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TOPOLOGY VERSUS SEQUENTIAL CONVERGENCE

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Abstract. We mention some situations in probability where sequential continuity is much more natural than continuity and which lead to the notion of sequential envelope. We present a survey of constructions in the realm of sequential convergence which are analogous to well-known topological constructions. From the categorical viewpoint the constructions are very similar, but the properties of the resulting object are strikingly different. In particular, we compare the Čech-Stone compactification, the Hewitt realcompactification, topological group and ring completions, and the Stone duality with their sequential counterparts. We study equivalence of the sequential convergence spaces of events with respect to probability measures. Finally, we deal with the relationship between an s-perfect field of sets and the induced 0-dimensional topological space. We show that the category of N-compact topological spaces is isomophic with the dual category of a subcategory of s-perfect fields of sets and sequentially continuous homomorphisms.

0. Introduction

The Čech-Stone compactification and the Hewitt realcompactification are apparently the most familiar constructions dealing with the extension of continuous maps in general topology. From the viewpoint of categorical topology, these two constructions are just special cases of epireflection. In applications of general topology there are situations in which sequential convergence and sequential continuity of maps seems to be more natural than topology and continuity of maps. Our talk has two goals.

1. We survey some epireflections dealing with the extension of sequentially continuous maps and to compare their properties and the properties of their topological counterparts.

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2. We present some new results related to sequential convergence and sequential continuity and to the mathematical foundations of probability.

Let $X \neq \varnothing$. Recall that a **sequential convergence** on X can be considered as a subset $\mathbb{L}$ of $X^N \times X$ satisfying certain axioms of convergence; if $(\langle x_n \rangle, x) \in \mathbb{L}$, then we say that the sequence $\langle x_n \rangle$ converges to x (under $\mathbb{L}$). We always assume the following two axioms: (i) each constant sequence $\langle x \rangle$ converges to x; (ii) if $\langle x_n \rangle$ converges to x, then each subsequence $\langle x'_n \rangle$ of $\langle x_n \rangle$ converges to x. If $\mathbb{L}$ satisfies (i) and (ii), then $(X, \mathbb{L})$ is said to be an $\mathcal{L}$-space. Additional axioms of convergence are: (iii) the uniqueness of limits; (iv) the Urysohn axiom. If X carries algebraic operations, then $\mathbb{L}$ is said to be compatible if each operation is sequentially continuous. For example, let $\mathbb{A}$ be a field of subsets of $Y \neq \varnothing$. Then a sequence $\langle A_n \rangle$ converges to A whenever $A = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} A_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n$ and compatibility means: if $\langle A_n \rangle \to A$ and $\langle B_n \rangle \to B$, then $\langle A_n \cup B_n \rangle \to A \cup B$, $\langle A_n \cap B_n \rangle \to A \cap B$, $\langle Y \setminus A_n \rangle \to Y \setminus A$.

More information about sequential convergences and compatible sequential convergences can be found in NOVÁK [1965], FRIČ and KOUTNÍK [1989], [1992], and JAKUBÍK [1988]. Standard references on Boolean algebras and category theory are SIKORSKI [1962] and HERRLICH and STRECKER [1976], respectively.

1. A survey

Consider the Čech-Stone compactification βN of the discrete space N of natural numbers. Since no one-to-one sequence converges in βN, many important topological constructions are completely outside the sequential convergence realm. On the other hand, some important classes of functions are sequentially closed but not closed in product spaces $\{0,1\}^X$ and R^X. Indeed, let $\mathbb{A}$ be a field of susets of X. If we identify each $A \in \mathbb{A}$ with its characteristic function χ_A, $\chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 1$ otherwise, the usual sequential convergence in $\mathbb{A}$ is equivalent to the pointwise convergence of characteristic functions, and the generated σ-field $\sigma(\mathbb{A})$ is the smallest sequentially closed subset in $\{0,1\}^X$ containing $\mathbb{A}$. Similarly, if X is a completely regular topological space, then the set $B(X)$ of all Borel measurable functions is the smallest subset of R^X which contains the set $C(X)$ of all continuous functions and it is sequentially closed in the pointwise convergence. Except trivial cases, both $\sigma(A)$ and $B(X)$ fail to be closed.

