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SOME QUADRATURE FORMULAS FOR LINEAR DIFFERENTIAL EQUATIONS IN A WIDERSENSE

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Abstract. In this article it is proved that any analytical differential equation can be solved by quadratures in a wider sense. It means, the method for solving by series is completed by rearrangement of the numerical terms and condensation, so the solution is expressed via a series of an infinite number of integrals of the equation coefficients.

Introduction

For the general linear differentiel equation

$$y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y = f(x) \tag{1}$$

if the following conditions are fulfilled:
$\mathbf{H_1}$: $a_1(x)$, $(i = 1, 2, \ldots, n)$ and $f(x)$ are defined on interval $I \subseteq$R, which is symmetrical to the coordinate origin;
$\mathbf{H_2}$: $a_1(x)$, $(i = 1, 2, \ldots, n)$ and $f(x)$ are analytical functions on I,
a unique solution can be found by using the method of power series. This solution is expandend in the form

$$y(x) = \sum_{k=0}^{\infty} c_k x^k \tag{2}$$

which converges inside a cycle with center $x = 0$ and the radius of convergence R is determined by

$$\frac{1}{R} = \lim_{i \to \infty} \sup \sqrt[i]{|a_i|}\,.$$

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The coefficients c_k, $(k = 0, 1, \dots)$ from (2) are constants and they can be determined by the method of undetermined coefficients and they depend on the terms from the expansion of the coefficients. By using the method of power series, we can do one step more, because the solution (2) can be expressed via the coefficients $a_i(x)$, not via these terms.

1. Canonical homogeneous differential equations

The canonical equations are equations of the simplest form in which only two terms exist: the highest derivation of the function and the term with coefficient $a(x)$ multiplied by the function.

1.1. Canonical differential equation of the II order

Let us consider the canonical homogeneous differential equation of the II order which has the form

$$y'' + a(x)y = 0 \tag{3}$$

in which the coefficien $a(x)$ satisfies the conditions H_1 and H_2. So it can be expanded in power series

$$a(x) = \sum_{k=0}^{\infty} a_k x^k \tag{4}$$

and there is a unique solution in the form of power series (2). By differentiating (2) two times and after substituting of this value by (2) and (4) in equation (3), the following is obtained

$$\sum_{k=2}^{\infty} k(k-1)c_k x^k + \sum_{k=0}^{\infty} a_k x^k \cdot \sum_{k=0}^{\infty} c_k x^k = 0\,.$$

By using the method of unknown coefficients, we get

$$\begin{aligned}
c_2 &= -\frac{1}{2\cdot 1}\, a_0 c_0 \\
c_3 &= -\frac{1}{3\cdot 2}\, (a_0 c_2 + a_1 c_1 + a_2 c_0) \\
c_4 &= -\frac{1}{4\cdot 3}\left[a_0\left(-\frac{1}{2\cdot 1}\, a_0 c_0\right) + a_1 c_1 + a_2 c_0\right] \\
&\ \vdots
\end{aligned}$$

and the solution is

$$
\begin{aligned}
y(x) = c_0 \Bigg[& 1 - \frac{1}{1\cdot 2}\, a_0 x^2 - \frac{1}{2\cdot 3}\, a_1 x^3 - \frac{1}{3\cdot 4}\, a_2 x^4 - \frac{1}{4\cdot 5}\, a_3 x^5 - \cdots - \\
& + a_0 \left(\frac{a_0}{1\cdot 2\cdot 3\cdot 4}\, x^4 + \frac{a_1}{2\cdot 3\cdot 4\cdot 5}\, x^5 + \frac{a_2}{3\cdot 4\cdot 5\cdot 6}\, x^6 + \cdots \right) + \\
& + a_1 \left(\frac{a_0}{1\cdot 2\cdot 4\cdot 5}\, x^5 + \frac{a_1}{2\cdot 3\cdot 5\cdot 6}\, x^6 + \cdots \right) + \\
& + a_2 \left(\frac{a_0}{1\cdot 2\cdot 5\cdot 6}\, x^6 + \frac{a_1}{2\cdot 3\cdot 6\cdot 7}\, x^7 + \cdots \right) + \cdots \Bigg] + \\
& + c_1 \Bigg[x - \frac{1}{2\cdot 3}\, a_0 x^3 - \frac{1}{3\cdot 4}\, x^4 - \frac{1}{4\cdot 5}\, a_2 x^5 - \cdots \\
& + a_0 \left(\frac{a_0}{2\cdot 3\cdot 4\cdot 5}\, x^5 + \frac{a_1}{3\cdot 4\cdot 5\cdot 6}\, x^6 + \cdots \right) + \\
& + a_1 \left(\frac{a_0}{2\cdot 3\cdot 5\cdot 6}\, x^6 + \frac{a_1}{3\cdot 4\cdot 6\cdot 7}\, x^7 + \cdots \right) + \cdots \Bigg].
\end{aligned}
$$

