FILOMAT (Niš) **12:2** (1998), 53–61

SUBORDINATION PROPERTIES FOR A CLASS OF ANALYTIC INTEGRAL OPERATORS

Teodor Bulboacă

Abstract. The present paper is a survey on some new results of the author concerning the images of different subclasses of univalent functions by a class of integral analytic operators.

1 Introduction

Let $H(U)$ be the space of all analytic functions in the unit disk U and let $h \in \mathcal{A} = \{\langle \in \mathcal{H}(\mathcal{U}) : \langle(/) = /, \langle'(/) \neq /, \langle(\ddagger)\langle'(\ddagger) \neq /, \{\imath\nabla / < |\ddagger| < \infty\}$. For $f \in \mathcal{K}_{(\beta,\gamma),\langle} \subset H(U)$ let $F = A_{(\beta,\gamma),h}(f)$ where

$$F(z) = \left[\frac{\beta+\gamma}{h^{\gamma}(z)}\int_0^z f^{\beta}(t)h^{\gamma-1}(t)h'(t)dt\right]^{1/\beta}, \ \beta,\gamma \in \mathbf{C}. \qquad (1.1)$$

In the particular case $\beta = \gamma = 1$ and $g(z) = z$, R.J. Libera [8] showed that $A_{(1,1),z}$ preserves the starlikeness, the convexity and the close-to-convexity.

For $\beta = 1$ and $\gamma > 0$, in some previous papers (see [13], [14]) the authors presented conditions on h such that $A_{(1,\gamma),h}$ is a convexity and a close-to-convexity preserving operator and in the same case the author determined the images of alpha-convex functions by $A_{(1,\gamma),h}$ operator [2].

This type of integral operators and different particular cases were studied in several papers like [1], [6], [8], [9], [10], [11] and others.

If $A = \{f \in H(U) : f(0) = f'(0) - 1 = 0\}$, a function $f \in A$ is called a *starlike function of order* α, where $\alpha < 1$, if $Re\, zf'(z)/f(z) > \alpha$, $z \in U$

Presented at the *Short Conference "Topology and Analysis"*, Mataruška Banja, June 4–7, 1998

1991 *Mathematics Subject Classification.* Primary 30C80; Secondary 30E20, 30C45.

Key words. Integral operator, univalent function, differential subordination, alpha-convex function, starlike function, convex function

and we denote by $S^*(\alpha)$ the class of all these functions. A function $f \in A$ is said to be an *α-convex function* if

$$Re\left\{(1-\alpha)\frac{zf'(z)}{f(z)}+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)\right\}>0,\ z\in U$$

and we denote this class by M_α (see [12]);note that $M_0 \equiv S^*$ and $M_1 \equiv S^c$ where S^* and S^c represents the class of *starlike* and respectively the class of *convex* functions in U.

2 Alpha-convex functions images

Our first result gives us sufficient conditions on h function such that the integral operator (1.1) is well defined.

Theorem 2.1 *[3]. The operator $A_{(\beta,\gamma),h}$ is well defined in the next two cases :*

1. For $\beta = 1$ and $Re\ \gamma > 0$ it is sufficient that $h \in H(U)$ with $h(0) = 0$, $h'(0) \neq 0$, $h(z) \neq 0$ in $\dot{U} = U \setminus \{0\}$ and $f \in H(U)$.

2. For $\beta > 0$, $\beta \neq 1$ or $\beta = 1$ and $\gamma \leq 0$ it is sufficient that $\beta\sigma + \gamma \geq 0$, $h \in H(U)$ with $h(0) = 0$, $h'(0) = 1, h(z)\cdot h'(z) \neq 0$ on $\dot{U}$,

$$f \in S^*(\sigma) \text{ where } \sigma < 1,$$

and

$$Re\left\{(\gamma-1)\frac{zh'(z)}{h(z)}+1+\frac{zh''(z)}{h'(z)}\right\}\geq-\beta\sigma,$$

$z \in U$.

The next result presents conditions on h function such that $A_{(\beta,\gamma),h}(M_\delta) \subset M_{1/\beta}$ where $\beta > 0$.

