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BASIC HOMOGENEITY IN THE CLASS OF ZERO-DIMENSIONAL SPACES

M. V. Matveev

Abstract. The relationships between homogeneity, h-homogeneity and B-homogenety in the class of zero-dimensional spaces are considered.

A topological space X is *basically homogeneous* (B-homogeneous) [MS], if it has a base every element of which can be mapped onto every other one by a homeomorphism of the whole space (a base having this property is called a homogeneous base). An equivalent definition: a space X is B-homogeneous if there is an open set $U \subset X$ such that the family $\{f(U) : f \in \mathcal{A}ut(X)\}$ is a base of X ($\mathcal{A}ut(X)$ denotes the group af all autohomeomorphisms of X). Basic homogeneity is different from the classical notion of topological homogeneity (recall that a space X is *homogeneous* if every point can be moved to every other one by an autohomeomorphism of X): there are simple examples of homogeneous spaces that are not B-homogeneous and vice versa [MS]. On the other hand, basic homogeneity seems to be no less natural than "usual" homogeneity because both these properties generalize properties of Euclidean spaces (the existense of translations and of shrinks and expansions, respectively).

In this paper we consider B-homogeneity in the class of zero-dimensional (in the sense *ind*) spaces, that is the spaces having bases consisting of clopen sets. Special consideration of this case is reasonable by the following reasons:

First, the combinatorics of clopen sets is simpler than that of the open sets; clopen sets behave more regular with respect to autohomeomorphisms. For example, if two non-intersecting clopen sets U and V in a space X are homeomorphic to each other then there is an autohomeomorphism of X that maps U onto V and does not move points of $X \setminus (U \cup V)$. Of course, this is not true for arbitrary open sets. Another example: a topological space X is SLB-*homogeneous* [MS] if it has a base $\mathcal{B}$ such that for any U, V, $W \in \mathcal{B}$

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such that $\overline{U}$, $\overline{V} \subset W$ there exists a homeomorphism $f_{UVW} \in \mathcal{A}ut(X)$ such that $f_{UVW}(U) = V$ and $f_{UVW}(t) = t$ as soon as $t \in X \setminus W$. Not every B-homogeneous space is SLB-homogeneous (the real line is a counterexample). On the other hand, if a homogeneous base consists of clopen sets then it fits the definition of SLB-homogeneity.

Second, in the class of zero-dimensional spaces, the following notion of *h-homogeneity* which is quite close to B-homogeneity was considered by many authors: a zero-dimensional space is h-homogeneous if all its nonempty clopen sets are homeomorphic to each other. This property was introduced independently by Ostrowskii and by van Mill [Ost], [vM1] and was examined in the papers [Mot1 - Mot3, Med1 - Med4, BD, Ter]. It is clear that h-homogeneity implies B-homogeneity while the converse is not in general true. In this paper we consider some extra conditions under which B-homogeneity implies h-homogeneity or local h-homogeneity.

It is interesting that most of the results on h-homogeneity concern one of the two opposite cases: the space under consideration is assumed to be either compact (often it is also first-countable) or non-pseudocompact (recall that a space is pseudocopact provided every continuous function on it is bounded; for Tychonoff spaces this property is equivalent to non-existense of infinite families of non-empty open sets [Eng]; for zero-dimensional spaces this property is equivalent to impossibility to represent the space as an infinite discrete sum of clopen sets). Thus the problem whether arbitrary product of h-homogeneous spaces is h-homogeneous has been solved to the positive in the cases when this product is either compact or non-pseudocompact [Ter]. A construction from [Mot1] gives affirmative answer also in the case when the product is first-countable. Most of the results on B-homogeneity below use similar assumptions.

We start with some technical lemmas. All zero-dimensional spaces are assumed to be Hausdorff. A B-space is a triple $(X, \mathcal{B}, \mathcal{F})$, where X is a topological space, $\mathcal{B}$ is its homogeneous base, and $\mathcal{F} : \mathcal{B} \times \mathcal{B} \to \mathcal{A}ut(X)$ is such a mapping that $\mathcal{F}(U,V)(U) = V$ for all $U, V \in \mathcal{B}$. If the base $\mathcal{B}$ consists of clopen sets, then we call the triple $(X, \mathcal{B}, \mathcal{F})$ a B_0-space. Two subsets of X one of which can be mapped onto another by an autohomeomorphism of X are called strongly homeomorphic. It follows from the remark above that if two non-intersecting clopen sets are homeomorphic then they are strongly homeomorphic. The following proposition shows that "non-intersecting" is not a very strong restriction.

