FILOMAT (Niš) **12:2** (1998), 21–29

THE BOLZANO-WEIERSTRASS THEOREM AND SET-THEORY

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Abstract. We analyse one of the classical proofs of the Bolzano-Weierstrass theorem. Cardinal numbers are associated with crucial steps of the proof. These are intended to measure, in some sense, the mathematical power of the statement proved in that step. By comparing the cardinal numbers we gather some information on which implications in are not provably reversible.

Mathematicians often define numbers from the mathematical objects which they study and then try to obtain information about the object by studying the properties of these numbers. The number π is one of the most ancient examples of this. In the case of Set Theory the numbers are usually infinite ordinals or infinite cardinals. One of the typical questions about such a cardinal number is then where it lies in the list of alephs, and another is how it compares with other cardinals obtained in this way. F

rom this one then obtains mathematically important information about the objects of study. n this paper I illustrate this practice by taking as object of study one of the classical proofs of the following theorem:

Theorem 1 (Bolzano-Weierstrass) *Every bounded sequence of real numbers has a convergent subsequence.*

One of the standard proofs of this theorem proceeds as follows: Let a bounded sequence of real numbers be given.

1. Extract a subsequence which is monotonic;

Presented at the *Short Conference "Topology and Analysis"*, Mataruška Banja, June 4–7, 1998

1991 *Mathematics Subject Classification.* 03E99

Supported in part by NSF grant DMS 95-05375

Key words: Ramsey's theorem, Bolzano Weierstrass theorem, Splitting family, cardinal number.

2. Since the terms of this subsequence is a bounded set, determine the least upper bound and greatest lower bound of this set;
3. If the sequence is increasing, then the least upper bound is its limit – else, the greatest lower bound is its limit.

Steps 2 and 3 are straightforward observations. The crucial step lies in the extraction of the monotonic subsequence, and several textbooks state this as a separate lemma.

Theorem 2 (Monotonic Subsequence Theorem) *Every sequence of real numbers has a monotonic subsequence.*

Thus the described proof of the Bolzano-Weierstrass theorem is really a proof of the implication

Monotonic Subsequence Theorem $\Rightarrow$ Bolzano-Weierstrass Theorem.

Could one also derive from the Bolzano-Weierstrass theorem the Monotonic Subsequence theorem?

The area of "reverse mathematics" and has developed powerful tools to treat these sorts of questions. Here I'll instead give a typical set theoretic approach to the problem.

1 Associating cardinal numbers with the two theorems.

First, observe that the Monotonic Subsequence Theorem is equivalent to the statement that every sequence of real numbers has a subsequence which, except for finitely many terms, is monotonic; we call these "eventually monotonic" sequences. Next, observe that though the two theorems speak about only a single sequence at a time, one could with minor modifications prove:

> Let $(x_n : n = 1, 2, 3, \ldots)$ and $(y_n : n = 1, 2, 3, \ldots)$ be sequences of real numbers. Then there is an infinite set A of natural numbers such that both $(x_n : n \in A)$ and $(y_n : n \in A)$ are eventually monotonic.

and

> Let $(x_n : n = 1, 2, 3, \ldots)$ and $(y_n : n = 1, 2, 3, \ldots)$ be bounded sequences of real numbers. Then there is an infinite set A of natural numbers such that both $(x_n : n \in A)$ and $(y_n : n \in A)$ are convergent.

We shall use the ability of extracting convergent subsequences or monotonic subsequences "simultaneously" from a large number of given sequences as a measure of the strength of these two theorems. To this end, let κ be a cardinal number. Define the statements $\mathsf{M}(\kappa)$ and $\mathsf{BW}(\kappa)$ as follows:

> $\mathsf{M}(\kappa)$ *For every sequence* $(f_n : n = 1, 2, 3, \ldots)$ *of real-valued functions defined on* κ *there is a subsequence* $(f_n : n \in A)$ *such that for each* $\alpha \in \kappa$ *the sequence* $(f_n(\alpha) : n \in A)$ *is eventually monotonic.*

> $\mathsf{BW}(\kappa)$: *For every sequence* $(f_n : n = 1, 2, 3, \ldots)$ *of pointwise bounded real-valued functions defined on* κ *there is a subsequence* $(f_n : n \in A)$ *which is pointwise convergent.*

Then we have for each cardinal number κ:

$$\mathsf{M}(\kappa) \Rightarrow \mathsf{BW}(\kappa). \tag{1}$$

For typographical convenience let the symbol $\mathbf{c}$ denote $2^{\aleph_0}$, the cardinality of the real line. The following two lemmas are set-theoretic folklore; I include proofs in the interest of self-contained reading.