The idea that the extension of probability measures from a field of events to the generated σ-field is of a topological nature was presented by J. Novák in 1954 at a meeting on Probability and Statistics in Berlin and in 1956 at the III. Mathematical Congress in Moscow (see NOVÁK [1958]). The outcome is the theory of sequential envelopes outlined at the First Prague Topological

Symposium in 1961 (see NOVÁK [1962]) and presented in NOVÁK [1965]. It was M. Hušek who in 1971 (an unpublished manuscript) pointed out the categorical background of the sequential envelope: it is an epireflection of a sequential convergence space belonging to the category simply generated by the closed interval $[0,1]$ or the real line R to the subcategory of absolutely sequentially closed spaces. The construction has been generalized to quite arbitrary classes of real-valued functions in NOVÁK [1968] and applied to the extension of probabilities. It is known that if m is an additive measure on a ring $\mathbb{A}$ of subsets, then σ-additivity is equivalent to the monotone continuity of m at $\varnothing$. As proved by J. NOVÁK in [1958], if m is bounded, then σ-additivity implies the sequential continuity of m: $A_n \to A$ implies $m(A_n) \to m(A)$. Remember, each field of subsets carries the convergence which is initial with respect to probabilities. J. Novák proved that if we start with a field $\mathbb{A}$ of subsets, then the generated σ-field $\sigma(\mathbb{A})$ has the following properties:

1. $\sigma(\mathbb{A})$ is the smallest sequentially closed subset of $\sigma(\mathbb{A})$ containing $\mathbb{A}$;

2. Each probability p on $\mathbb{A}$ can be uniquely extended to a probability p' on $\sigma(\mathbb{A})$;

3. $\sigma(\mathbb{A})$ is absolutely sequentially closed with respect to the extension of probabilities.

Let us explain the notion of the absolute sequential closedness. Let X be a set carrying an initial sequential convergence with respect to a class $\mathcal{C}_0$ of sequentially continuous real-valued functions on X. If $\mathcal{C}_0$ separates points of X, then the $\mathcal{L}$-space $(X,\mathcal{L})$ is said to be **$\mathcal{C}_0$-sequentially regular**. In extending $\mathcal{C}_0$ we are looking for a set X' carrying a sequential convergence $\mathbb{L}'$ such that:

(e1) $X \subset X'$, $\mathbb{L}'$ restricted to X is equal to $\mathbb{L}$, and X' is the smallest sequentially closed subset of X' containing X;

(e2) Each $f \in \mathcal{C}_0$ can be extended to a sequentially continuous function on X' and $(X',\mathcal{L}')$ is $\overline{\mathcal{C}_0}$-sequentially regular, where $\overline{\mathcal{C}_0} = \{f \in \mathcal{C}(X); f \restriction X \in \mathcal{C}_0)$.

If there are no such X' and $\mathbb{L}'$, $X \neq X'$, then $(X,\mathbb{L})$ is said to be $\mathcal{C}_0$-sequentially complete (cf. FRIČ [1976], FRIČ and KOUTNÍK [1992]). We shall call such spaces **absolutely sequentially closed** with respect to the extension of $\mathcal{C}_0$. If no confusion can arise, we shall speak of absolutely sequentially closed spaces.

Recall the definition of a **$\mathcal{C}_0$-sequential envelope** of a $\mathcal{C}_0$-sequentially regular $\mathcal{L}$-space $(X,\mathbb{L})$. It is an $\mathcal{L}$-space $(X',\mathbb{L}')$ satisfying conditions (e1), (e2) and

(e3) $(X',\mathbb{L}')$ is absolutely sequentially closed with respect to $\overline{\mathcal{C}_0}$.

The $\mathcal{C}_0$-sequential envelope always exists (it can be constructed via the evaluation map into the $\mathcal{L}$-power R^{C_0} and taking the smallest sequentially closed set containing the image of the space in question) and it is unique up to a homeomorphism fixed on the original space.