Theorem 2.2 *[3] . Let $\beta > 0$, let δ be real with $\beta\delta \geq 1$ and let γ be real , with $\gamma \geq \max\{0; \beta - 1\}$. If $\theta \in [\theta_0, \gamma]$, where $\theta_0 = \max\left\{0; \dfrac{\gamma+\beta-1}{2}\right\}$ and $h \in H(U)$ with $h(0) = 0$, $h'(0) = 1$, $h(z) \neq 0$ on $\dot{U}$ suppose that*

$$\gamma\, Re\ \frac{zh'(z)}{h(z)}\geq\theta,\ z\in U$$

and

$$Re\left\{(\gamma+1)\frac{zh'(z)}{h(z)}-\left(1+\frac{zh''(z)}{h'(z)}\right)\right\}\leq\frac{\gamma+\beta}{{}_2F_1\left(1,2\left(\gamma+\beta-\theta\right),\gamma+\beta+1;1/2\right)}.$$

If $\beta = 1$ *suppose that* $\gamma > 0$. *For* $\beta \neq 1$ *or* $\beta = 1$ *and* $\gamma = 0$ *we suppose in addition that*

$$Re \left\{ (\gamma - 1)\frac{zh'(z)}{h(z)} + 1 + \frac{zh''(z)}{h'(z)} \right\} \geq -\beta\,\Delta(\delta),\ z \in U$$

and $h'(z) \neq 0$ *on* $\dot{U}$, *where*

$$\Delta(\delta) = \frac{\Gamma\left(\frac{1}{\delta} + \frac{1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{1}{\delta} + 1\right)} \text{ for } \delta \geq 1 \quad \text{and} \quad \Delta(\delta) = 0 \text{ for } 0 < \delta < 1.$$

Then $A_{(\beta,\gamma),h}(M_\delta) \subset M_{1/\beta}$.

Taking $\beta = 1$ and $\gamma > 0$ in Theorem 2.2 we obtain:

Corollary 2.1 *Let* $\delta \geq 1$, $\gamma > 0$ *and let* $\theta \in [\gamma/2, \gamma]$. *If* $h \in H(U)$ *with* $h(0) = h'(0) - 1 = 0$, $h(z) \neq 0$ *on* $\dot{U}$ *suppose that*

$$Re\frac{zh'(z)}{h(z)} \geq \frac{\theta}{\gamma}, z \in U$$

and

$$Re \left\{ (\gamma + 1)\frac{zh'(z)}{h(z)} - \left(1 + \frac{zh''(z)}{h'(z)}\right) \right\} \leq \frac{\gamma + 1}{{}_2F_1(1, 2(\gamma + 1 - \theta), \gamma + 2; 1/2)},\ z \in U.$$

Then $A_{(1,\gamma),h}(M_\delta) \subset M_1 \equiv S^c$.

Remark that for $\delta = 1$, if we take $\theta := \gamma + \beta$ where $\beta \in [-\frac{\gamma}{2}, 0]$, this corollary represents Theorem 1 from [14].

3 Bazilevič functions images

The function $f \in H(U)$ is called to be a *close - to - convex* function, if there exists a function $\phi \in S^c$ such that $Re\,\dfrac{f'(z)}{\phi'(z)} > 0, z \in U$ and we denote this class by C. Note that every close - to - convex function is univalent [7] , [16] .

Let $g \in S^*$ and $h \in H(U)$ with $h(0) = 1$ and $Re\ \{e^{i\gamma}h(z)\} > 0$, $z \in U$ for some real γ. For $\alpha > 0$ and β real , the function f defined by

$$f(z) = \left[\int_0^z g^\alpha(t)h(t)t^{i\beta - 1}\,dt\right]^{1/(\alpha + i\beta)}$$

is called *Bazilevič function of type* (α,β), this class being denoted by $M(\alpha+i\beta)$ (powers are meant as principal values) . For $\beta=0$ let $M(\alpha)\equiv M(\alpha+i0)$ be the class of *Bazilevič functions of type* α . We denote by $B(\alpha)$, $\alpha>0$, the class of *Bazilevič functions of type* α *usually normalized in* U, i. e. $f\in B(\alpha)$ if

$$f(z)=\left[\alpha\int_0^z g^{\alpha}(t)h(t)t^{-1}dt\right]^{1/\alpha},$$

where $g\in S^*$, $h\in H(U)$ with $h(0)=1$ and $Re\ \{e^{i\gamma}h(z)\}>0$, $z\in U$ for some real γ .