This is the end of the well known method for determining a solution via power series. As the series in brackets behind c_0 and c_1 are, in fact, double integrals of the terms of series (4), by rearranging of the coefficients, the solution is

$$
\begin{aligned}
y(x) = c_0 \Bigg[& 1 - \int_0^x dx \int_0^x a_0\, dx - \int_0^x dx \int_0^x a_1 x\, dx - \int_0^x dx \int_0^x a_2 x^2\, dx - \cdots \\
& + a_0 \left(\int_0^x dx \int_0^x dx \int_0^x dx \int_0^x a_0\, dx + \int_0^x dx \int_0^x dx \int_0^x dx \int_0^x a_1 x\, dx + \cdots \right) + \\
& + a_1 \left(\int_0^x dx \int_0^x x\, dx \int_0^x dx \int_0^x a_0\, dx + \int_0^x dx \int_0^x x dx \int_0^x dx \int_0^x a_1 x\, dx + \cdots \right) + \cdots \Bigg] \\
& + c_1 \Bigg[x - \int_0^x dx \int_0^x x a_0\, dx - \int_0^x dx \int_0^x a_1 x^2\, dx - \int_0^x dx \int_0^x a_2 x^3\, dx - \cdots \\
& + a_0 \left(\int_0^x dx \int_0^x dx \int_0^x dx \int_0^x a_0 x\, dx + \int_0^x dx \int_0^x dx \int_0^x dx \int_0^x a_1 x^2\, dx + \cdots \right) + \\
& + a_1 \left(\int_0^x dx \int_0^x x\, dx \int_0^x dx \int_0^x a_0 x\, dx + \int_0^x dx \int_0^x x\, dx \int_0^x dx \int_0^x a_1 x^2\, dx + \cdots \right) + \cdots \Bigg].
\end{aligned}
$$

By condensation and by using (4), the solution is

$$\begin{aligned} y(x) =c_0 \Bigg[& 1 - \int_0^x\int_0^x a(x)\,dx^2 + \int_0^x\int_0^x a(x)\,dx^2 \int_0^x\int_0^x a(x)\,dx^2 - \\ & - \int_0^x\int_0^x a(x)\,dx^2 \int_0^x\int_0^x a(x)\,dx^2 \int_0^x\int_0^x a(x)\,dx^2 + \cdots \Bigg] + \\ +c_1 \Bigg[& x - \int_0^x\int_0^x xa(x)\,dx^2 + \int_0^x\int_0^x a(x)\,dx^2 \int_0^x\int_0^x xa(x)\,dx^2 - \\ & - \int_0^x\int_0^x a(x)\,dx^2 \int_0^x\int_0^x a(x)\,dx^2 \int_0^x\int_0^x xa(x)\,dx^2 + \cdots \Bigg] \end{aligned}$$

or

$$y(x) = c_0 y_1 + c_1 y_1 \tag{5}$$

where

$$y_1(x) = \sum_{k=0}^{\infty} (-1)^k \underbrace{\int_0^x\int_0^x a(x)\,dx^2 \int_0^x\int_0^x a(x)\,dx^2 \cdots \int_0^x\int_0^x a(x)\,dx^2}_{k-\text{double fold integrals}} \tag{6}$$

and

$$y_2(x) = x + \sum_{k=1}^{\infty} (-1)^k \underbrace{\int_0^x\int_0^x a(x)\,dx^2 \int_0^x\int_0^x a(x)\,dx^2 \cdots \int_0^x\int_0^x xa(x)\,dx^2}_{k-\text{double fold integrals}} \tag{7}$$

and c_0 and c_1 are arbitrary constants. So we can give the following:

Theorem 1. *For the canonical linear differential equation of the* II *order* (3), *for which coefficient $a(x)$ the hypotheses H_1 and H_2 are valid,* (5) *is a general solution where the power series* (6) *and* (7) *are particular solutions*

Proof. First of all, we prove that (6) and (7) are solutions. If we differentiate (6) two times, we get

$$\begin{aligned} y_1''(x) &= - a(x) + a(x) \int_0^x\int_0^x a(x)\,dx^2 - a(x) \int_0^x\int_0^x a(x)\,dx^2 \int_0^x\int_0^x a(x)\,dx^2 + \cdots \\ &= - a(x) \left(1 - \int_0^x\int_0^x a(x)\,dx^2 + \int_0^x\int_0^x a(x)\,dx^2 \int_0^x\int_0^x a(x)\,dx^2 - \cdots \right) \\ &= - a(x) y_1(x) \,, \end{aligned}$$

which proves the theorem. The proof that $y_2(x)$ is a solution of (3) is the same. The solutions (6) and (7) can be considered as solutions in a wider sense, because they are expressed via series of an infinite number of integrals of the equation coefficient.

As the Wronskian determinant

$$W(y_1, y_2) = \begin{vmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1,$$

the solutions (6) and (7) are two independent solutions and their linear combination (5) is a general solution in a wider sense of equation (3).

Remark. The solutions (6) and (7) can be considered as generalized trigonometric functions of the II order, and the solution (6) is a generalized sine with a base $a(x)$, and (7) is a generalized cosine with a base $a(x)$. If base $a(x) = 1$, the well-known trigonometric functions

$$\sin(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$
$$\cos(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

are obtained.

1.2. Canonical differential equation of the III order

Consequently, analogous to the case of the canoncal equation of the II order, we can give the theorem for the solutions of the canonical equation of the III order.

Theorem 2. *The canonical linear dierential equation of the* III *order*

$$y''' + a(x)\, y = 0 \tag{8}$$

(in which the hypotheses H_1 and H_2 are valid for the coeficient $a(x)$), has a general solution

$$y = c_1 y_1 + c_2 y_2 + c_3 y_3 \tag{9}$$

where

$$y_1 = 1 - \int_0^x\!\int_0^x\!\int_0^x a(x)\, dx^3 + \int_0^x\!\int_0^x\!\int_0^x a(x)\, dx^3 \int_0^x\!\int_0^x\!\int_0^x a(x)\, dx^3 - \cdots \tag{10}$$

$$y_2 = x - \int_0^x\!\int_0^x\!\int_0^x xa(x)\, dx^3 + \int_0^x\!\int_0^x\!\int_0^x a(x)\, dx^3 \int_0^x\!\int_0^x\!\int_0^x xa(x)\, dx^3 - \cdots \tag{11}$$

$$y_3 = x^2 - \int_0^x\int_0^x\int_0^x x^2 a(x)\,dx^3 + \int_0^x\int_0^x\int_0^x a(x)\,dx^3 \int_0^x\int_0^x\int_0^x x^2 a(x)\,dx^3 - \cdots \tag{12}$$

are particular solutions and c_i, $(i = 1, 2, 3)$ *are arbitrary constants.*

Proof. First we prove that (10), (11) and (12) are solutions. If we differentiate (10) three times, we get

$$\begin{aligned} y_1'''(x) &= -\,a(x) + a(x)\int_0^x\int_0^x\int_0^x a(x)\,dx^3 - a(x)\int_0^x\int_0^x\int_0^x a(x)\,dx^3 \int_0^x\int_0^x\int_0^x a(x)\,dx^3 + \cdots \\ &= -\,a(x)\left(1 - \int_0^x\int_0^x\int_0^x a(x)\,dx^3 + \int_0^x\int_0^x\int_0^x a(x)\,dx^3 \int_0^x\int_0^x\int_0^x a(x)\,dx^3 - \cdots\right) \\ &= -\,a(x)\,y_1(x)\,, \end{aligned}$$

which proves the theorem. The proof that $y_2(x)$ and $y_3(x)$ are solutions of (8) is the same. The solutions (10), (11) and (12) can be considered as solutions in a wider sense, because they are expressed via series of an infnite number of integrals of the equation coefficient.