Thus the sequential envelope with respect to all sequentially continuous functions and all bounded sequentially continuous functions is of the same nature as the Hewitt realcompactification and the Čech-Stone compactification of a completely regular topological space, respectively. Observe that the two sequential envelopes (unlike in the case of ν and β) are identical. In fact, a stronger result holds true.

For $E \subset R, |E| \geqq 2$, let C_E be the set of all sequentially continuous functions on X into E. If E does not contain an interval, then the C_E-sequential envelope and the $C_{\{0,1\}}$-sequential envelopes coincide. If E contains an interval, than the C_E-sequential and the $C_{[0,1]}$-sequential envelopes coincide. Hence each space can have at most two different C_E-sequential envelopes and there are spaces with two such different envelopes, cf. FRIČ [1972], [1976]. Generalizations of the sequential envelope appeared in KENT and RICHARDSON [1979] and the categorical background can be found in SCHRODER [1995].

The natural objection to $\sigma(\mathbb{A})$ - as the sequential envelope of $\mathbb{A}$ with respect to probabilities - is that the algebraic structure is not explicitly involved. Further, probabilities are not a true invariant in this otherwise categorical construction. As it has turned out, the intrinsic completion theory of rings carrying a compatible sequential convergence does not provide a solution. In different categories of convergence rings the intrinsic categorical completions (as epireflections into various subcategories of complete convergence rings - recall that a convergence ring is complete if each Cauchy sequence converges) do exist but, when applied to a field $\mathbb{A}$ of subsets, they do not yield $\sigma(\mathbb{A})$. In contradistinction to the topological case, a convergnce ring can have many nonhomeomorphic completions and it can happen that the categorical completion of the underlying group is not compatible with the underlying ring structure, cf. FRIČ and KOUTNÍK [1989], [1992].

Is $\sigma(\mathbb{A})$ a categorical construction? The ansver is positive, cf. FRIČ [1997].

Consider the category FS the objects of which are fields of subsets, carrying the Boolean operations $\bigcup, \bigcap, 0, 1, {}^c$, and the usual convergence of subsets, and morphisms of which are the sequentially continuous homomorphisms. Observe that each morphism is a homomorphism with respect to the symmetric difference $\triangle$ and hence a sequentially continuous ring homomorphism.

Let $\mathbb{A}$ be a field of subsets of X. Then $\mathbb{A}$ carries the initial sequential convergence with respect to $\mathrm{Hom}(\mathbb{A}, \mathbf{2})$, the set of all morphisms of $\mathbb{A}$ into the two-element Boolean algebra $\mathbf{2}$ and the evaluation map $\varphi : \mathbb{A} \to \mathbf{2}^{\mathrm{Hom}(\mathbb{A},\mathbf{2})}$

is an isomorphism (i.e. an algebraic isomorphism and a sequential homeomorphism) into.

Proposition 1.1. (FRIČ [1997]) *Let* $\mathbb{A}$ *be a field of subsets of* X. *Then* $\mathbb{A} = \sigma(\mathbb{A})$ *iff* $\mathbb{A}$ *is absolutely sequentially closed with respect to* $\hom(\mathbb{A}, \mathbf{2})$.

Denote $SCFS$ the (full and isomorphism closed) subcategory of FS consisting of σ-fields.

Proposition 1.2. (FRIČ [1997]) $\sigma : FS \to SCFS$ *is an epireflector.*

Now, let us turn to the relationship between Boolean algebras and fields of sets.

Usually, events in probability form a Boolean algebra. Each Boolean algebra is isomorphic to a reduced field of sets and each field of sets can be considered as a Boolean algebra. Further, each measurable map induces a Boolean homomorphism (going the opposite direction) and each Boolean homomorphism of the Borel subsets of the real line R into a field of sets is induced by a measurable function; in probability such function is called a random variable. These facts lead to a duality for fields of events. Homomorphisms preserve the structure of events, but are less useful when calculations are needed. The random variables preserve the structure of events only indirectly (via preimage), but provide a freeway to caclulus.

