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# A SURVEY ON NON-ARCHIMEDEAN IMMERSIONS

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**Abstract.** The paper is a survey on the non-archimedean analysis with a selection of those notions and results published in the last thirty years that are involved, as important tools of study, in the recent researches of local and global analysis on manifolds, and also in many other mathematical and physical domains.

Because of the length limitation,the survey does not contain proofs. These can be found in the indicated references.

The original results of the author in the last sections are placed. These are focused on the structure equations, the ultrametric volume formula on a typical subbundle, and the absolute curvature of a p-adic manifold immersed in a finite dimensional Banach space.

## 1. Ultrametric normed fields

The ultrametric normed (or non-archimedean valued ) fields arised in Mathematics in relationship with the congruences in the Number Theory. But there are more reasons to connect them with Analysis because of the existence of strong triangle inequality for distances, which provides the possibility to make many new analogous constructions as those in the classical analysis. The results so obtained confer a new point of view on the Analysis itself, but these are of interest for many other domains, such as : the Quantum Mechanics, the Differential Geometry and, recently, the Numerical Analysis. Each of them seems to be interested of the non-archimedean completion of the rational number field.

It is well known the Analysis over the fields $R$, $C$, $H$ is of archimedean type, but the Analysis, and therefore all other mathematical domains based on it, over any field different from the former is by all means non-archimedean.

There are some fields which cannot be non-trivially normed, such as finite fields (the field $Z_p$, for each prime number p, and the field $\mathbf{F}_q$, for $q = p^n, (n \in N^*)$).

The only one absolute value that can be considered on such fields is the *trivial* one :

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$$\mid x \mid_0 = \begin{cases} 1, x \neq 0 \\ 0, x = 0 \end{cases} \tag{1}$$

which is a non-archimedean absolute value.

Thus, the fundamental topological structures defined on these normed fields will be naturally of non-archimedean type.

The trivial absolute value on every field K can be defined. The pair ( K, $\mid \cdot \mid_0$) is an improper ultrametric field.

But the interest of the mathematicians on the non-trivial cases is focused.

Let us recall the following definition :

A field K (commutative or not) is said to be a *non-archimedean valued ( or ultrametric normed) field* if one can consider the mapping

$$\varphi : \mathrm{K} \rightarrow R_+ \cup \{0\}, (R_+ = (0, +\infty)) \tag{2}$$

such that the following conditions

(KN$_1$) $/ \ \varphi(x) = 0 \ / \Longleftrightarrow / \ \mathrm{x} = 0 \ (\in K) \ /,$

(KN$_2$) $\varphi(x \cdot y) = \varphi(x) \cdot \varphi(y), \ (x, y \in \mathrm{K})$

(**KN$_3$**) $\varphi(x + y) \leq \max ( \ \varphi(x), \varphi(y) \ ), \ (x, y \in \mathrm{K})$

hold.

The mapping $\varphi$ is called *non-archimedean valuation* on K, and the image $\varphi(x) \doteq \mid x \mid (\geq 0)$ is said to be the *non-archimedean absolute value (n.a.v.)* of the element x. The condition (**KN$_3$**) for $\varphi$ is called " the ultrametric inequality".

We remark the fact that every mapping of type (2), which satisfies the ultrametric inequality, naturally, also satisfies the weaker inequality

(KN$_3$) $\varphi(x + y) \leq \varphi(x) + \varphi(y) \ , \ (x, y \in \mathrm{K}) \ ,$

called "the triangle inequality". If moreover, $\varphi$ satisfies also (KN$_1$−KN$_2$), then this mapping will be a "valuation" on K, and the image $\varphi(x) \doteq |x| \ (\geq 0)$ - an "absolute value" (a.v.) of $x$. In this case K is said to be a *valued field* (or a *normed field)* .

However, those valuations which are not at the same time n.a.v. too, i.e. verify the conditions (KN$_1$−KN$_3$) and not verify (**KN$_3$**), are called "archimedean valuations" , and the fields so-valued are called "archimedean valued fields".

Therefore, we distinguish the archimedean valued fields from those which are non-archimedean valued and we will make use of the notations : $(K, |.|) \doteq K$ , and $(K, \mid \cdot \mid) \doteq \mathbf{K}$ , respectively.

Examples of proper archimedean valued fields, $K$ : the rational number field $Q$ , the real number field $R$ , the complex number field $C$ , the quaternion number field $H$ , on which the a.v. by the usual *modulus* is given :

$$|x| = \sqrt{x \cdot \bar{x}} \tag{3}$$

where $\bar{x}$ denotes the conjugate of $x$ ($\in K$ ) .

Here we remark (3) is not the unique a.v. on $K$ . For instance it can be shown that, in the case of the field $Q$ , $|.|^{\alpha}$ , $(0 < \alpha \leq 1)$ , is also an a.v. (BOREVICI, ŞAFAREVICI,[1972]).

The most important examples of proper non-archimedean valued fields, **K**, are :

$1^o$. The rational number field $Q$ , normed with a p-adic absolute value

$$|x|_p = \begin{cases} p^{-m} , & x \neq 0 \\ 0 , & x = 0 \end{cases} \tag{4}$$

where p is a prime number, $x = p^m(a/b)$ , $m \in Z$ , $a, b \in Z \setminus \{0\}$ , $(a,p) = (b,p) = 1$ . This is a n.a.v. and $(Q , |x|_p) \doteq \mathbf{Q}_p$ is an ultrametric field .

$2^o$. The p-adic fields $\mathbf{R}_p$ .

The p-adic fields $\mathbf{R}_p$ (for each prime number p), as well as the real field $R$, by completion of $Q$ are obtained .

We will not recall none of the known ways to construct the p-adic fields; these can be found in (ARTIN, [1969] ; BACHMAN, [1964]) , but instead we will indicate the canonical representation of the elements of such a field .

Thus, every $x \in \mathbf{R}_p$ , $x \neq 0$ , admits a unique representation of the form

$$x = p^m \cdot \varepsilon \tag{5}$$

where $m \in Z$ , and $\varepsilon$ is the sum of a convergent power series as

$$\varepsilon = \sum_{k=0}^{\infty} a_k p^k, \ a_k \in \{0, 1, ..., p-1\} \ , a_0 \not\equiv 0 \pmod{p}) \tag{6}$$

The set $\mathbf{Z}_p = \{p^m \cdot \varepsilon \mid m \in N\}$ is the p-adic integer ring, and U($\mathbf{Z}_p$) consisting in p-adic integers corresponding to $m = 0$ is the multiplicative group of invertible elements in $\mathbf{Z}_p$, that is of the so-called *p-adic units* .

In comparison with U($Z$)= $\{-1, 1\}$ , the mentioned above group is not finite .

On the field $\mathbf{R}_p$ one defines the p-adic absolute value

$$|x|_\rho = \begin{cases} \rho^m , & x \neq 0 \\ 0 , & x = 0 \end{cases} \tag{7}$$

where $\rho \in (0,1)$ and $m \in Z$ is the exponent of $p$ in the canonical expression of $x$ , (5) . This is a n.a.v. on $\mathbf{R}_p$, that becomes in such a manner an ultrametric field .