Proposition 1. *If a Hausdorff space X has a base $\mathcal{B}$ for every two non-intersecting elements U and V of which there exists an autohomeomorphism of X that maps U onto V then the base $\mathcal{B}$ contains a homogeneous base of X*

Proof. Denote $\mathcal{B}_1 = \{U \in \mathcal{B} : \overline{U} \neq X\}$. Since X is Hausdorff, $\mathcal{B}_1$ is a base of X. Let $U, V \in \mathcal{B}_1$. We pick points $x \in X \setminus \overline{U}$ and $y \in X \setminus \overline{V}$. This points can be assumed to be different. Indeed, this is not possible only in the case when $X \setminus \overline{U} = X \setminus \overline{V} = \{x\}$ is an one-point set; but then the point x is isolated, hence so is any other point; the space is discrete and there is nothing to prove. So, let $x \neq y$. We pick $O, W \in \mathcal{B}_1$ so that $x \in O$, $y \in W$, $O \cap U = \varnothing = O \cap W = W \cap V$. By our assumption there are autohomeomorphisms φ_{UO}, φ_{OW} and φ_{WV} such that $\varphi_{UO}(U) = O$, $\varphi_{OW}(O) = W$ and $\varphi_{WV}(W) = V$. Then $f = \varphi_{UO} \circ \varphi_{OW} \circ \varphi_{WV}$ also is an autohomeomorphism of X and $f(U) = V$. So $\mathcal{B}_1$ is a homogeneous base of X.

Proposition 2. *Let $(X, \mathcal{B}, \mathcal{F})$ be a B-space, O an open set in X, $O \subset U \in \mathcal{B}$. Then $\{\mathcal{F}(U, V)(O) : V \in \mathcal{B}\}$ is a π-base of X.*

The proof follows imediately from the definition of a π-base. Thus a zero-dimensional B-homogeneous space has a homogeneous π-base consisting of clopen sets. However, the author does not know the answer to the following questions:

Question 1. Does every zero-dimensional B-homogeneous space X have a homogeneous base consisting of clopen sets?

Question 2. Let U be an element of homogeneous base of a space X and $O \subset U$ a nonempty clopen subset of X. Do the sets strongly homeomorphic to O form a base of X?

Question 3. Let U_1 and U_2 be clopen subsets of X, and the sets strongly homeomorphic to U_1 form a) a base, b) a π-base of X as well as the sets strongly homeomorphic to U_2. Do this imply that the sets U_1 and U_2 are homeomorphic?

The following result is crutial for the next:

Proposition 3. *[Mot1, Ter] Suppose a zero-dimensional space X has a π-base $\mathcal{B}$ consisting of clopen sets homeomorphic to X, and either a) X is not pseudocompact or b) X (and thus every element of $\mathcal{B}$) containes a point and a sequence of open sets converging to this point. Then X is h-homogeneous.*

Case a) was proved in [Ter]. In [Mot1], the proposition was proved in a stronger assumption than b): X is a first-countable compact space. However, compactness is not in fact used in the proof from [Mot1] while instead of first-countability, the construction from [Mot] works, as we shall see in the assumption b).

Property b) is essentially weaker than first-countability: it follows from the Preiss-Symon property which is known to be weaker than first-countability but stronger than Frechet-Urysohn property (recall that a space has the Preiss-Symon property provided in every closed subspace for every point there is a sequence of open sets converging to this point [PS], [AT]; a sequence of sets is said to converge to a point provided every neighbourhood of this point contains all but finitely many elements of this sequence; "closed" is not essential in the definition of this property). Below we reproduce (with minor modifications) the construction from [Mot] to witness that it works in case b) and give the proof for the case a) which is somewhat different from the original [Ter]. By $\approx$ we denote homeomorphism of spaces, $\oplus$ denotes the discrete sum.