The usual nontopological Stone duality is not exactly what is needed in probability. On the one hand, the perfectness of the domain guarantees that each Boolean homomorphism is induced by a measurable map but, on the other hand, the perfectness amounts to the compactness and hence each additive probability measure is countably additive. Since in the probability theory some natural fields of sets are not perfect and some additive measures are not countably additive, it is natural to seek a duality where the perfectness is replaced by some weaker property. In FRIČ [1997] and [19∞] it was shown that s-perfectness of fields of sets is exactly what is needed.

Not to destroy completeness and cocompleteness of the categories to be dealt with, we do not exclude from our considerations the Boolean algebra for which $0 = 1$ and likewise the field of subsets for which the carrier set is empty. Denote by MM the category whose objects are reduced fields of sets (each two points can be separated by a measurable set; in what follows, all fields of sets will be reduced) and whose morphisms are measurable maps (MM stands for measurable sets and measurable maps, while the category of fields of subsets and sequentially continuous Boolean homomorphisms - as arrows going the opposite direction - it is denoted by FS). The nontopological version of the Stone duality asserts that the subcategory PMM of perfect fields and the category BA of all Boolean algebras and Boolean homomorphisms are dual (cf. SIKORSKI [1962], JOHNSTONE [1982]).

Let $\mathcal{A}$ be a Boolean algebra. Denote by $\hom(\mathcal{A}, \mathbf{2})$ the set of all Boolean homomorphisms of $\mathcal{A}$ into $\mathbf{2}$. By a Stone family of $\mathcal{A}$ we understand a subset H of $\hom(\mathcal{A}, \mathbf{2})$ such that if $a, b \in \mathcal{A}$ and $a \neq b$, then there exists $h \in H$ such that $h(a) \neq h(b)$.

Let $\mathbb{A}$ be a field of subsets of X. Each $A \in \mathbb{A}$ is represented by its characteristic function $\chi_A : X \longrightarrow \{0,1\}$, $\chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 0$ otherwise. The pointwise sequential convergence of characteristic functions is compatible with the Boolean and field structure of $\mathbb{A}$. Each point $x \in X$ represents a Boolean homomorphism $x : \mathbb{A} \longrightarrow \mathbf{2}$, $x(A) = 1$ if $x \in A$ and $x(A) = 0$ otherwise. Clearly, the homomorphism x is sequentially continuous and X is a Stone family of $\mathbb{A}$.

Let $\mathcal{A}$ be a Boolean algebra carrying a sequential convergence $\mathbb{L}$ such that $\langle x_n \rangle$ converges under $\mathbb{L}$ to x iff for each sequentially continuous homomorphism h of $\mathcal{A}$ into $\mathbf{2}$ the sequence $\langle h(x_n) \rangle$ converges in $\mathbf{2}$ to $h(x)$. Then $\mathcal{A}$ carrying $\mathbb{L}$ is said to be **2-generated** (cf. FRIČ [19∞]).

Denote by $\mathcal{B}(\mathbf{2})$ the category whose objects are $\mathbf{2}$-generated Boolean algebras and whose morphisms are sequentially continuous Boolean homomorphisms. Some of the properties of $\mathcal{B}(\mathbf{2})$ are described in FRIČ [19∞] and in FRIČ and JAKUBÍK [19∞].

Let $\mathbb{A}$ be a field of subsets of X. If each ultrafilter $\mathcal{F}$ of elements of $\mathbb{A}$ having the countable intersection property (CIP) is fixed (i.e. there exists $x \in X$ such that $\mathcal{F} = \{A \in \mathbb{A}; x \in A\}$), then $\mathbb{A}$ is said to be s-**perfect**.

Denote by SPMM the full subcategory of MM consisting of all s-perfect fields (s-perfect fields have been presented at the BBFEST in 1996 in Cape Town). Some of the properties of SPMM are described in FRIČ [1997].

Denote by SPFS the full subcategory of FS consisting of s-perfect fields.