There are many specific properties of an arbitrary n.a.v. field $\mathbf{K}$, such those which follow from the theorem of characterization of the non-archimedean valuation :

**Theorem 1.** *If $K$ is a commutative field and $\varphi : K \to R_+ \cup \{0\}$ satisfies the condition ($KN_1$) and ($KN_2$) , then the following assertions are equivalent :*

*(i) $\varphi$ is a non-archimedean valuation on $K$ .*

*(ii) $\varphi$ satisfies ($KN_3$) and the set $A = \{\varphi(m \cdot \mathbf{1}) \mid m \in N\}$ is bounded .*

*(iii) $\varphi$ satisfies ($KN_3$) and $\varphi(m \cdot \mathbf{1}) \leq 1$ for any $m \in N$ .*

*(iv) $(\forall)\ x \in K$ : $/\varphi(x) \leq 1/ \Rightarrow /\varphi(x + \mathbf{1}) \leq 1/$ .*

*(v) $(\forall)\ a \in (0, \infty)$ , the mapping $\varphi^a : K \to R_+ \cup \{0\}$ is a valuation on $K$ .*

As a corollary it results that the ultrametric fields does not contain integers whose n.a.v. is $> 1$ .

Moreover, in an arbitrary n.a.v. field $\mathbf{K}$ the following relations hold :

**Proposition 2.** *If $x, y \in \mathbf{K}$, then*

$$|x + y| = \max(|x|, |y|), \quad if \ |x| \neq |y| \tag{8}$$

$$|x - y| = |x + y|, \quad if \ |x| \neq |y| \tag{9a}$$

$$|x - y| \leq |x + y|, \ if \ \ x = y \tag{9b}$$

$$|m \cdot x| \leq |x|, \quad if \ m \in Z, \tag{10}$$

$$|x + m \cdot \mathbf{1}| \leq 1, if \ \ |x| \leq 1, \ m \in Z \tag{11}$$

*and others.*

The proofs can be seen, for instance, in (BOJA, DĂIANU,[1993]).

Finally, we will put in evidence the relationship of these valuations and those of the ordered fields, more exactly, between the archimedean valued fields and the so-called "archimedean fields" , which are ordered fields .

## 2. The norm on ordered fields

First of all we recall that any n.a.v. field $\mathbf{K}$ cannot be ordered.

Any ordered field $(K, \geq)$ admits the disjoint decomposition $K = K_- \cup \{0\} \cup K_+$, where $K_+ = \{a \in K \mid a > 0\}$ .

The *absolute value* on an ordered field is the mapping

$$|\cdot| : K \to K_+ \cup \{0\}, \tag{12}$$

defined by

$$|a| = \max(-a, a) \tag{13}$$

.

It is easy to check that (13) satisfies the conditons $(KN_1 - KN_3)$ of an archimedean absolute value ; however , $(K, |.|)$ is not an archimedean valued field because of (12) that does not coincide with (2) for any field K . If K is a *numerical field* (i.e. a subfield of $C$ , for instance : $Q$ , $Q(\sqrt{2}), ..., R$ , $C$ ), then the absolute value defined above will coincide with the modulus function because, in this case, the function values lie in $R_+ \cup \{0\}$..

An ordered field K is said to be *archimedean* if the axiom of Archimedes

$$(\forall)\ a, b \in K,\ \exists\, n \in N\ :\ n \cdot a = a + ... + a > b. \tag{14}$$

holds .

Any archimedean field is commutative .

It results that the class of archimedean valued fields does not coincide with the class of archimedean fields : there are noncommutative archimedean valued fields (such as $H$ ) , and, conversely, there are nonnumerical archimedean fields (such as the field K(t) of rational functions) .

We recall that the order structure of K(t) is obtained by putting

$$K(t)_+ = \{f \in K(t) | \ \frac{a_0(f)}{b_0(f)} > 0\}, \tag{15}$$

where $a_0, b_0$ are the dominant coefficients of the polynomials $p, q \in K[t]$ for which $f = p/q$ .

### 3. The completion of topological non-archimedean valued fields

Let $(K, |\cdot|)$ be a valued field endowed with a non-trivial norm.

The set $G_K = \{\ |x| \mid x \in K \setminus \{0\}\}$ is the multiplicative group of different from zero values; that is the so-called *value group* of K. It is a subgroup of the multiplicative group $R_+^*$ of positive real numbers.

The surjection $|\cdot| : K \to G_K \cup \{0\}$, satisfying the conditions written above for norms, is a valuatin on K.

If the conditions are $(KN_1)$, $(KN_2)$ and $(\mathbf{KN_3})$ the norm $|\cdot|$ will be of non-archimedean type and only one of the following two assertions is possible :

(a). the norm is *discrete* $\Leftrightarrow$ $G_K$ is a cyclical group,

and, in this case, there is a number $\rho > 1$ such that

$$G_K = \{\rho^m | m \in Z\}, \tag{16}$$

or,

(b). the norm is *dense* $\Leftrightarrow$ $1 \in \overline{\{a \mid a \in G_K, a > 1\}}$, i.e. 1 is an adherent point of the specified set.

*Examples* 1) The p-adic norms on $K=Q$ are discrete.

2) The norm on the field $\mathbf{K}$ of formal power series

$$x = \sum_{n=1}^{\infty} a_i t^{\alpha_i}, \tag{17}$$

defined by

$$|x| = \begin{cases} 2^{-a_1} & , \quad x \neq 0 \text{ and } \mathrm{a}_1 \neq 0 \\ 0 & , \quad x = 0 \end{cases} \tag{18}$$

where $a_i \in K$ (a numerical field) and $(\alpha_i)_{i \in N}$ is an increasing sequence of rational numbers which tends to $+\infty$ , is dense (A. F. MONNA).

For $a \in \mathrm{K}$ and $\varepsilon \in \mathrm{G}_K$ the following sets are important:

$$B_K(a, \varepsilon) = \{x \in \mathrm{K} \,|\, |x - a| \leq \varepsilon\} \quad , \tag{19}$$

$$B_K^-(a, \varepsilon) = \{x \in \mathrm{K} \,|\, |x - a| < \varepsilon\} \quad , \tag{20}$$

these are called the *closed ball* and the *open ball* of center $a$ and radius $\varepsilon$ , respectively.

The **topology** induced naturally on the n.a. valued field $\mathbf{K}$ by the distance

$$d(x, y) = |x - y|, \quad (x, y \in \mathbf{K}) \tag{21}$$

makes $\mathbf{K}$ into a topological field. The field operations are continuously with respect to the topology defined by the distance.