The Wronskian determinant

$$W(y_1, y_2, y_3) = \begin{vmatrix} y_1(0) & y_2(0) & y_3(0) \\ y_1'(0) & y_2'(0) & y_3'(0) \\ y_1''(0) & y_2''(0) & y_3''(0) \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2! \end{vmatrix} = 2! \cdot 1!\,,$$

so the solutions (10), (11) and (12) are three independent solutions and their linear combination (9) is a general solution in a wider sense of equation (8).

1.3. Canonical differential equation of the n–th order

Theorem 3. *The canonical differential equation*

$$y^{(n)} + a(x)y = 0 \tag{13}$$

in which the hpothesis H_1 *and* H_2 *are valid for the coefficient* $a(x)$, *has a general solution*

$$y(x) = \sum_{k=1}^{n} c_k y_k \tag{14}$$

where the series

$$y_1 = 1 - \int_0^x\int_0^x \cdots \int_0^x a(x)\,dx^n + \int_0^x\int_0^x \cdots \int_0^x a(x)\,dx^n \int_0^x\int_0^x \cdots \int_0^x a(x)\,dx^n - \cdots$$

$$y_2 = x - \int_0^x\int_0^x \cdots \int_0^x xa(x)\,dx^n + \int_0^x\int_0^x \cdots \int_0^x a(x)\,dx^n \int_0^x\int_0^x \cdots \int_0^x xa(x)\,dx^n - \cdots$$

$$y_3 = x^2 - \int_0^x\int_0^x \cdots \int_0^x x^2a(x)\,dx^n + \int_0^x\int_0^x \cdots \int_0^x a(x)\,dx^n \int_0^x\int_0^x \cdots \int_0^x x^2a(x)\,dx^n - \cdots$$

$$\vdots$$

$$y_n = x^{n-1} - \int_0^x\int_0^x \cdots \int_0^x x^{n-1}a(x)\,dx^n + \int_0^x\int_0^x \cdots \int_0^x a(x)\,dx^n \int_0^x\int_0^x \cdots \int_0^x x^{n-1}a(x)\,dx^n - \cdots \tag{15}$$

are n particular solutions (the integrals in series (15) *are n-fold integrals) and c_k, $(k = 1, 2, \dots, n)$ are arbitrary consants.*

Proof. If the series from (15) are diferentiated n–times, for example the series $y_1(x)$, we obtain

$$\begin{aligned} y_1^{(n)} &= \\ &= -a(x) + a(x)\int_0^x\int_0^x \cdots \int_0^x a(x)\,dx^n - \int_0^x\int_0^x \cdots \int_0^x a(x)\,dx^n \int_0^x\int_0^x \cdots \int_0^x a(x)\,dx^n + \cdots \\ &= -a(x)\left(1 - \int_0^x\int_0^x \cdots \int_0^x a(x)\,dx^n + \int_0^x\int_0^x \cdots \int_0^x a(x)\,dx^n \int_0^x\int_0^x \cdots \int_0^x a(x)\,dx^n - \cdots\right) \\ &= -a(x)\,y_1(x)\,, \end{aligned}$$

which proves that $y_1(x)$ is a solution of (13). In the same way, using the same procedure, we can prove that every series from (15) is a solution of (13).

As the Wronskian determinant for $x = x_0 = 0$ is

$$W(x_0) = \begin{vmatrix} y_1(0) & y_2(0) & \cdots & y_n(0) \\ y_1'(0) & y_2'(0) & \cdots & y_n'(0) \\ \cdot & \cdot & \cdot & \cdot \\ y_1^{(n-1)}(0) & y_2^{(n-1)}(0) & \cdots & y_n^{(n-1)} \end{vmatrix} = 1! \cdot 2! \cdot \ldots \cdot (n-1)! \neq 0\,,$$

the solutions $y_k(x)$, $(k = 1, 2, \dots, n)$ are linearly independent, the system (15) is a fundamental system of particular solutions and (14) is a general solution in a wider sense of equation (13).