Proposition 1.3. (FRIČ [19∞]) *The dual category $SPFS^{op}$ of the category SPFS and the category SPMM are isomorphic.*

Proposition 1.4. (FRIČ [19∞]) *The categories $\mathcal{B}(\mathbf{2})$ and SPMM are dual.*

Let $\mathcal{A}$ be a Boolean algebra. Consider the initial sequential convergence $\mathbb{L}_{\mathcal{A}}$ on $\mathcal{A}$ with respect to $\hom(\mathcal{A}, \mathbf{2})$: $\langle a(n) \rangle$ converges to a under $\mathbb{L}_{\mathcal{A}}$ iff $\langle h(a(n)) \rangle$ converges to $h(a)$ in $\mathbf{2}$ for all $h \in \hom(\mathcal{A}, \mathbf{2})$. Clearly, $\mathbb{L}_{\mathcal{A}}$ is $\mathbf{2}$-generated and it is finer than any other $\mathbf{2}$-generated convergence $\mathbb{L}$ on $\mathcal{A}$ (i.e., $\mathbb{L}_{\mathcal{A}} \subseteq \mathbb{L}$).

Let $\mathcal{A}$ be a Boolean algebra. Then the initial sequential convergence $\mathbb{L}_{\mathcal{A}}$ with respect to $\mathrm{Hom}(\mathcal{A}, \mathbf{2})$ is said to be **fine** and $(\mathcal{A}, \mathbb{L}_{\mathcal{A}})$ is said to be a fine $\mathbf{2}$-generated Boolean algebra.

Denote by $\mathcal{FB}(\mathbf{2})$ the full subcategory of $\mathcal{B}(\mathbf{2})$ consisting of fine objects.

Proposition 1.5. (FRIČ [19∞]) *The categories BA and $\mathcal{FB}(\mathbf{2})$ are isomorphic.*

Corollary 1.6. (The nontopological Stone duality.) *The categories BA and PMM are dual.*

2. Equivalence and maximality

Let $(X, \mathbb{L})$ be a $\mathcal{C}_0$-sequentially regular $\mathcal{L}$-space. Clearly, $(X, \mathbb{L})$ is $\mathcal{C}_1$-sequentially regular for each $\mathcal{C}_1$ such that $\mathcal{C}_0 \subset \mathcal{C}_1 \subset \mathcal{C}(X)$. There are nontrivial interesting situations such that $\mathcal{C}_0 \neq \mathcal{C}_1$ and the $\mathcal{C}_0$-sequential envelope of $\sigma_0(X, \mathbb{L})$ of $(X, \mathbb{L})$ is also its $\mathcal{C}_1$-sequential envelope.

The idea can be illustrated by the following two examples.

Example 2.1. Let $\mathbb{A}$ be a field of subsets of Y. For each $y \in Y$, let δ_y be the Dirac measure on $\mathbb{A}$ generated by y ($\delta_y(A) = 1$ if $y \in A$ and $\delta_y(A) = 0$ otherwise, $A \in \mathbb{A}$) and let $\mathcal{D}(\mathbb{A}) = \{\delta_y; y \in Y\}$ be the set of all Dirac measures on $\mathbb{A}$. It is known that $\mathbb{A}$ is $\mathcal{D}(\mathbb{A})$-sequentially regular and that the sequential envelope of $\mathbb{A}$ with respect to $\mathcal{D}(\mathbb{A})$ coincides with the generated σ-field $\sigma(\mathbb{A})$. Further, $\sigma(\mathbb{A})$ is also the sequential envelope of $\mathbb{A}$ with respect to each subset $\mathcal{P}_0$ of the set $\mathcal{P}(\mathbb{A})$ of all probability measures on $\mathbb{A}$ such that $\mathcal{D}(\mathbb{A}) \subset \mathcal{P}_0 \subset \mathcal{P}(\mathbb{A})$. Recall that the sequential envelope of $\mathbb{A}$ with respect to $\mathcal{P}_0$ is constructed as follows:

(i) we embed $\mathbb{A}$ into $[0,1]^{\mathcal{P}_0}$ via the evaluation map $ev_{\mathcal{P}_0}(A) = (p(A); p \in \mathcal{P}_0), A \in \mathbb{A}$;

(ii) we take the smallest sequentially closed subset $t - cl(ev_{\mathcal{P}_0}(\mathbb{A}))$ of $[0,1]^{\mathcal{P}_0}$ containing $ev_{\mathcal{P}_0}(\mathbb{A}) = \{ev_{\mathcal{P}_0}(A); A \in \mathbb{A}\}$.