**Theorem 3.** *Two equivalent norms $|\cdot|_1$ and $|\cdot|_2$ define one and the same topology on* $\mathbf{K}$ .

We recall that the considered above norms are equivalent , if the following implication holds :

$$(\forall) x \in \mathbf{K} \text{, with} |x|_1 < 1 \quad \Rightarrow \quad |x|_2 < 1. \tag{22}$$

In this case there exists a real number $v > 0$ such that

$$|x|_1 = |x|_2^{\nu}\ , \quad (\forall)x \in \mathbf{K}\ . \tag{22'}$$

A *basis* of this topology is formed with sets which are simultaneously open and closed. So are the balls in $\mathbf{K}$ : $B_K(a,\varepsilon)$ , $B_K^-(a,\varepsilon)$ and the complementary set $B_K(a,\varepsilon)\backslash B_K^-(a,\varepsilon) = \{x \in \mathbf{K}\ | |x-a| = \varepsilon\}$.

Thus a field endowed with such a topology is of topological dimension zero.

Moreover , if

(*i*) $b \in B_K(a,\varepsilon)$ , then $B_K(b,\varepsilon) = B_K(a,\varepsilon)$ , and

(*ii*) $b \in B_K^-(a,\varepsilon)$ , then $B_K^-(b,\varepsilon) = B_K^-(a,\varepsilon)$ ,

which means : every point of a ball can be taken as center of the ball.

Let us denote by $B_K$ and $B_K^-$ the balls $B_K(0,1)$ and $B_K^-(0,1)$, respectively.

$B_K$ is a ring with unity, called the *valuation ring* . Indeed, we have $|x \cdot y| = |x| \cdot |y| \leq 1$ and $|x-y| \leq \max(|x|,|y|) \leq 1$.

$B_K^-$ is its maximal ideal , because , moreover , if $x \in B_K$ and $y \in B_K^-$ , then $|x \cdot y| = |x| \cdot |y| < 1$ and , so , $xy \in B_K^-$ .

The factorization $B_K \ /\ B_K^- = k$ is a field, called the *residue class field* of $\mathbf{K}$ . The *index* of $\mathbf{K}$ is the natural number $q$ defining the cardinal of $k$ , that is the order of the residue class field . It can be finite or not.

An ultrametric normed field $\mathbf{K} = (K, |\cdot|)$ is said to be *complete* in the usual meaning , that is if every Cauchy sequence in $\mathbf{K}$ converges .

*Example :* $(Q\ ,\ |\cdot|_p)$ is not complete any be the p-adic norm on it. However, as it is well-known, the field $Q$ can be completed in two different ways.

**Theorem 4.** *If the ultrametric normed field* $\mathbf{K}$ *is complete , then the series* $\sum_{n=1}^{\infty} x_n$ , $(x_n \in \mathbf{K})$ , *converges if and only if* $\lim_{n\to\infty} x_n = 0$ .

As it is known, for the archimedean case the last conditon is only necessary, but it is not also sufficient.

**Theorem 5.** *An ultrametric normed field* $\mathbf{K}$ *is* **locally compact** *if and only if it is complete, with a discrete norm, and its index* $q$ *is finite.*

A locally compact field is a **local field** .

A very important and specific notion in relationship with n.a.v. fields is the spherical completeness.

## 4. Spherical completeness

An ultrametric field $\mathbf{K} = (K, |\cdot|)$ is said to be *spherically complete* if every shrinking sequence of its closed balls

$B_{\mathbf{K}}(a_1, r_1) \supset B_{\mathbf{K}}(a_2, r_2) \supset ... \supset B_{\mathbf{K}}(a_n, r_n) \supset ...$

has non-empty intersection.

Note that the property of the sequence $(B_{\mathbf{K}}(a_n, r_n))_{n\in N}$ to be "shrinking" means to have a decreasing order of radii in $R_+^*$ : $r_1 > r_2 > ... > r_n > ...$

This notion can be easily extended to the non-archimedean metric spaces $(X, \mathbf{d})$, whose distances satisfy the strong triangle inequality:

$$\mathbf{d}(x, y) \leq \max(\mathbf{d}(x, z), \mathbf{d}(z, y)), \quad (x, y, z \in X), \tag{23}$$

The best characterizations of the completeness and spherical completeness of metric spaces by CANTOR,[1967], and INGLETON,[1952], respectively, are made

In order to compare them let us recall the following :

(i). A metric space $(X, d)$ is complete iff every shrinking sequence of closed non-empty sets

$$F_1 \supset F_2 \supset ... \supset F_n$$

whose diameters tend to zero has non-empty intersection (and it consists in only one point).

(ii). A non-archimedean metric space $(X, \mathbf{d})$ is spherically complete iff the intersection of every shrinking sequence of its balls is non-empty.

This type of completeness can be equivalently characterized by means of the following result (see KAKOL, [1995]):

**Theorem 6.** *Let $(X, \mathbf{d})$ be a non-archimedean metric space. Then $(X, \mathbf{d})$ is spherically complete iff every family of balls in $X$, with the property that no which two balls are disjoint, has non-empty intersection.*

The spherical completeness implies completeness, both for metric spaces and for normed fields, clearly, only in the non-archimedean cases.

A proof of this assertion for the n.a.v. fields can be obtained immediately:

It results by association to every Cauchy sequence $(x_n)_{n\in N}$ of the ultrametric normed field $\mathbf{K}$ the family of balls $(B_{\mathbf{K}}(x_n, r_n))_{n\in N}$, where

$$B_{\mathbf{K}}(x_n, r_n) = \{x \in \mathbf{K} |\ |x - x_n| \leq \max_{k\geq n} |x_{k+1} - x_k|\}.$$

For a non-archimedean metric space $(X, \mathbf{d})$, the assertion can be obtain analogously.

The converse assertion fails: there are complete n.a.v. fields or non-archimedean metric spaces which are not spherically complete.

For instance, the real number field $R$ is complete with respect to the natural metric defined by the usual absolute value on it, but it is not spherically complete, while any p-adic field $\mathbf{R}_p$ (for each prime number $p$ ) is both complete and spherically complete with respect to the metric defined by the p-adic norm $|\cdot|_p$ .

The Ostrovski's theorem represents the best characterization of the completion of the rational number field $Q$, which states:

$i)$ every archimedean norm on $Q$ is of the form $|.|^\alpha$, $0 < \alpha \leq 1$ ,

$ii)$ every non-archimedean norm on $Q$ is one of the p-adic norms $|\cdot|_p$ , with

$$|x|_\rho = \rho^{v_p(x)} , \tag{24}$$

where $$v_p(x) = \begin{cases} m , & x \neq 0 \text{ and } x = p^m \cdot \varepsilon \\ +\infty , & x = 0 \end{cases}$$

So, $(\mathbf{R}_p, |\cdot|_\rho)$ becomes a complete ultrametric field.

From the Ostrovski's theorem it follows that every complete field K , with respect to a non-trivial metric and of zero characteristic, contains as a topological subfield either $Q$ , endowed with the discrete topology, or one of the p-adic fields $\mathbf{R}_p$ .