Lemma 3 *The statement* $\mathsf{BW}(\mathbf{c})$ *is false.*

Proof: Let the collection $\mathcal{A}$ of infinite subsets of the natural numbers with infinite complements represent $\mathbf{c}$. Define sequence $(f_n : n = 1, 2, 3, \ldots)$ of functions from $\mathcal{A}$ to the real line so that for each $A \in \mathcal{A}$ and for each n,

$$f_n(A) = \begin{cases} 1 & \text{if } n \in A, \\ 0 & \text{otherwise.} \end{cases}$$

Then no subsequence of $(f_n : n = 1, 2, 3, \ldots)$ is pointwise convergent. For let A be any infinite set of natural numbers. Choose a $Y \in \mathcal{A}$ such that $A \cap Y$ and $A \setminus Y$ are infinite. Then $(f_n(Y) : n \in A)$ has both values 0 and 1 infinitely many times.

Lemma 3 indicates that there is a least cardinal number κ for which $\mathsf{BW}(\kappa)$ is false, and there is a least cardinal number λ for which $\mathsf{M}(\lambda)$ is false. We define:

> $\mathbf{m}$ is the least cardinal number κ such that $\mathsf{M}(\kappa)$ is false.
> $\mathbf{bw}$ is the least cardinal number κ such that $\mathsf{BW}(\kappa)$ is false.

Lemma 4 **m** *is uncountable.*

Proof: To see this, consider a sequence $(f_n : n = 1, 2, 3, \ldots)$ of functions from a countable set, say the natural numbers, to the real line. Using the Monotonic Subsequence Theorem recursively choose a sequence

$$Y_1 \supset Y_2 \supset Y_3 \supset \ldots \supset Y_n \supset \ldots$$

of infinite subsets of the natural numbers such that for each n the sequence $(f_m(n) : m \in Y_n)$ is monotonic. Then choose $n_1 < n_2 < n_3 < \ldots$ such that for each k we have $n_k \in Y_k$, and put $Y_\infty = \{n_k : k = 1, 2, 3, \ldots\}$. The subsequence $(f_n : n \in Y_\infty)$ is a pointwise-eventually monotonic subsequence of the original.

On account of implication (1) and Lemmas 3 and 4 we have the following inequalities among cardinal numbers:

$$\aleph_1 \leq \mathbf{m} \leq \mathbf{bw} \leq \mathbf{c}. \qquad (2)$$

The Continuum Hypothesis implies that these four cardinal numbers are equal. Since the Continuum Hypothesis is consistent relative to the consistency of classical mathematics, one cannot on the basis of classical mathematics prove that these cardinals are distinct. However, the negation of the Continuum Hypothesis is also consistent relative to the consistency of classical mathematics. Should it be the case that it is consistent that $\mathbf{m} < \mathbf{bw}$, this could be taken as evidence that the implication in (1) is not reversible, and thus that the Monotonic Subsequence Theorem is formally stronger statement than the Bolzano-Weierstrass theorem.

2 The cardinal number m.

Towards analysing the relationship between **m** and **bw** further, we introduce a third cardinal number relevant to the task. For f and g functions from the natural numbers to the natural numbers, write

$$f \prec g$$

to denote that $\lim_{n \to \infty}(g(n) - f(n)) = \infty$.

Then $\prec$ is a partial order. For a cardinal number κ let $\mathsf{B}(\kappa)$ denote the statement:

> If $\mathcal{F}$ is a family of at most κ functions from the natural numbers to the natural numbers, then there is such a function g such that for each $f \in \mathcal{F}$ we have $f \prec g$.

It is evident that $\mathsf{B}(\mathbf{c})$ is false; a standard diagonalization argument shows that $\mathsf{B}(\aleph_0)$ is true. Define

$$\mathbf{b} = \min\{\kappa : \mathsf{B}(\kappa) \text{ is false }\}.$$

The following Lemma, also folklore, is useful in constructions regarding $\mathbf{b}$.

Lemma 5 *Let f and g be strictly increasing functions from the natural numbers to the natural numbers. If for all but finitely many n there is a k such that*

$$g(n) < f(k) < f(k+1) < g(n+1)$$

then $f \prec g$.

Theorem 6 *For each cardinal number κ the following are equivalent:*

1. $\mathsf{M}(\kappa)$;
2. $\mathsf{B}(\kappa)$ *and* $\mathsf{BW}(\kappa)$.