Lemma 3.1 *[4]. Let* $\alpha>0$ *and* $f\in A$ *. Then* $f\in B(\alpha)$ *if and only if there exists a function* $\phi\in M_{\frac{1}{\alpha}}$ *such that*

$$Re\ \frac{f^{\alpha-1}(z)f'(z)}{\phi^{\alpha-1}(z)\phi'(z)}>0,\ z\in U.$$

Theorem 3.1 *[4]. Suppose that the assumptions of Theorem 2.1 are satisfied for* $\beta=1$ *and for* $\beta>0$, $\beta\neq1$ *suppose in addition that*

$$Re\left\{(\gamma-1)\frac{zh'(z)}{h(z)}+1+\frac{zh''(z)}{h'(z)}\right\}\geq\max\left\{-\beta\delta;\ -\beta\Delta\left(\frac{1}{\beta}\right)\right\},\ z\in U$$

where $\delta<1$ *and*

$$\Delta\left(\frac{1}{\beta}\right)=\begin{cases}0,\text{for}\beta>1\\ \dfrac{\Gamma\left(\beta+\frac{1}{2}\right)}{\sqrt{\pi}\Gamma\left(\beta+1\right)},\text{for}0<\beta\leq1\end{cases}$$

and $h\in H(U)$ *with* $h(0)=0,\ h'(0)=1,\ h(z)\cdot h'(z)\neq0$ *on* $\dot{U}$*.*

If

$$Re\left\{\gamma\frac{zh'(z)}{h(z)}\right\}>0,\ z\in U\ \text{where}\ Re\,\gamma>0$$

and

$$A_{(\beta,\gamma),h}\left(M_{\frac{1}{\beta}}\right)\subset M_{\frac{1}{\beta}}$$

then $A_{(1,\gamma),h}(\mathcal{C})\subset\mathcal{C}$ *if* $\beta=1$ *and* $A_{(\beta,\gamma),h}\left(B(\beta)\cap S^*(\delta)\right)\subset B(\beta)$ *if* $\beta>0$, $\beta\neq1$.

Considering $\beta=1$ in the above theorem we obtain :

Corollary 3.1 *Let γ be a complex number with $Re\,\gamma > 0$ and let $h \in H(U)$ with $h(0) = 0$, $h'(0) \neq 0$ and $Re\left\{\gamma \frac{zh'(z)}{h(z)}\right\} > 0$, $z \in U$. If ; $A_{(\beta,\gamma),h}(S^c) \subset S^c$ then $A_{(\beta,\gamma),h}(C) \subset C$.*

Note that this result was previously obtained in [13] .

Definition 3.1 *[4]. Let $\alpha \geq 1$ and $f \in A$. We say that f is a strongly close - to - convex function of order α , denoted $f \in SC_\alpha$, if $f \in C$ for $\alpha = 1$ or if $f \in M_\alpha$ and there exists $h \in M_\alpha$ such that $Re\, \frac{f^{\alpha-1}(z)f'(z)}{h^{\alpha-1}(z)h'(z)} > 0$, $z \in U$ for $\alpha > 1$.*

Theorem 3.2 *[4]. If $\alpha > 1$, then $SC_\alpha \subset B(\alpha) \cap M_\alpha$.*

The next theorem deals with the images of the strongly close - to - convex functions by the $A_{(\beta,\gamma),h}$ operator .

Theorem 3.3 *[4]. Let $\beta \geq 1$ and $\gamma \geq \beta - 1$, such that $\beta + \gamma$ is a pozitive integer. Let $\theta \in \left[\frac{\gamma+\beta-1}{2}, \gamma\right]$ and $h \in H(U)$ with $h(0) = 0$, $h'(0) = 1$, $h(z) \neq 0$ in $\dot{U}$.*

Suppose that

$$\gamma Re \frac{zh'(z)}{h(z)} \geq \theta z \in U$$

and

$$0 \leq Re\left\{(\gamma+1)\frac{zh'(z)}{h(z)} - \left(1 + \frac{zh''(z)}{h'(z)}\right)\right\} \leq \frac{\gamma+\beta}{{}_2F_1(1, 2(\gamma+\beta-\theta), \gamma+\beta+1; 1/2)}$$

and for $\beta > 1$ or $\beta = 1$ and $\gamma = 0$ suppose in addition that

$$Re\left\{(\gamma-1)\frac{zh'(z)}{h(z)} + 1 + \frac{zh''(z)}{h'(z)}\right\} \geq -\beta\,\Delta(\beta),\ z \in U \text{ and } h'(z) \neq 0 \text{ in } \dot{U}.$$

Then $A_{(1,\gamma),h}(C) \subset C$ if $\beta = 1$, $\gamma > 0$ and $A_{(\beta,\gamma),h}(SC_\beta) \subset B(\beta) \cap M_{\frac{1}{\beta}}$ if $\beta > 1$ or if $\beta = 1$, $\gamma = 0$.