Proof. *Case b).* Let us fix a point $x^0 \in X$ and a sequence of open sets $\{U_n^0 : n \in \omega\}$ converging to x_0. Without loss of generality we suppose that the sets U_n^0 are elements of $\mathcal{B}$ and that they are disjoint. Denote $H^0 = X \setminus \cup\{U_n^0 : n \in \omega\}$. Then the set H^0 is closed, and x^0 is the only point of H^0 that is not in its interior. Let A be a clopen set in X. We show that $A \approx X$. Pick $V \in \mathcal{B}$, $V \subset A$ and fix a homeomorphism $f : X \to V$. Denote $x = f(x^0)$, $U_n = f(U_n^0)$ $(n \in \omega)$, $H = f(H^0)$, $U_{-1} = A \setminus V$. The sequence $\{U_{-1}, U_0, U_1, ...\}$ converges to the point x and all its elements but may be the first are homeomorphic to X. If $U_{-1} = \varnothing$, then there is nothing to prove. Suppose $U_{-1} \neq \varnothing$. In U_0, there is a homeomorphic to $X \setminus U_{-1}$, proper clopen subset V_0; denote $W_0 = U_0 \setminus V_0$; in U_1, there is a homeomorphic to $X \setminus W_0$, proper clopen subset V_1, etc.

Then $A = U_{-1} \oplus U_0 \oplus U_1 \oplus U_2 \oplus ... \cup H = U_{-1} \oplus (V_0 \cup W_0) \oplus (V_1 \cup W_1) \oplus (V_2 \cup W_2) \oplus ... \cup H = (U_{-1} \oplus V_0) \oplus (W_0 \oplus V_1) \oplus (W_1 \oplus V_2) \oplus ... \cup H$.

Denote $U_{-1} \oplus V_0 = O_0$, $W_0 \oplus V_1 = O_1$, etc. Then $A = O_0 \oplus O_1 \oplus O_2 \oplus ... \cup H$ whence $O_0 \approx O_1 \approx O_2 \approx ... \approx X$ and the sequence $\{O_n : n \in \omega\}$ converges to the point x since $O_n \subset U_{n-1} \cup U_n$ for each n. This provides the desired homeomorphism between $X = U_0^0 \oplus U_1^0 \oplus U_2^0 \oplus ... \cup H_0 \approx X \oplus X \oplus ... \cup H$ and A.

Case a). Let A be a nonempty clopen subset of X. It contains an element of $\mathcal{B}$ which, on its turn, containes a discrete (in X) family $\{U_n : n \in \omega\}$ of clopen sets homeomorphic to X. Denote $K = A \setminus \cup\{U_n : n \in \omega\}$ and $H = X \setminus A$. Then the sets K and H also are clopen and $X = H \oplus A = H \oplus K \oplus \oplus_{n \in \omega} U_n \approx H \oplus K \oplus X \times \omega \approx H \oplus K \oplus (H \oplus K \oplus X \times \omega) \times \omega \approx H \times \omega \oplus K \times \omega \oplus X \times \omega$. Also, we have $A = K \oplus \oplus_{n \in \omega} U_n \approx K \oplus X \times \omega \approx K \oplus (H \oplus K \times X \times \omega) \times \omega \approx H \times \omega \oplus K \times \omega \oplus X \times \omega$. So $A \approx X$.

The proposition just proved provides partial affirmative answers to Questions 1–3.

Proposition 4. *Let a zero-dimensional B-homogeneous space X either a) is not locally pseudocompact or b) contains a sequence of open sets that converges to some point. Then:*

1) X has a homogeneous base consisting of clopen sets,

2) X is locally h-homogeneous,

3) for every nonempty clopen set O which is contained in some element of a homogeneous base of X, the sets strongly homeomorphic to O form a (homogeneous) base of X,

4) every clopen subspace of X is B-homogeneous.