Observe that $ev_{\mathcal{P}_0}(\mathbb{A})$ is homeomorphic (as a convergence space) to $\mathbb{A}$ and its t-closure is homeomorphic to $\sigma(\mathbb{A})$ for all $\mathcal{P}_0$ such that $\mathcal{D}(\mathbb{A}) \subset \mathcal{P}_0 \subset \mathcal{P}(\mathbb{A})$. The interesting case is when $\mathcal{P}_0$ is the set Y^* of all $\{0,1\}$-valued probabilities on $\mathbb{A}$. In fact, then $ev_{Y^*}(\mathbb{A})$ is a field of subsets of Y^* and its t-closure is the generated σ-field; denote it by $\mathbb{A}^*$ and $\sigma(\mathbb{A}^*)$, respectively. Moreover, we can consider Y as a subset of Y^* and then $\mathbb{A} = \mathbb{A}^* \cap Y, \sigma(\mathbb{A}) = \sigma(\mathbb{A}^*) \cap Y$. The most important fact is that $\mathbb{A}$ and $\mathbb{A}^*$, resp. $\sigma(\mathbb{A})$ and $\sigma(\mathbb{A}^*)$, are (algebraically) isomorphic in the natural way, the isomorphism is a (convergence) homeomorphism, and $\mathbb{A}$ and $\sigma(\mathbb{A}^*)$ "have the same probabilities" (cf. FRIČ [19∞]).

The next example shows that in general $Y \neq Y^*$.

Example 2.2. For $Y = \omega_1$ let $\mathbb{A}$ be the field of al finite subsets of Y and their complements. The generated σ- field $\sigma(\mathbb{A})$ consists of all countable subsets and their complements. It is known (cf. SIKORSKI [1964]) that there is a one-to-one correspondence between $\{0,1\}$-valued additive measures on $\mathbb{A}$, ultrafilters on $\mathbb{A}$, and homomorphisms of $\mathbb{A}$ into the two-element Boolean algebra **2**. Let m be a $\{0,1\}$-valued additive measure on $\mathbb{A}$. Then (cf. FRIČ [19∞]) m is σ-additive iff the corresponding ultrafilter on $\mathbb{A}$ has te CIP (countable intersection property). It is easy to see that the only free ultrafilter on $\mathbb{A}$ having the CIP consists of the complements of finite subsets of Y and its only extension to an ultrafilter on $\sigma(\mathbb{A})$ consists of the complements of countable subsets of Y and has the CIP, too. Put $Y^* = \omega_1 + 1$ and let $\mathbb{A}^*$ be the field of subsets of Y^* consisting of all finite subsets of $Y^* \setminus \{\omega_1\}$ and their complements in Y^*. Then the generatd σ-field $\sigma(\mathbb{A}^*)$ consits of all countable subsets of Y and their complements in Y^*. Observe that each ultrafilter on $\mathbb{A}^*$ having the CIP is fixed and hence each $\{0,1\}$-valued σ-additive measure m on $\mathbb{A}^*$ is a Dirac measure. It follows from Example 2.1 that $\sigma(\mathbb{A}^*) \cap Y$ yields an isomorphism and a homeomorphism of $\sigma(\mathbb{A})$ onto $\sigma(\mathbb{A}^*)$.

For $i = 1,2$, let $\mathbb{A}_i$ be a reduced field of subsets of Y_i and let $\sigma(\mathbb{A}_i)$ be the generated σ-field.

Definition 2.3. We say that $\mathbb{A}_1$ and $\mathbb{A}_2$ are $\mathcal{P}$-equivalent if there is a set Y and a reduced σ-field $\mathbb{A}$ of subsets of Y such that, for $i = 1,2$, there is a one-to-one map $f_i \colon Y_i \to Y$ satisfying the following conditions:

(e_1) $\{f_i(A); A \in \sigma(\mathbb{A}_i)\} = \mathbb{A} \cap f_i(Y_i)$;

(e_2) the correspondence $A \mapsto A \cap f_i(Y_i), A \in \mathbb{A}$, yields an isomorphism φ_i of $\mathbb{A}$ onto $\sigma(\mathbb{A}_i)$ such that both φ_i and $\varphi_i^{\leftarrow}$ are sequentially continuous.