It is easy to see that Theorem 3.3 may be rewritten in the following form :

Theorem 3.4 *Let $\beta \geq 1$ and $\gamma \geq \beta - 1$,such that $\beta + \gamma$ is a pozitive integer . Let $\theta \in \left[\frac{\gamma+\beta-1}{2}, \gamma\right]$ and $h \in H(U)$ with $h(0) = 0$, $h'(0) = 1$, $h(z) \neq 0$ in $\dot{U}$.*

Suppose that

$$\gamma\, Re\, \frac{zh'(z)}{h(z)} \geq \theta,\ z \in U$$

and

$$0 \leq Re\left\{(\gamma+1)\frac{zh'(z)}{h(z)} - \left(1 + \frac{zh''(z)}{h'(z)}\right)\right\} \leq \frac{\gamma+\beta}{{}_2F_1(1, 2(\gamma+\beta-\theta), \gamma+\beta+1; 1/2)}$$

and for $\beta > 1$ or $\beta = 1$ and $\gamma = 0$ suppose in addition that

$$Re\left\{(\gamma-1)\frac{zh'(z)}{h(z)} + 1 + \frac{zh''(z)}{h'(z)}\right\} \geq -\beta\,\Delta(\beta),\ z \in U \text{ and } h'(z) \neq 0 \text{ in } \dot{U}.$$

a) For $\beta = 1, \gamma > 0$, if $f \in A$ such that there exists $\phi \in S^c$ with $Re\,\frac{f'(z)}{\phi'(z)} > 0$ in U , then $Re\,\frac{F'(z)}{\Phi'(z)} > 0$ in U where $F = A_{(1,\gamma),h}(f)$, $\Phi = A_{(1,\gamma),h}(\phi)$ and $F, \Phi \in S^c$.

b) For $\beta > 1$ or $\beta = 1, \gamma = 0$ if $f \in M_\beta$ such that there exists $\phi \in M_\beta$ with $Re\,\frac{f^{\beta-1}(z)f'(z)}{\phi^{\beta-1}(z)\phi'(z)} > 0$ in U , then $Re\,\frac{F^{\beta-1}(z)F'(z)}{\Phi^{\beta-1}(z)\Phi'(z)} > 0$ in U , where $F = A_{(\beta,\gamma),h}(f)$, $\Phi = A_{(\beta,\gamma),h}(\phi)$ and $F, \Phi \in M_{\frac{1}{\beta}}$.

Some particular cases of Theorem 3.4 obtained for different choices of β, γ and $h(z)$ may be found in [4].

4 Univalent images

The $A_{(\beta,\gamma),h}$ integral operator may be defined on a larger subset of $H(U)$ as follows.

Like in [15], let $c \in \mathbf{C}$ with $Re\, c > 0$ and let

$$N = N(c) = \frac{|c|\sqrt{1 + 2\,Re\,c} + Im\,c}{Re\,c}.$$

Considering the univalent function $k(z) = \frac{2Nz}{1-z^2}$, we define the *"open door"* function

$$R_c(z) = k\left(\frac{z+b}{1+\bar{b}z}\right),\ z \in U.$$

Note that R_c is univalent in U, $R_c(0) = c$ and $R_c(U) = k(U)$ is the complex plane slit along the half lines $Re\, w = 0, Im\, w \geq N$ and $Re\, w = 0, Im\, w \leq -N$.

For f, $g \in H(U)$ we say that f *is subordinate to* g, written $f(z) \prec g(z)$, if g is univalent in U, $f(0) = g(0)$ and $f(U) \subset g(U)$.