Proof. Let $U \in \mathcal{B}$, $V \subset U$, V is nonempty and clopen in X. The sets strongly homeomorphic to U form a base of V, and each of them contains a clopen subset homeomorphic to V. Since the properties a) and b) (the later in the presense of B-homogeneity) are preserved with respect to clopen subspaces, Proposition 3 implies that V is h-homogeneous. So we see that the clopen subsets of the elements of the homogeneous base are h-homogeneous which implies 2).

Next, since all clopen (in X) subsets of U are h-homogeneous and the property of being clopen is preserved under finite unions, we conclude that all clopen (in X) subsets of U are strongly homeomorphic to each other. This implies 3) and 1).

Last, 4) follows from 1), Proposition 1 and the simple fact that non-intersecting homeomorphic clopen sets are strongly homeomorphic.

Proposition 5. *Let a zero-dimensional space X either a) is not locally psudocompact or b) contains a dense subspace each point of which is a limit of a converging sequence of open sets in X. Let U_1 and U_2 be clopen sets in X and the sets strongly homeomrphic to U_1 form a π-base of X, as well as the sets strongly homeomorphic to U_2. Then $U_1 \approx U_2$ and the space X contains a dense, open B-homogeneous subspace*

Proof. By Proposition 3, U_1 is h-homogeneous and thus U_1 contains a clopen (in X) subspace homeomorphic to U_2, we have $U_1 \approx U_2$. The subspace $Y = \{U : U$ is strongly homeomorphic to $U_1\}$ is open, dense in X and h-homogeneous. □

Corollary. *Let a zero-dimensional space X either a) is not locally pseudocompact or b) contains a sequence of open sets converging to some point. If X has a homogerneous base such that some (hence every) element of which contains a clopen subset of X which is homeomorphic to X, then X is h-homogeneous*

The problem when B-homogeneity implies h-homogneity is especially interesting in the case of compact spaces.

Question 4. Is every B-homogeneous zero-dimensional compact space h-homogeneous?

Under certain extra assumptions the answer is affirmative. The following statement is obvious.

Lemma 1. *Every more than one-point Hausdorff h-homogeneous space X is srongly divisable by n for $n = 2, 3, 4, \ldots$, that is it can be represented as the discrete sum of n subspaces each of which is homeomorphic to X*

Proposition 6. *An infinite, locally h-homogeneous, B-homogeneous, zero-dimensional compact space is h-homogeneous*

Proof. Let $\mathcal{B}$ be a homogeneous base of a h-homogeneous compact space X. Every point $x \in X$ has a clopen (in X), h-homogeneous neighbourhood which is a subset of some element of $\mathcal{B}$. These neighbourhoods form a cover of X which has a refinement which is a decomposition of X into finitely many clopen h-homogeneous subspaces: $X = H_1 \oplus \ldots \oplus H_n$.

We show that the subspaces $H_1, \ldots, H_n$ are homeomorphic to each other. Indeed, let $1 \le i, j \le n$. There are $U, V \in \mathcal{B}$ such that $H_i \subset U$ and $H_j \supset V$. Denote $K = \varphi_{UV}(H_1)$ (where φ_{UV} is an autohomeomorphism of X that maps U onto V). Then $K \approx H_i$ (since φ_{UV} is a homeomorphism) and $K \approx H_j$ (since K is a clopen subspace of an h-homogeneous space H_j). Therefore $H_i \approx H_j$.

So, X is homeomorphic to the sum of n copies of H_1. Then H_1, being infinite and h-homogeneous, by Lemma 1 also is homeomorphic to the sum of n copies of H_1. Therefore X is homeomorphic to H_1 and thus h-homogeneous. $\square$

Proposition 7. *If a B-homogeneous zero-dimensional compact space X containes a sequence of open sets that converges to a point, then X is h-homogeneous*

Proof. This follows from Proposition 4 and Proposition 6. $\square$

Corollary. *A B-homogeneous, first-countable compact space is h-homogeneous and homogeneous.*

Indeed, every first countable, h-homogeneocus compact space is homogeneous by [Mot1].