In the mentioned above paper of KAKOL also can be found the following very interesting example of non-archimedean metric space which is complete but it is not spherically complete :

$$(N,\mathbf{d}), \quad \mathbf{d}\,(\mathrm{m,n}) = \begin{cases} 1 + \max(1/m, 1/n) , & m \neq n \\ 0 , & m = n \end{cases} \tag{25}$$

Indeed, every sequence of balls in $N$ whose radii tend to zero has non-empty intersection, because every ball whose radius is smaller than 1 contains exactly one point, and, so, the Cantor's theorem holds; but the balls $B_{1+\frac{1}{1}}(1), B_{1+\frac{1}{2}}(2), \ldots$ form a decreasing (shrinking) sequence with empty intersection, and, so, the Ingleton's assertion does not hold.

It also can be shown (see Gh.ISAC, Gh.MARINESCU,[1976] ), that:

**Theorem 7.** *Every ultrametric normed field* $\mathbf{K} = (K, |\cdot|)$, *which is complete and whose norm* $|\cdot|$ *is discrete, is spherically complete.*

Moreover, every local field is spherically complete.

But, there exist ultrametric fields whose norm are dense and these also are spherically complete. Such an example is given by the field $\mathbf{K}$ whose elements are the power series (17) with the n.a.v. defined by (18).

The spherical completeness is of crucial importance for the study of Banach spaces over non-archimedean valued fields, especially in relationship with the Mackey topology and the Hahn- Banach extension property for locally convex spaces.

## 5. Non-archimedean Banach spaces

In the sequel by $\mathbf{K} = (\mathrm{K}, |\cdot|)$is always meant a complete n.a.v. field, whose norm is non-trivial.

Let E be a Banach space over **K**.

The norm $\|.\|$ on E satisfies the usual properties, excepting the triangle inequality, which is replaced by the strong triangle inequality $(\forall) x, y \in E \ : \ \|x+y\| \leq \max (\|x\|, \|y\|)$,

A set $\mathrm{X} \subset \mathrm{E} \setminus \{0\}$ is said to be an *orthogonal system* of E if

$$\left\| \sum_{i=1}^{m} \xi_i x_i \right\| = \max_i |\xi|_i \cdot \|x_i\| \ , \qquad (26)$$

for all $m \in N$ , such that $\{x_1,...,x_m\} \subset \mathrm{X}$ , with $x_i \neq x_j$ , when $i \neq j$ , and $\xi_i \in \mathbf{K}$ , $(i \in \overline{1,m})$. More precisely, having the property (26), X is said to be a *norm-orthogonal* set in E .

If the closed linear hull $\langle X \rangle$ spanned by an orthogonal system X coincides with E itself, then X is an *orthogonal basis* of E .

Let us denote by $\mathbf{K}^n$ $(n \in N^*)$, the normed $\mathbf{K}-$linear space of n-tuples $x = (x^1, ..., x^n)$, $(x^i \in \mathbf{K}$ , $i \in \overline{1,n})$. Then $\mathrm{E}=\mathbf{K}^n$ is a Banach space, because finite dimensional normed spaces over **K** are always Banach spaces, whith the norm of vectors, defined by

$$\|x\| = \max_i |x_i| \ , (x \in \mathbf{K}^n), \qquad (27)$$

like that one can obtain from (26).

The canonical basis $\{e_1, ..., e_n\} \subset \mathbf{K}^n$ forms obviously an orthogonal set in the previous meaning.

Analogously, as in the case of ultrametric fields, the spherical completness of a n.a. Banach space by means of the shrinking sequences of balls, $(B_E(a_n, x_n))_{n \in N}$ , is defined.

Functional Analysis also makes use of the so-called "t-orthogonality" in a non-archimedean Banach space E.

A system $\mathrm{X} \subset E \setminus \{0\}$ is said to be *t-orthogonal* if (26) is replaced by (see C.Perez-Garcia,[1995])

$$\left\| \sum_{i=1}^{m} \xi_i x_i \right\| \geq t \cdot \max_i |\xi_i| \cdot \|x_i\| \ , \qquad (26')$$

with $t \in (0,1]$ and all the previous conditions remain.

For $t = 1$ , (26') becomes (26).

If E and F are two n.a.Banach spaces over the ultrametric field , and T:E→F is a continuous linear map then, as usually, we have:

- $\|T\| = \sup\{\|Tx\| \,/\, \|x\| \mid x \in E\,,\, x \neq 0\}$,
- T is an isomorphism iff it is a bijection and $T^{-1}$ is continuous.
- T is an isometry iff $\|Tx - Ty\| = \|x - y\|$, $(\forall x, y \in E)$ . Note that this isometry need not to be necessarily additive or homogeneous.

The following well-known properties of every linear operator $T(\in L(E))$ on a n.a.Banach space $E=\mathbf{K}^n$ are important to define the differentiability on the non-archimedean normed spaces (see: SCHIKHOF,[1970]):

$(i)$. T is an isometry$\Leftrightarrow |\det T| = 1$ and $|a_{ij}| \leq 1$ for all entries $a_{ij}(\in\mathbf{K})$ of the operator matrix.

$(ii)$. If $\mathbf{K}$ is a local field and $\tilde{T}$ is bijective, then $\tilde{T}=T\cdot D\cdot T_1$ , where T is a linear isometry, D is the diagonal map, and $T_1$ maps the unit ball $B_E(0,1)$ onto itself and $|\det T_1| = 1$. Note that all operators belong to $L(\mathbf{K^n})$ .

The proof can be seen in (MONNA,[1948]).

$(iii)$.If T is a linear isomorphism and S∈L(E) is continuous and $\|T - S\| < \|T^{-1}\|^{-1}$ , then S is an isomorphism and $T(B_E(O,r))=S(B_E(O,r))$, for every ball with center $O(\in E)$.

When the n.a.Banach spaces E and F are spherically complete a very interesting property of semi-Fredholm and Tauberian mappings T∈L(E,F) is proved (see: J.MARTINEZ-MAURICA,T.PELLÓN,[1987]):

T is semi-Fredholm⇔T is Tauberian.

Note that if $\mathbf{K}$ is not spherically complete (as in the case of $R$, $C$ ) the properties : "T is semi-Fredholm"(i.e. KerT is finite dimensional and T(E)⊂F is closed) and "T is Tauberian"(i.e. $(T'')^{-1}(F)\subset E$ , where $T' : F' \to E'$ denots the adjoint between the corresponding topological duals) are independent.

A subset U⊂E is called *absolutely convex* if

$$/(\forall)x, y \in U \;;\; \alpha, \beta \in B_{\mathbf{K}}/ \Rightarrow /\alpha x + \beta y \in U/ \tag{28}$$

where $B_{\mathbf{K}}$ denotes the unit ball of $\mathbf{K}$ with center at O.

The absolutely convex hull of an arbitrary system S⊂E will be denoted by **co** S,
(or [S]).