Proof: $\mathsf{M}(\kappa) \Rightarrow \mathsf{B}(\kappa)$ and $\mathsf{BW}(\kappa)$: We already know that $\mathsf{M}(\kappa) \Rightarrow \mathsf{BW}(\kappa)$. Assume $\mathsf{M}(\kappa)$ and let $\mathcal{F}$ be a family of κ functions from the natural numbers to the natural numbers. For each $f \in \mathcal{F}$ define a new function $\Psi(f)$ so that on intervals of integeers of the form

$$[f(4 \cdot n), f(4 \cdot n + 3)]$$

the values of $\Psi(f)$ are strictly decreasing, but all larger than the values of $\Psi(f)$ on any earlier intervals of this form. Since $\mathsf{M}(\kappa)$ holds fix an infinite set A of natural numbers such that for each $f \in \mathcal{F}$ the sequence $(\Psi(f)(n) : n \in A)$ is eventually monotonic. Then by the construction of the $\Psi(f)$'s we see that for each $f \in \mathcal{F}$ and for all but finitely many n the set

$$Y \cap [f(4 \cdot n), f(4 \cdot n + 3)]$$

has at most one element. Define g so that for each n $g(n)$ is the $2 \cdot n$-th element of Y in the increasing listing of Y. Then for each $f \in \mathcal{F}$, for all but finitely many n there are k with $g(n) < f(k) < f(k+1) < g(n+1)$. By Lemma 5 we see that $f \prec g$ whenever f is in $\mathcal{F}$. This shows that $\mathsf{B}(\kappa)$ is true.

$\mathsf{B}(\kappa)$ and $\mathsf{BW}(\kappa) \Rightarrow \mathsf{M}(\kappa)$: Assume that both $\mathsf{B}(\kappa)$ and $\mathsf{BW}(\kappa)$ are true. Let $(f_n : n = 1, 2, 3, \ldots)$ be a sequnce of real-valued functions defined on κ. Temporarily replace each f_n by $g_n = arctan \circ f_n$. Then the sequence $(g_n : n =$

$1,2,3,\dots)$ is pointwise bounded. An application of $\mathsf{BW}(\kappa)$ gives an infinite set A of natural numbers such that $(g_n : n \in A)$ is pointwise convergent.

For each $\alpha \in \kappa$ let s_α be the limit of the sequence $(g_n(\alpha) : n \in A)$. Define

$$T = \{\alpha \in \kappa : (g_n(\alpha) : n \in A) \text{ not eventually constant}\}.$$

For each $\alpha \in T$ and for each n define

$$A_\alpha(n) = \{m \in A : \frac{1}{n} \geq |g_m(\alpha) - s_\alpha| > \frac{1}{n+1}\}$$

and then define f_α as follows: $f_\alpha(1)$ is so large that if $f_\alpha(1) \leq n$, then $1 < \min(A_\alpha(n))$ and $\max(Y_\alpha(1)) < f_\alpha(1)$. Recursively define it further so that $f_\alpha(k+1)$ is so large that if $n \geq f_\alpha(k)$ then $f_\alpha(k) < \min(A_\alpha(n))$, and for each $j \leq f_\alpha(k)$ we have $\max(A_\alpha(j)) < f_\alpha(k+1)$. Then each such f_α is increasing.

Next, for each $\alpha \in T$ and for each n define $f^*_\alpha(n) = f_\alpha(4 \cdot n)$. Since $\mathsf{B}(\kappa)$ is true, select an increasing sequence h of natural numbers such that for each $\alpha \in T$ we have $f^*_\alpha \prec h$, such that $h(1) > 1$, such that the range of h is a subset of A, and such that for each k $h^{2^k}(1) + 2 < h^{2+k+1}(1)$ (Here, h^m denotes the m-th iterate of h under composition). Then for each k define

$$\ell(k) = h^{2_k}(1).$$

Since h is increasing, $h \prec \ell$. Let $Z \subset A$ be the range of ℓ. On account of the definition of ℓ it is also true that for each $\alpha \in T$ and for each but finitely many m the set Z has at most one element in common with $A_\alpha(m)$.

Now define for each α in T the sequence $(j_n(\alpha) : n \in Z)$ so that for each $n \in Z$,

$$j_n(\alpha) = \begin{cases} 0 & \text{if } g_n(\alpha) > s_\alpha \\ 1 & \text{otherwise} \end{cases}$$

Apply $\mathsf{BW}(\kappa)$ to the sequence $(j_n : n \in Z)$: we find an infinite set $B \subset Z$ such that $(j_n : n \in B)$ is pointwise convergent. It follows that on κ $(g_n : n \in B)$ is pointwise either eventually constant, or else eventually monotonically convergent.

Finally, since tan is monotonic on $(-\frac{\pi}{2}, \frac{\pi}{2})$, and since for each n $f_n = \tan \circ g_n$, we see that $(f_n : n \in B)$ is pointwise either eventually constant, or else eventually monotonic.