We shall say that a homogeneous base $\mathcal{B}$ consisting of clopen sets is *combinatorically homogeneous* if for every $U, V \in \mathcal{B}$ the set $U \setminus V$ is either empty or is homeomorphic to the elements of $\mathcal{B}$. It is easy to see that every set which is obtained from finitely many elements of $\mathcal{B}$ via finitely many operations of union, intersection and supplement also is homeomorphic to the elements of $\mathcal{B}$. This explaines the name.

It is clear also that in a non-discrete T_0-space every element U of a combinatoricaly homogeneous base can be represented as the disjoint union of any finite number of clopen pices which are homeomorphic to U.

Proposition 8. *Every infinite compact space X having a combinatorically homogeneous base $\mathcal{B}$ is h-homogeneous*

Proof. It is easy to see that X can be represented as the discrete sum of finitely many pices which are homeomorphic to the elements of $\mathcal{B}$. The same is true for every non-empty clopen set A. If the number of pices for X and for A is not the same it can be improved by cuting one of the pices into the necessary number of parts each of which is homeomorphic to the elements of $\mathcal{B}$. $\square$

The known examples of homogeneous compact spaces that are not h-homogeneous [vD, Mot3] can not give the negative answer to Question 4 since each of them has a measure which is preserved under autohomeomorphisms; such spaces can not be even B-homogeneous.

We finish the paper with an example of a pseudocompat, zero-dimensional B-homogeneous space which is not h-homogeneous (this space has even a combinatorically homogeneous base). This space is similar to the well-known Isbel-Mrowka space [GJ, Mro]: the only difference is that instead of the isolated points one takes the Cantorsets $\mathcal{C} = D^\omega$.

Example. We denote $\mathcal{M}$ the discrete sum of countably many copies of $\mathcal{C}$: $\mathcal{M} = \cup\{C_n : n \in \omega\}$. Let $\mathcal{R}$ be a maximal almost disjoint family of subsets of ω, $|\mathcal{R}| = c$ (recall that a family of sets is called almost disjoint if the intersection of any two its elements is finite). We put $\xi = \mathcal{M} \cup \mathcal{R}$. The topology of ξ is defined as follows: the subspace $\mathcal{M}$ is open in ξ while the basic neighbourhood of the point $r \in \mathcal{R}$ takes the form

$$O_{r,K} = \{r\} \cup \cup\{C_n : n \in r \setminus K\}$$

where K is arbitrary finite subset. It is easy to check that ξ is a pseudocompact, zero-dimensional, Hausdorff (hence Tychonoff), locally compact, non-compact space. Therefore X is not h-homogeneous. Let us verify that it is B-homogeneous. Every subspace C_n is clopen in ξ and has the base $\mathcal{B}_n$ (the canonical base of the Tychonoff product $D^\omega \approx C_n$) all elements of which are homeomorphic to $\mathcal{C}$. The basic neighbourhoods of the points $r \in \mathcal{R}$ also are clopen in ξ and homeomorphic to $\mathcal{C}$. Indeed, the subspace O_{rK} is homeomorphic to the one-point compactification of the discrete sum of countably many copies of $\mathcal{C}$. Let us verify that the same is true for the basic neighbourhoods of the points from $\mathcal{M}$. Indeed, let $x \in C_n \approx \mathcal{C}$. Since $\mathcal{C}$ is zero-dimensional, h-homogeneous and has countable character, we can represent $\mathcal{C}$ in the form $\mathcal{C} = \oplus\{A_n : n \in \omega\} \cup \{x\}$ where the sets A_n are clopen in $\mathcal{C}$, homeomorphic to $\mathcal{C}$ and $\mathcal{C}$ is the one-point compactification of $\oplus\{A_n : n \in \omega\}$ by point x. To do this we choose $x \in \mathcal{C}$ arbitrary and choose a local base $U_n : n \in \omega\}$ of $\mathcal{C}$ at x consisting of clopen sets so that $\mathcal{C} = U_0 \supset U_1 \supset U_2 \supset ...$ and $U_n \setminus U_{n+1} \neq \varnothing$ and denote $A_n = U_n \setminus U_{n+1}$.

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Moscow "N. E. Bauman" State Technical University Chair FN-11 Moscow 107005, 2-ja Baumanskaja ul., 5, RUSSIA