A subset A⊂E is said to be a *compactoid* of E if for every zero-neighbourhood U (i.e. $B_E(O,r) \subset U$ , $r > 0$) there exists a finite system $S=\{x_1, ..., x_n\}\subset E$ such that

$$A\subset U+\mathbf{co}\,S\,, \tag{29}$$

where $B_E(O,r)$ denotes the closed ball of radius r in E, with respect to the norm $\|.\|$ on the $\mathbf{K}$-space E.

The compactoidity was introduced in the non-archimedean Analysis in connection with n.a.Fredholm theory of compact operators.

This theory depends on the fact that the Banach space E is defined over n.a.v.field $\mathbf{K}$ , which is locally compact or not.

We recall that, in the first case, an operator $T \in L(E)$ is *compact* if $T(B_E)$ is precompact in E. The study of compact operators was developed in the sixties by many researchers, such as : R.Ellis, J.P.Serre, J.van Tiel, J.Martinez-Maurica, and others, and a very important result states:

**Theorem 8.** *If E is a non-archimedean Banach space over a locally compact field* $\mathbf{K}$ *and* $T \in L(E)$ *is compact, then* $I+T$ *is a semi-Fredholm operator, where I denotes the identity map on E.*

But when $\mathbf{K}$ is not locally compact (such as in some of the cases in p-adic Analysis), the convex precompact sets are reduced to a single point, and so the previous result does not have sense.

Thus, a solution to going out from this difficulty was found in replacing the concept of precompactness by that of compactoidity, when E is defined over a field $\mathbf{K}$ which is not locally compact.

So, defining compact operators $T \in L(E)$ by the condition that $T(B_E)$ is a compactoid in E, W.H.Schikhof proved in 1989 that Theor. 8 is also valid in the cases of non-locally compact fields.

But J.Martinez-Maurica and N.de Grande-de Kimpe, [1989] shown the previous assertion is true only if E is a complete normed space over $\mathbf{K}$ . In the cases of non-complete spaces a new notion is needed, that of *semicompact operators* . As was defined in the mentioned above paper, $T \in L(E,F)$ is called *semicompact* if there exists a compactoid and completing subset D of F, such that $T^{-1}(D)$ is a zero-neighbourhood in E, where E and F are locally convex spaces(i.e., the topologies are defined by families of non-archimedean seminorms).

## 6. Differentiation in non-archimedean Banach spaces

Throughout the section, $(E,\|.\|)$ and $(F,\|.\|)$ are Banach spaces over $\mathbf{K}$ , where $\mathbf{K}$ is an ultrametric normed field, which is complete and its valuation is supposed to be non-trivial.

The definitions and results that will be in our interest follow almost exclusively the papers of Schikhof [1970] and Tison [1964] .

A function $f:U(\subset E)\to F$ is said to be *differentiable* at an interior point $\mathbf{x} \in U$ if there exists a continuous linear mapping $l_{\mathbf{x}} : E \to F$ and a function $\varepsilon_{\mathbf{x}} : U \setminus \{\mathbf{x}\} \to F$, such that for all $\mathbf{h} \in E$ for which $\mathbf{x}+\mathbf{h} \in U$

$$f(\mathbf{x}+\mathbf{h})=f(\mathbf{x})+l_{\mathbf{x}}(\mathbf{h})+\varepsilon_{\mathbf{x}}(\mathbf{h}) \ , \tag{30}$$

where $\lim_{\mathbf{h}\to 0} \|\varepsilon_{\mathbf{x}}(\mathbf{h})\| \cdot \|\mathbf{h}\|^{-1} = 0$ .

$l_{\mathbf{x}} \in L(E,F)$ is called the *derivative* of f in **x**, being uniquely determined by (30).

If E and F are finite-dimensional spaces, the mapping $l_{\mathbf{x}}$ can be represented by means of a functional matrix whith respect to a pair of bases of E×F.

Note that the previous notions are translated from real and complex analysis.

But other properties are essentially different, especially the proofs in a proper manner being made.

So is "the local invertibility theorem":

**Theorem 9.** *Let $f: U(\subset E)\to E$ be a differentiable function at an interior point $\mathbf{x}\in U$ and continuous in a neighbourhood $V_{\mathbf{x}} \subset U$. If its derivative $l_{\mathbf{x}}$ $(\in L(E))$ is an isomorphism, then, for a sufficiently small ball $B_E(O,r)$,*

$$f:\mathbf{x}+B_E(O,r) \mapsto f(\mathbf{x})+l_{\mathbf{x}}(B_E(O,r)) \ , \tag{31}$$

*homeomorphically.*

This result can be wholly extended to a neighbourhood of **x**, $V_{\mathbf{x}} \subset U$, in the case when U is an open set and, moreover, $\mathbf{x}\mapsto l_{\mathbf{x}}$ $(\mathbf{x}\in U)$ is continuous, by replacing **x** with any $\mathbf{y}\in V_{\mathbf{x}}$ in (31).

In this case f is an open mapping.

**Theorem 10.** *Let $\sigma : U\to V$ be a homeomorphism between the open sets $U, V\in E$. If $\sigma$ is differentiable at **x** and the derivative $l_{\mathbf{x}}$ is an isomorphism, then $\sigma^{-1}: V\to U$ is differentiable at $\sigma(\mathbf{x})$ with derivative $l_{\sigma(\mathbf{x})} = l_{\mathbf{x}}^{-1}$.*

Note that by omitting the condition "$l_{\mathbf{x}}$ is an isomorphism" the two previous theorems cannot be true, because of the existence of locally constant functions, i.e. of the functions which are constant on some sets that are open and closed simultaneously.

It is important to remark that there exist homeomorphisms with everywhere vanishing derivatives.

*Example :* $f:\mathbf{Z}_p \to \mathbf{Z}_p$ , $f(\mathbf{x}):=\sum_{i=0}^{\infty} a_i p^{2i}$, for every integer $\mathbf{x}=\sum_{i=0}^{\infty} a_i p^i$ , $a_i \in \{0,1,...,p-1\}$. It results $|f(\mathbf{y}) - f(\mathbf{x})|_p = |\mathbf{y} - \mathbf{x}|_p^2$ , $(\mathbf{y}=\mathbf{x}+\mathbf{h}\in\mathbf{Z}_p)$, which implies that $l_{\mathbf{x}} = f'(\mathbf{x})=0$. There one can remark that f is a homeomorphism of $\mathbf{Z}_p$ onto $f(\mathbf{Z}_p)\neq\mathbf{Z}_p$ , and it is known that homeomorphisms of $\mathbf{Z}_p$ onto $\mathbf{Z}_p$ with vanishing derivative does not exists.

Let $\mathbf{K}$ be a local field and $\Omega(\mathbf{K}^n)$ denotes the class of all compact open subsets of $\mathbf{K}^n$ , $(n \in N)$.

A map $\sigma$ :U→V, (U,V∈ $\Omega(\mathbf{K}^n)$), is said to be a *diffeomorphism* if $\sigma$ is a homeomorphism and $\sigma$ and $\sigma^{-1}$ are differentiable.