Clearly, $\mathbb{A}$ is $\mathcal{P}$-equivalent both to $\mathbb{A}_1$ and $\mathbb{A}_2$. We claim that for each reduced field of subsets there exists a maximal $\mathcal{P}$-equivalent reduced field of subsets.

Proposition 2.4. *Let $\mathbb{A}_1$ and $\mathbb{A}_2$ be $\mathcal{P}$-equivalent. Then:*

(i) There is an isomorphism φ of $\sigma(\mathbb{A}_i)$ onto $\sigma(\mathbb{A}_2)$ such that both φ and $\varphi^{\leftarrow}$ are sequentially continuous;

(ii) φ induces a natural one-to-one correspondence between the sets $\mathcal{P}(\mathbb{A}_i)$ of all probability measures on $\mathbb{A}_i, i = 1,2$.

Proof. (i) Clearly, $\varphi = \varphi_2 \circ \varphi_i^{\leftarrow}$ and $\varphi^{\leftarrow} = \varphi_1 \circ \varphi_2^{\leftarrow}$ have the desired properties.

(ii) It is proved in NOVÁK [1958] that a bounded additive measure on a ring of sets is σ-additive iff it is sequentially continuous. If p is a probability on $\sigma(\mathbb{A}_2)$, then $p \circ \varphi$ is a probability on $\sigma(\mathbb{A}_1)$ and if q is a probability on $\sigma(\mathbb{A}_1)$, then $q \circ \varphi^{\leftarrow}$ is a probabiliy on $\sigma(\mathbb{A}_2)$. Clearly, this yields a one-to-one correspondence between the sets $\mathcal{P}(\sigma(\mathbb{A}_i))$ of all probability measures on $\sigma(\mathbb{A}_i), i = 1,2$. It is known that a σ-additive measure on a field of sets has a unique extension on the generated σ-field. Consequently, φ induces a natural one-to-one correspondence between $\mathcal{P}(\mathbb{A}_1)$ and $\mathcal{P}(\mathbb{A}_2)$.

Let $(X, \mathbb{L})$ be a subspace of an $\mathcal{L}$-space $(X', \mathbb{L}')$ and let $\mathcal{C}_0 \subset \mathcal{C}(X)$. Recall that $(X, \mathbb{L})$ is said to be $\mathcal{C}_0$-**embedded** in $(X', \mathbb{L}')$ if each $f \in \mathcal{C}_0$ can be extended to an $f' \in \mathcal{C}(X')$, i.e. $f = f' \restriction X$ for some $f' \in \mathcal{C}(X')$. If $\mathbb{A}$ is a subfield of a field $\mathbb{A}'$ of subsets of Y and each probability on $\mathbb{A}$ can be extended to a probability on $\mathbb{A}'$, then $\mathbb{A}$ is said to be $\mathcal{P}$-**embedded** in $\mathbb{A}'$. Motivated by Definition 2.3 and Proposition 2.4, we introduce the following generalization.

Definition 2.5. Let $\mathbb{A}'$ be a reduced field of subsets of Y'. Let Y be a subset of Y' such that the correspondence $A \mapsto A \cap Y, A \in \mathbb{A}'$, yields an isomorphism φ of $\mathbb{A}'$ onto $\mathbb{A}' \cap Y$ and let both φ and $\varphi^{\leftarrow}$ be sequentially continuous. Let $\mathbb{A}$ be a subfield of $\mathbb{A}' \cap Y$. If $\mathbb{A}$ is $\mathcal{P}$-embedded in $\mathbb{A}' \cap Y$, then $\mathbb{A}$ is said to be weakly $\mathcal{P}$-embedded in $\mathbb{A}'$.

Remark 2.6. Observe that in the proof of Proposition 2.4 it is actually proved that both $\mathbb{A}_1$ and $\mathbb{A}_2$ are weakly $\mathcal{P}$-embedded in $\mathbb{A}$.