As a result we see that

$$\mathbf{m} = \min\{\mathbf{b}, \mathbf{bw}\}. \tag{3}$$

3 The cardinal number bw.

A collection $\mathcal{A}$ of subsets of the natural numbers is said to be a splitting family if, for each infinite set X of natural numbers there is an $A \in \mathcal{A}$ such that $X \cap A$ and $X \setminus A$ are both infinite; A is said to "split" X.

For a cardinal number κ let $\mathsf{S}(\kappa)$ denote the statement:

> For each family of κ subsets of the natural numbers there is an infinite set of natural numbers not split by any member of the family.

Nowadays a family $\mathcal{S}$ which is a counterexample to the statement $\mathsf{S}(\kappa)$ is said to be a *splitting family*. It is evident that $\mathsf{S}(\mathbf{c})$ is false; one can show that $\mathsf{S}(\aleph_0)$ is true. Let $\mathbf{s}$ denote the minimal κ for which $\mathsf{S}(\kappa)$ is false. Booth proved in Theorem 2 of [3] a result which amounts to:

Theorem 7 (Booth) *For each cardinal number,*

$$\mathsf{BW}(\kappa) \Leftrightarrow \mathsf{S}(\kappa).$$

Consequently, $\mathbf{s} = \mathbf{bw}$. Set theorists have proved that any of the following relations

$$\mathbf{b} < \mathbf{s} \tag{4}$$

$$\mathbf{b} = \mathbf{s} \tag{5}$$

$$\mathbf{b} > \mathbf{s} \tag{6}$$

is consistent relative to the consistency of classical mathematics.

Consequently it is consistent relative to the consistency of classical mathematics that $\mathbf{m} < \mathbf{bw}$. This gives evidence that the Monotonic Subsequence Theorem is a formally stronger statement than the Bolzano Weierstrass Theorem.

4 Proving the Monotonic Subsequence Theorem: a conjecture.

It is also interesting to analyse in the same way the usual proofs of the Monotonic Subsequence Theorem. One of the more sophisticated proofs makes use of Ramsey's theorem:

Theorem 8 (Ramsey) *For each natural number n and each function f defined on the n–element subsets of the set of natural numbers and with finite range, there is an infinite set S of natural numbers such that f is constant on the n–element subsets of S.*

One proof of the Monotonic Subsequence Theorem from Ramsey's theorem proceeds as follows: Let a sequence $(x_n : n = 1, 2, 3, \ldots)$ of real numbers be given. Define f so that for $m < n$ we have

$$f(\{m,n\}) = \begin{cases} 0 & \text{if } x_m < x_n \\ 1 & \text{if } x_m = x_n \\ 2 & \text{if } x_m > x_n \end{cases}$$

For an infinite set A of natural numbers on which f is constant one has that $(x_n : n \in A)$ is monotonic or constant.

In the sense described above Blass has considered the strength of Ramsey's theorem as follows: For a function f with finite range, defined on the n–element subsets of the natural numbers, say that an infinite subset A of the natural numbers is "almost homogeneous for f" if there is a finite subset $F \subset A$ such that f is constant on the set of n–tuples of $A \setminus F$. Let κ be a cardinal number. Let $\mathsf{R}(\kappa)$ denote the statement:

> For each family $\mathcal{F}$ of functions with finite ranges and with domain the two-element subsets of the natural numbers there is an infinite set A of natural numbers which is almost homogeneous for each $f \in \mathcal{F}$.

Standard techniques show that $\mathsf{R}(\kappa) \Rightarrow \mathsf{M}(\kappa)$.

Most familiar with the proof of Ramsey's Theorem and with an elementary proof of the Monotonic Subsequence Theorem (see for example Theorem 3.4.6 of [1]) would argue that resorting to Ramsey's Theorem to prove the Monotonic Subsequence Theorem is overkill. It is somewhat surprising that the sort of analysis for comparing the Monotonic Subsequence Theorem and the Bolzano Weierstrass Theorem is inconclusive when comparing the Monotonic Subsequence Theorem with Ramsey's Theorem. The reason is as follows:

Following Blass, let **par** denote the least κ for which $\mathsf{R}(\kappa)$ is false. In[2] Blass shows that

Theorem 9 (Blass) $\mathbf{par} = \min\{\mathbf{b}, \mathbf{s}\}$.

On account of our work above this means that $\mathbf{par} = \mathbf{m}$. I doubt that this equation shows that the two theorems are formally of equal strength; it more likely shows that the comparing these cardinal numbers is not the tool for the task. I would conjecture that using the more refined tools of reverse mathematics it can be shown that Ramsey's Theorem is formally stronger than the Monotonic Subsequence Theorem.

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