2. Canonical nonhomogeneous differential equations

2.1. Canonical nonhomogeneous equation of the II order

For the nonhomogeneous canonical differential equation of the II order

$$y'' + a(x)\,y = f(x)\,, \tag{16}$$

in which coeficients $a(x)$ and $f(x)$ satisfy the hypothesis H_1 and H_2, the following theorem exists:

Theorem 4. *The power series*

$$Y = F(x) - \int_0^x\!\!\int_0^x F(x)\,a(x)\,dx^2 + \int_0^x\!\!\int_0^x a(x)\,dx^2 \int_0^x\!\!\int_0^x F(x)\,a(x)\,dx^2 - \cdots \tag{17}$$

is a particular solution of (16), *where* $F(x) = \int_0^x\!\!\int_0^x f(x)\,dx^2$.

Proof. By differentiating (17) two times

$$\begin{aligned} Y'' =& F''(x) - F(x)a(x) + a(x)\int_0^x\!\!\int_0^x F(x)a(x)\,dx^2 - \cdots = \\ =& F''(x) - a(x)\Bigg[F(x) - \int_0^x\!\!\int_0^x F(x)\,dx^2 \\ & + \int_0^x\!\!\int_0^x a(x)\,dx^2 \int_0^x\!\!\int_0^x F(x)a(x)\,dx^2 - \cdots\Bigg] = \\ =& f(x) - a(x)\,Y \end{aligned}$$

the theorem is proved.

Without any difficulties in proving, we can give the following theorem:

Theorem 5. *The power series*

$$Y_1 = [1 + F(x)] - \int_0^x\!\!\int_0^x [1 + F(x)]\,a(x)\,dx^2 + \int_0^x\!\!\int_0^x a(x)\,dx^2 \int_0^x\!\!\int_0^x [1 + F(x)]\,a(x)\,dx^2 - \cdots$$

and

$$Y_2 = [x + F(x)] - \int_0^x\!\!\int_0^x [x + F(x)]\,a(x)\,dx^2 + \int_0^x\!\!\int_0^x a(x)\,dx^2 \int_0^x\!\!\int_0^x [x + F(x)]\,a(x)\,dx^2 - \cdots$$

are particular solutions of (16) *as well, where* $F(x) = \int_0^x\!\!\int_0^x f(x)\,dx^2$.

2.2. Canonical nonhomogeneous equation of the n-th order

For the nonhomogeneous canonical differential equation

$$y^{(n)} + a(x)\,y = f(x) \tag{18}$$

where cofficients $a(x)$ and $f(x)$ satisfy the hpothesis H_1 and H_2, we can give following theorem:

Theorem 6. *The function*

$$Y = F(x) - \int_0^x\int_0^x \dots \int_0^x F(x)a(x)\,dx^n + \int_0^x\int_0^x \dots \int_0^x a(x)\,dx^n \int_0^x\int_0^x \dots \int_0^x F(x)a(x)\,dx^n - \dots \tag{19}$$

is a particular solution of nonhomogeneous equation (18) *where* $F(x) = \int_0^x\int_0^x \dots \int_0^x f(x)\,dx^n$. (*The integrals in* (19) *are n-fold integrals*).

The proof of theorem 6 is the same as the proof of theorem 5. The presented solution (19) can be considered as solution in a wider sense.

3. General differential equation of the II order

It is known that the general differential equation of the II order

$$y'' + a_1(x)y' + a_2(x)y = f(x) \tag{20}$$

in which the hipothesis H_1 and H_2 are valid for coefficients $a_1(x)$, $a_2(x)$, $f(x)$, can be transformed by substitution

$$y = \exp\left(-\frac{1}{2}\int a_1(x)\,dx\right) z$$

into a canonical form

$$z'' + A(x)z = f(x)\exp\left(-\frac{1}{2}\int a_1(x)\,dx\right),$$

where

$$A(x) = a_2(x) - \frac{1}{4}\,a_1^2(x) - \frac{1}{2}\,a_1'(x) \tag{21}$$

is an analytical function. According to the previous theorems 1, 4 and 5, there exist the following:

Theorem 7. *The power series*

(22)

$$y(x)=\exp\left(-\frac{1}{2}\int a_1(x)dx\right)\left\{c_1\left[1-\int_0^x\int_0^x A(x)\,dx^2 + \int_0^x\int_0^x A(x)\,dx^2\int_0^x\int_0^x A(x)\,dx^2-\cdots\right]+ + c_2\left[x-\int_0^x\int_0^x xA(x)\,dx^2+\int_0^x\int_0^x A(x)\,dx^2\int_0^x\int_0^x xA(x)\,dx^2-\cdots\right]+ + F(x)-\int_0^x\int_0^x F(x)\,A(x)\,dx^2+\int_0^x\int_0^x A(x)\,dx^2\int_0^x\int_0^x F(x)\,A(x)\,dx^2-\cdots\right\}$$

is a general solution of equation (20), where $A(x)$ is given by (21) and

$$F(x)=\int_0^x\int_0^x f(x)\exp\left(-\frac{1}{2}\int a_1(x)\,dx\right)dx^2\,.$$

Hence the solution of every linear differential equation of the II order with analytical coefficients can be expressed via series of an infinitive number of integrals of the coefficients, it is considered that it can be solved by quadratures in a wider sense.

4. Example

The nonlinear differential equations that can be transformed into linear equations, are considered as a quadrature solved in a wider sense. For example, such is the Riccati equation

$$y'=-y^2-xy-2 \tag{23}$$

which by substitution

$$y=\frac{u'}{u}$$

is transformed into a linear differential equation of the II order

$$u''+xu'+2u=0\,. \tag{24}$$

By substituting

$$u=\exp\left(-\frac{1}{2}\int x\,dx\right)w=\exp\left(-\frac{x^2}{4}\right)w$$

it is transformed into

$$w'' + \left(\frac{3}{2} - \frac{x^2}{4}\right) w = 0$$

and according (7), it has a particular solution

$$w_2 = x - \int_0^x\int_0^x x\left(\frac{3}{2} - \frac{x^2}{4}\right) dx^2$$

$$+ \int_0^x\int_0^x \left(\frac{3}{2} - \frac{x^2}{4}\right) dx^2 \int_0^x\int_0^x x\left(\frac{3}{2} - \frac{x^2}{4}\right) dx^2 - \cdots$$

$$= x - \left(\frac{x^3}{4} - \frac{x^5}{5\cdot 4}\right) + \left(\frac{3x^5}{2^3\cdot 2^2\cdot 5} - \frac{13x^7}{7\cdot 3\cdot 2^6\cdot 5} + \frac{x^9}{9\cdot 8\cdot 4^3\cdot 5}\right)$$

$$- \left(\frac{9x^7}{7\cdot 6\cdot 8\cdot 4\cdot 5\cdot 2} + \cdots\right) + \cdots$$

$$= x\left[1 - \frac{x^2}{2^2} + \frac{x^4}{2^4}\left(\frac{1}{5} + \frac{3}{2\cdot 5}\right) - \frac{x^6}{2^6}\left(\frac{13}{7\cdot 3\cdot 5} + \frac{9}{7\cdot 3\cdot 5\cdot 2}\right) + \cdots\right]$$

$$= x\left[-\left(\frac{x}{2}\right)^2 + \frac{\left[\left(\frac{x}{2}\right)^2\right]^3}{2} - \frac{\left[\left(\frac{x}{2}\right)^2\right]^3}{3} + \cdots\right] = x\cdot\exp\left[-\left(\frac{x}{2}\right)^2\right]$$

$$= x\cdot\exp\left(-\frac{x^2}{4}\right).$$

So

$$u_2 = \exp\left(-\frac{x^2}{4}\right) w_2 = \exp\left(-\frac{x^2}{4}\right) x\cdot\exp\left(-\frac{x^2}{4}\right) = x\cdot\exp\left(-\frac{x^2}{2}\right)$$

is the particular solution of linear equation (24), and

$$y = \frac{u_2'}{u_2} = \frac{1 - x^2}{x}$$

is the particular integral of the Riccati equation (23).

Remark. The particular solution in this example has been condensed, and the solution is done in a closed form. Condensations of such type can be done only in the case, when the series can be summed up.

5. Conclusion

For the linear differential equations with analytical coefficients, first we can present the formulas for the solutions. So, the homegeneous canonical equation of the n-th order (13) has n particular solutions (15), the non-homogeneous equation (18) has a particular solution (19), and the general linear differential equation of the II order (20) has a general solution (22). The solutions presented as series of integrals via the equation coefficients are considered as solutions in a wider sense. Such solutions are convenient for numerical treatment, because they explicitly express the dependence of the solution on the equation coefficients.

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