Lemma 4.1 *[5]. Let $\beta, \gamma \in \mathbf{C}$ with $Re\,(\beta + \gamma) > 0$ and let $h \in \mathcal{A}$. Then the integral operator given by (1.1) is well defined on the subset*

$$\mathcal{K}_{(\beta,\gamma),\langle} = \left\{ f \in H(U) : f(0) = 0, f'(0) \neq 0, \beta\frac{zf'(z)}{f(z)} + J(\gamma, h)(z) \prec R_{\beta+\gamma}(z) \right\},$$

where $$J(\gamma, h)(z) = (\gamma - 1)\frac{zh'(z)}{h(z)} + \frac{zh''(z)}{h'(z)} + 1.$$

Theorem 4.1 *[5].Let $\beta, \gamma \in \mathbf{C}$ with $\beta + \gamma > 0$. For a function $h \in \mathcal{A}$ we denote by*

$$m = inf \left\{ Re\, \gamma\frac{zh'(z)}{h(z)} : z \in U \right\} \quad \text{andby} \quad M = sup \left\{ Re\, \gamma\frac{zh'(z)}{h(z)} : z \in U \right\}.$$

Let δ be a real number such that

$$m - (\beta + \gamma) < \delta \leq min \left\{ m; m - \frac{\beta + \gamma - 1}{2} \right\}$$

and

$$max\{0; M\} \leq \frac{\beta + \gamma}{{}_2F_1(1, 2(\beta + \gamma + \delta - m), \beta + \gamma + 1; \frac{1}{2})}$$

and suppose that h function satisfies the inequality

$$Re\, J(-\gamma, h)(z) \geq -\frac{1}{2} - \frac{\beta + \gamma}{{}_2F_1(1, 2(\beta + \gamma + \delta - m), \beta + \gamma + 1; \frac{1}{2})},\ z \in U.$$

If $f \in \mathcal{K}_{(\beta,\gamma),\langle}$ and $Re\, J(\beta, f)(z) > -\delta$, $z \in U$, then $F = A_{(\beta,\gamma),h}(f)$ given by (1.1) is univalent in U. In addition f is also univalent in U.

Different particular cases of the above result, obtained for appropriate choices of $h(z)$, β and γ are given in [5].

Refrences

[1] S.D. Bernardi, *Convex and starlike univalent functions*, Trans. Amer. Math. Soc., **135** (1969), 337–351

[2] T. Bulboacă, *Alpha-convex function images by certain integral operators*, Mathematica (Cluj), **36 (59), 2** (1994), 127–136

[3] T. Bulboacă, *A class of integral operators for alpha-convex functions*, Mathematica (Cluj), **38 (61), 1-2** (1996), 27–33

[4] T. Bulboacă, *Classes of integral operators on Bazilevič functions*, Mathematica (Cluj), (to appear)

[5] T. Bulboacă, *A special class of univalent integral operators*, Glasnik Matematički, (to appear)

[6] W.M. Causey and W.L. White, *Starlikeness of certain functions with integral representations*, J. Math. Anal. Appl., **64** (1978), 458–466

[7] W. Kaplan, *Close-to-convex schlicht functions*, Michigan Math. J., **1** (1952), 169–185

[8] R.J. Libera, *Some classes of regular univalent functions*, Proc. Amer. Math. Soc., **16** (1965), 755–758

[9] S.S. Miller and P.T. Mocanu, *Classes of univalent integral operators*, J. Math. Anal. Appl., **157, 1** (1991), 147–165

[10] S.S. Miller and P.T. Mocanu, *Integral operators on certain classes of analytic functions*, Univalent Functions, Fractional Calculus and their Applications, Halstead Press, J. Willey (1989), 153-166

[11] S.S. Miller, P.T. Mocanu and M.O. Reade, *Starlike integral operators*, Pacific J. Math., **79** (1978), 157–168

[12] P.T. Mocanu, *Une propriété de convexité generaliseé dans la théorie de la representation conforme*, Mathematica (Cluj), **11 (34)** (1969), 127–133

[13] P.T. Mocanu, *On a close-to-convexity preserving integral operator*, Studia Univ. Babeş-Bolyai, Math., **32, 2** (1981), 49–52

[14] P.T. Mocanu, *Convexity and close-to-convexity preserving integral operators*, Mathematica (Cluj), **25 (48), 2** (1983), 177–182

[15] P.T. Mocanu, *Some integral operators and starlike functions*, Rev. Roum. Math. Pures Appl., **31** (1986), 231–235

[16] S. Ozaki, *On the theory of multivalent functions*, Sci. Rep. Tokyo Bunrika Daigaku, A2, **40** (1935), 167–188

Faculty of Mathematics and Computer Science,
Babeş-Bolyai University, 3400 Cluj-Napoca, Romania
e-mail: bulboaca@math.ubbcluj.ro