The necessary and sufficient conditions in order that $\sigma$ :U→V be a diffeomorphism are:

(*i*) $\sigma$ is differentiable and injective

(*ii*) $\sigma$(U) is dense in V

(*iii*) the Jacobian of $\sigma$ be nonzero everywhere.

On $\Omega(\mathbf{K}^n)$ can be defined an equivalence relation by "U∼V ⇔ there exists a diffeomorphism of U onto V".

It is known that there are exactly $(q-1)$ equivalence classes if $q=|k|$, where $k$ is the residue class field of $\mathbf{K}$ . The open sets which belong to such a class of equivalence are said to be *of the same type* .

If U and V are disjoint union of $m$ and $n$ open balls in E, respectively, the sufficient condition that U and V be of the same type is : $m \equiv n$(mod q-1).

A linear isomorphism A:$\mathbf{K}^n \to \mathbf{K}^n$ does not change the type.

A map $\sigma$ :U(⊂E)→E is called *locally linear function* if each point $\mathbf{x}$ of U has a neighbourhood $\mathrm{V}_{\mathbf{x}}$(⊂U) such that for al $\mathbf{y} \in \mathrm{V}_{\mathbf{x}}$ : $\sigma(\mathbf{y}) = \mathbf{x} + \mathrm{A}\mathbf{y}$ , for some linear map A:E→E.

**Theorem 11.** (W.H.Schikhof, [1970]) *Let U, V∈ $\Omega(\mathbf{K}^n)$. The following assertions are equivalent:*

*(a) U and V are of the same type.*

*(b) There is a differentiable homeomorphism of U onto V, for which the functional determinant vanishes nowhere.*

*(c) There is a locally linear homeomorphism of U onto V.*

For analytic functions the above result was obtained by J.P.Serre [1965].

The local behaviour of a p-adic function ($\mathbf{K}=\mathbf{R}_p$, for some prime number $p$) was studied by F.Tison [1964]. He used the notion of *uniformly regular function* (u.r.f.) on U.

A function f:U($\subset \mathbf{R}_p^n$) $\to \mathbf{R}_p^n$ is said to be *u.r.f.* if : $\exists b(\mathbf{x}) > 0$ and $r(\mathbf{x}) > 0$ such that

$$\left\|\mathbf{x}' - \mathbf{x}\right\| \le r(\mathbf{x}) \;\Rightarrow\; \left\|\frac{f(\mathbf{x}')-f(\mathbf{x})}{\mathbf{x}'-\mathbf{x}}\right\| \ge b(\mathbf{x}) \;, \tag{32}$$

for all interior points $\mathbf{x} \in$U.

**Theorem 12.** *A differentiable mapping which is* **regular** *and* **strictly derivable** *at every point* $\mathbf{x}_0 \in U$*, is uniformly regular on* $U$.

A u.r.f. on U is invertible on every ball of radius $r$ , $B_E(\mathbf{x}, r) \subset U$ ; the inverse $g=f^{-1}$ , defined on $f(B_E(\mathbf{x}, r))$ , is also u.r.f.

The mapping f in the previous example is *singular* at every point $\mathbf{x} \in \mathbf{Z}_p$, so it is not a u.r.f., but, $\mathbf{x} \mapsto e^{\mathbf{x}}$ is u.r.f. on the ball $B_E$ $(\subset \mathbf{R}_p)$.

## 7. The immersion of a differentiable manifold in an ultrametric Banach space

Consider the Banach space $(E, \|.\|)$ over an ultrametric field $\mathbf{K}$ of zero characteristic and assume $\dim_{\mathbf{K}} E = n$. Then, E is isomorphic with the linear normed space $\mathbf{K}^n$. Because $\mathbf{K}$ is complete with respect to a non-trivial metric, according to the Ostrovski's theorem, it results $\mathbf{K}$ is an extension of one of the p-adic fields $\mathbf{R}_p$.

In order to simplify the manner of writing we will consider $\mathbf{K}=\mathbf{R}_p$, for a fixed prime number $p$, and define its norm by the formula (7).

On the linear space $(\mathbf{R}_p)^n$ we define the following nondegenerate symmetric bilinear form (Boja [1994])

$$b_k^n(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} \tilde{e}_i x^i y^i \tag{33}$$

where

$$\tilde{e}_i = p^z \cdot \varepsilon_i \quad , \quad z = \begin{cases} o \ , & i \leq k \\ 1 \ , & i > k \end{cases} \quad , \ \varepsilon_i \in U(\mathbf{Z}_p). \tag{33'}$$

The number $k\ (\leq n)$ of coefficients which are not divisible by $p$ is called "the p-adic index" of the space and the space of index $k$ will be denoted by $(\mathbf{R}_p)_k^n$ .

A basis $\{\mathbf{v}_i\}_n$ of this space is said to be "$p(k)$−orthonormal" if

$$b_k^n(\mathbf{v}_i, \mathbf{v}_j) = \tilde{e}_i \delta_{ij} \ , \tag{34}$$

where $\delta_{ij}$ are Kronecker symbols. With respect to such a basis the quadratic form $q(\mathbf{x}) := b_k^n(\mathbf{x}, \mathbf{x})$ will have a canonical form. It is known (Browkin [1966]) that the p-adic index depends only on $q$ , and if $k < n$ we will say that such a quadratic form is "indefinite".

Thus, identifying E with the space $(\mathbf{R}_p)_k^n$ it becomes a finite dimensional Banach space of $p$-signature $(k, n-k)$, whose norm is defined by

$$\|\mathbf{x}\|_p = |b_k^n(\mathbf{x}, \mathbf{x})|_\rho^{\frac{1}{2}} \tag{35}$$

Also it can be endowed in a natural way (see Boja [1990] ) with a differentiable manifold structure : it is a complete flat manifold, with the metric (a differentiable tensor field $\bar{g}_k^n$ , or simply, $\bar{g}$ ) obtained by parallel translation at each point from the bilinear form $b_k^n$ defined as (33). The pseudo-Riemannian metric on E one defines, globaly, by

$$d\bar{s}^2 = \sum_{i,j=1}^{n} \bar{g}_{ij}(\mathbf{x}) dx^i dx^j \ , \tag{36}$$

where $(x^i)$ are the coordinates of the point $\mathbf{x}$ with respect to the chosen basis and, in the case of in which this one is a $p(k)$ -orthonormal basis, we have

$$\bar{g}_{ij}(\mathbf{x}) = \tilde{e}_i \delta_{ij} \ (\in \mathbf{R}_p) \ , \ (i,j \in \overline{1,n}) \tag{36'}$$

Let $\chi = (\mathbf{x}; \mathbf{v}_1, ..., \mathbf{v}_n)$ be a moving frame in $(\mathbf{R}_p)_k^n$, where $(\mathbf{v}_i)_n$ is a $p(k)$-orthonormal basis of the tangent pseudo-Euclidean space $\mathrm{T}_x(\mathbf{R}_p)_k^n$, identified with E itself. Then we have $\bar{g}_x(\mathbf{v}_i, \mathbf{v}_j) = \bar{g}_{ij}(\mathbf{x})$ , given by (36').

