability. □

Finally, we formulate an open question.

Question. If (a_n) is a sequence of positive numbers, and (λ_n) is its sequence of exponents of slow variability, is it always true that sequence (b_n), $b_n = a_n^{\lambda_n}$ $(n \in \mathbf{N})$, belongs to the class RV_{-1}?

3. References

[1] D. D. Adamović, *O pojmu eksponenata konvergencije kod Mihaila Petrovića*, Mat. Vesnik **5 (20)** (1968), 447–458.

[2] N. H. Bringham, C. M. Goldie, J. L. Teugels, *Regular Variation*, Cambridge Univ. Pres, Cambridge, 1987.

[3] E. Borel, *Lecons sur les fonctions entieres*, Paris, 1921.

[4] G. Pólya, Szegö,*Aufgaben und Lehrshtze*, Bd. I, 1925, 19–20.

[5] A. Pringsheim, *Elementare Theorie der ganzen transcedenten Functionen von endlicher Ordnung*, Math. Ann. **58** (1904), 257–342.

[6] M. R. Tasković, *On parameter convergence and convergence in larger sense*, Mat. Vesnik **8 (23)** (1971), 55–59.

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