Let $\mathrm{F}(n;k)$ be the space of all frames on E. Then $\mathrm{F}(n;k)$ is a differentiable manifold of dimension $n(n+1)/2$. Also it is a fibre bundle over E with structural group $\mathrm{SO}(n;k)$ , having p-adic numbers as entries of the component matrices. On $\mathrm{F}(n;k)$ are defined in a natural way the 1-forms $\bar{\theta}^i$ , $\bar{\omega}_j^i$ , $(i,j \in \overline{1,n})$ as follows:

The coordinate form $\bar{\theta} = (\bar{\theta}^i)_n$ is defined by the relations $\bar{\theta}^i(\mathbf{v}_j) = \delta_j^i$, so that $\Theta = (\mathbf{x}; \bar{\theta}^1, ..., \bar{\theta}^n)$ will be the dual frame of $\chi$.

The connection form $\bar{\omega} = (\bar{\omega}_j^i)_{n^2}$ is defined by the relations $\bar{\nabla}_X \mathbf{v}_j = \sum_{i=1}^{n} \bar{\omega}_j^i(\mathrm{X}) \mathbf{v}_i$ , $(\mathrm{X} \in \mathrm{T}(\mathbf{R}_p)_k^n)$ , $\bar{\nabla}$ being the unique affine connection, satisfying the following two conditons: it has no torsion and the pseudo-Riemannian metric $\bar{g}$ is parallel, that is the well-known Riemannian connection.

These differential forms satisfy the structure equations (Boja [1990]) :

$$d\bar{\theta}_i = \sum_{j=1}^{n} \tilde{e}_j^{-1} \bar{\theta}_j \wedge \bar{\omega}_{ij} \tag{37}$$

$$d\bar{\omega}_{ij} = \sum_{k=1}^{n} \tilde{e}_k^{-1} \bar{\omega}_{kj} \wedge \bar{\omega}_{ik} \tag{38}$$

(without torsion and curvature), where, for $i \in \overline{1,n}$,

$$\tilde{e}_j^{-1} = \varepsilon_j' \ , \text{ if } \ i \le k \ , \text{ when } \tilde{e}_i = \varepsilon_i$$

$$\tilde{e}_j^{-1} = p^{-1} \varepsilon_j' \ , \text{ if } \ i > j \ , \ (\varepsilon_i, \varepsilon_i' \in U(\mathbf{Z}_p)).$$

We observe that with respect to the chosen frame $\chi$, denoting the sums $\sum_{j=1}^{n} \bar{g}_{ij} \bar{\theta}^j$ and $\sum_{k=1}^{n} \bar{g}_{kj} \bar{\omega}_j^k$ , respectively, by $\bar{\theta}_i$ and $\bar{\omega}_{ij}$ , we have

$$\bar{\theta}_i = \tilde{e}_i \bar{\theta}^i \ , \quad \bar{\omega}_{ij} = \tilde{e}_i \bar{\omega}_j^i \tag{39}$$

where $i \in \overline{1,n}$ is not a summation index. These also satisfy the relations of $p$-symmetry

$$\tilde{e}_i \bar{\omega}_j^i + \tilde{e}_j \bar{\omega}_i^j = 0 \ , \tag{40}$$

Let $\mathbf{M}$ be a differentiable manifold of class $\mathrm{C}^\infty$, whose modelling space (the coordinate space) is the non-archimedean Banach space $\mathrm{E}=\mathbf{K}^n$, where

$\mathbf{K}=\mathbf{R}_p$. We also consider E endowed with a metric $\|.\|_p$.

The necessary algebraic-topological properties of E in an extensive form were recalled in the above.

Consider $\mathbf{M}$ of dimension $m(< n)$ is immersed in E. If the immersion is $\phi$ and we denote by $\mathbf{M}_p = \phi(\mathbf{M})(\subset \mathrm{E})$ the image, then at every point $\mathrm{M} \in \mathbf{M}_p$ for the induced mapping of the tangent space we have: $\mathrm{rank}\phi_*(\mathrm{M}) = \mathrm{rank}[\frac{\partial x^i}{\partial u^\alpha}]_{\mathbf{x}} = m$ , where $\phi(\mathrm{M}) \doteq \mathbf{x} = (x^i)_n$ is a vector-valued function and $x^i = x^i(u^\alpha)$ , $(\alpha \in \overline{1,m})$, are the equations which, analytically, represent the mapping $\varphi' \circ \phi \circ \varphi^{-1}$, $(U, \varphi)$ being a local chart on $\mathbf{M}$ and $(\bar{U}, \varphi' = \mathrm{Id\ E})$ is a chart of $\mathrm{E} = (\mathbf{R}_p)^n_k$, such that $\mathbf{x} \in \bar{U}$. Certainly, $x^i$ are p-adic differentiable functions of $(u^\alpha(\mathrm{M}))$.

As well we can assume that, locally, for a neighbourhood $U \subset \mathbf{M}$ of the point M, $\phi$ is an embedding, case in which $\phi(U)$ is without multiple points and we can identify $U$ (and, also $\mathbf{M}$) with its image by inclusion $(\mathbf{M}_p)$. In E this one is a disjoint union of open balls, hence it is totally disconnected and we may consider injective nonconstant functions on $U$. Then, $U$ will be a submanifold in E of which equations are given above, where $(u^\alpha) \in \varphi(U)$ are coordinate functions in the neighbourhood $U$ of the point M, and $(x^i)$ are the coordinates of the same point in the neighbourhood $\bar{U}$ in E, such that $U = \bar{U} \cap \phi(\mathbf{M})$.

Now we associate to the immersion $\phi : \mathbf{M} \to \mathbf{M}_p (\subset \mathrm{E})$ the fibre bundle $\mathbf{B}$ , whose bundle space is a subspace of $\mathbf{M}_p \times \mathrm{F}(n;k)$, consisting of all elements $(\mathrm{M}, \chi)$ such that the first $m$ vectors $\mathbf{v}_\alpha \doteq \mathrm{X}_\alpha$ , $(\alpha \in \overline{1,m}$ ), of $\chi$ are tangent to an open set $\mathrm{U} \subset \phi(\mathbf{M})$ and the last $n-m$ vectors $\mathbf{v}_{m+1} \doteq \xi_{m+1}$, ,$\mathbf{v}_n \doteq \xi_n$ are normal vectors at $\phi(\mathrm{M})$ $(\subset \mathrm{U})$. If we consider the Chern's mapping

$$\mathrm{B} \xrightarrow{i} \mathbf{M}_p \times \mathrm{F}(n;k) \xrightarrow{\lambda} \mathrm{F}(n;k)$$

where "$i$" is the inclusion and "$\lambda$" is the projection into the second factor, and putting

$$\theta^i = (\lambda \circ i)^* \bar{\theta}^i, \quad \omega^i_j = (\lambda \circ i)^* \bar{\omega}^i_j$$

we can verify for $\theta^i$, $\omega^i_j$ the structure equations (37) and (38).

On the immersed submanifold $\mathbf{M}_p$ the induced metric $g$ by $\bar{g}$ is given by the formula

$$g_M(X_1, X_2) := \bar{g}_{\mathbf{x}}(\phi_* X_1, \phi_* X_2) , \qquad (X_1, X_2 \in \mathrm{T}_M \mathbf{M} ) \tag{41}$$

Endowed with this metric $g$ the manifold $(\mathbf{M}_p, \phi)$ becomes an isometric immersed submanifold into $(\mathbf{R}_p)^n_k$.

Thus from the definition of B it follows that $\theta^r = 0$ for all $r = m+1, ..., n$ and $\theta^1, ..., \theta^m$ are linearly independent 1-forms. Also, taking into account of (39), the first structural equation can be written under the form

$$\alpha \in \overline{1,m}: \quad d\theta^{\alpha} = \sum_{\beta=1}^{m} \theta^{\beta} \wedge \omega^{\alpha}_{\beta} \tag{42}$$

$$r \in \overline{m+1,n}: \ d\theta^{r} = 0 = \sum_{\alpha=1}^{m} \theta^{\alpha} \wedge \omega^{r}_{\alpha} \tag{43}$$

## 8. The ultrametric volume element and the total absolute curvature of the immersion

Let $T^{\perp}\mathbf{M}_p$ be the normal bundle of the submanifold whose bundle space consists of al points $(\mathbf{x}, \xi)$ such that $\mathbf{x} \in \phi(\mathrm{M})$ and $\xi$ is a normal vector at $\mathbf{x}$ ; this one is a vector bundle of $(n-m)$ -planes over $\phi(\mathrm{M})$.

If $\nu$ denotes a unit vector into direction of $\xi$, then in the real case there are only two unit normal vectors to the submanifold at a given point collinear with $\xi$ : $\pm\nu$ , usually the sign being chosen with respect to the orientation, while in a non-archimedean case there are an infinite number of related unit vectors at a point into a given direction. For instance in the case of p-adic manifolds if $\nu$ is a vector of norm 1, then any collinear vector of the norm $\nu' = \varepsilon \cdot \nu$, for all $\varepsilon \in \mathrm{U}(\mathbf{Z}_p)$ will have the same norm, that is

$$\left\| \nu' \right\|_p = |\varepsilon|_p \cdot \|\nu\|_p = \rho^0 \cdot 1 = 1.$$

We denote by $[\nu(\mathbf{x})]$ the class of all unit vectors collinear with the unit normal vector $\nu$ at the point $\mathbf{x} \in \mathrm{M}_p$.

Let us consider the subbundle $\mathrm{B}_1'$ of the normal bundle $T^{\perp}\mathbf{M}_p$ whose bundle space consists of all pairs $(\mathbf{x}, v')$ of $\mathbf{M}_p \times \mathrm{E}$ for all points of the immersed manifolds and towards all normal directions to it, such that $\left\| \nu' \right\|_p = 1$ . Thus, the fiber over $\mathbf{x}$ , defined by $\mathrm{S}_{\mathbf{x}}$, consists of all clases $[\nu(\mathbf{x})]$. Choosing an ordering of the moving frames $\chi$ such that at each point $\mathbf{x}$ the last vector $\xi_n$ of the local $p(k)$-frame coincides with one of the unit vectors $\nu'$ to $\phi(\mathrm{M})$ we obtain the following result (Boja [1994] ) :

**Theorem 13.** *The volume element $dV_1$ of $B_1'$ associated the m-submanifold $\mathbf{M}_p$ of p-signature $(k', m-k')$ isometric immersed into the Banach space $E{=}(\mathbf{R}_p)^n_k$ , $(k \geq k'+1)$ , is a $(n-1)$-form given by the formula*

$$dV_1 = (\varepsilon_n')^{n-m-1} \cdot \tilde{e}_1^{-1} \cdot \ldots \cdot \tilde{e}_m^{-1} dV_m \wedge d\sigma_{n-m-1} \tag{43}$$

*where $\varepsilon_n'$ is a p-adic "unit" such that $\varepsilon_n' \cdot \varepsilon_n = 1$ , and*

$$dV_m = \theta_1 \wedge \ldots \wedge \theta_m \quad , \quad d\sigma_{n-m-1} = \omega_{n(m+1)} \wedge \ldots \wedge \omega_{n(n-1)}.$$

In order to compute the curvature of the immersion $\mathbf{M}_p \hookrightarrow \mathrm{E}$ , let us denote by $\mathrm{S}_0^{n-1}(1)$ the unit hypersphere of $(\mathbf{R}_p)^n_k$ with the center in origin and by $d\Sigma$ its volume element. Consider the map: $\gamma : \mathrm{B}_1' \to \mathrm{S}_0^{n-1}(1)$, defined by $\gamma(\mathbf{x},\nu') = \nu'$ , $\nu' \in[\nu(\mathbf{x})]$ which generalizes the well-known normal mapping of Gauss in the classical theory of surfaces. Denoting by $\gamma^*$ the dual mapping on the differential forms induced by $\gamma$, the **total absolute curvature** in the sense of Chern and Lashof of an immersed compact manifold is defined (as in the real case) by the formula

$$\mathrm{TA}(\phi,\rho)=\frac{1}{c_{n-1}}\int_{B_1'} |\,\nu^* d\Sigma\,|_\rho \quad , \tag{44}$$

where $c_{n-1}$ denotes the volume of the unit $(n-1)$-sphere.

The main result of this section is to show the dependence of the absolute curvature on the submanifold and the space signatures with respect to an ultrametric normed field.

Thus, we have the following (Boja [1996] ):

**Theorem 14.** *Let* $\mathbf{M}_p$ *be a* $m$*-dimensional compact* $p$*-adic manifold isometric immersed in the space* $(\mathbf{R}_p)^n_k$ *of* $p$*-signature* $(k,n-k)$ *defined by the bilinear form* $b^n_k$ , $(n>m)$.

*Then the absolute curvature of the immersion* $\phi : \mathbf{M} \to (\mathbf{R}_p)^n_k$ *has the form*

$$TA(\phi,\rho)=\rho^{-m+k'}\cdot\int_{\phi(\mathbf{M})} |\det[h^n_{\alpha\beta}]\,|_\rho \; dV_m \quad , \tag{45}$$

*where* $h^n_{\alpha\beta}\,(\in\mathbf{R}_p)$ , $(\alpha,\beta=1,...,m)$, *are the coefficients of the second fundamental form of the immersion and* $dV_m$ *is the volume element of the immersed submanifold of* $p$*-signature* $(k',n-k')$.

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