Let $\mathbb{A}$ be a reduced field of subsets of Y and let $\sigma(\mathbb{A})$ be the generated σ-field. Let Y^* be the set of all $\{0,1\}$-valued probabilities on $\mathbb{A}$ (remember, Y^* can identified both with the set of all ultrafilters on $\mathbb{A}$ having the CIP and with the set of all sequentially continuous homomorphisms of $\mathbb{A}$ into $\mathbf{2}$). For $A \in \mathbb{A}$, define $A^* = \{y \in Y^*;\ y(A) = 1\}$ and put $\mathbb{A}^* = \{A^*;\ A \in \mathbb{A}\}$. Clearly (cf. Example 2.1), $\mathbb{A}$ is weakly $\mathcal{P}$-embedded both in $\mathbb{A}^*$ and in the generated σ-field $\sigma(\mathbb{A}^*)$.

Proposition 2.7. *(i) $\mathbb{A}$ and $\sigma(\mathbb{A}^*)$ are $\mathcal{P}$-equivalent.*

(ii) Let $\mathbb{A}'$ be a reduced field of subsets of Y' and let $\mathbb{A}$ and $\mathbb{A}'$ be $\mathcal{P}$-equivalent. Then there is a one-to-one map f of Y' into Y^* such that $f(\mathbb{A}')$ is a $\mathcal{P}$-embedded subfield of $\sigma(\mathbb{A}^*) \cap f(Y')$ and, finally, $\mathbb{A}'$ and $\sigma(\mathbb{A}^*)$ are $\mathcal{P}$-equivalent.

Proof. (i) The assertion can be proved by the same argument as used in Example 2.1.

(ii) Since $\mathbb{A}$ and $\mathbb{A}'$ are $\mathcal{P}$-equivalent, it follows from Proposition 2.4 that there is an isomorphosm φ of $\sigma(\mathbb{A}^*)$ onto $\sigma(\mathbb{A}')$ such that both φ and $\varphi^{\leftarrow}$ are sequentially continuous. Each CIP ultrafilter on $\mathbb{A}^*$ is fixed, hence $\mathbb{A}^*$ and, according to Proposition 2.4 in FRIČ [19∞], also $\sigma(\mathbb{A}^*)$ are s-perfect. Thus, by Proposition 2.6 in FRIČ [19∞], there is a uniquely determined measurable map f of $(Y', \sigma(\mathbb{A}'))$ into $(Y^*, \sigma(\mathbb{A}^*))$ such that the isomorphism φ is induced by f (i.e. $\varphi(A) = f^{\leftarrow}(A), A \in \mathbb{A}^*$). Clearly, f is one-to-one and straightforward calculations show that f has the desired properties.

Remark 2.8. Observe that the underlying set Y^* of $\sigma(\mathbb{A}^*)$ is maximal in the sense that $\mathbb{A}$ and $\sigma(\mathbb{A}^*)$ have "the same $\{0,1\}$-valued probabilities" and $\sigma(\mathbb{A}^*)$ is the maximal field of subsets of Y^* in which the isomorphic image of $\mathbb{A}$ is $\mathcal{P}$-embedded and sequentially dense.

Corollary 2.9. *$\sigma(\mathbb{A}^*)$ is the maximal reduced field of subsets which is $\mathcal{P}$-equivalent to $\mathbb{A}$.*

3. N-COMPACTNESS

This section is devoted to the topological background of Proposition 1.3.

Let $\mathbb{A}$ be a reduced field of subsets of X. Then $\mathbb{A}$ is a base of a 0-dimensional (Hausdorff) topology $\tau_{\mathbb{A}}$ on X. A subset M of X is clopen iff M is the union of all elements of $\mathbb{A}$ contained in M and at the same time the intersection of all elements of $\mathbb{A}$ containing M. Let Clopen($\mathbb{A}$) be the set of all clopen subsets. Recall, (cf. HERRLICH and STRECKER [1997]) that $\tau_{\mathbb{A}}$ is N-compact iff each clopen ultrafilter (i.e. a maximal centered family of clopen sets) is fixed.

Proposition 3.1. *Let $\mathbb{A}$ be s-perfect. Then $\tau_{\mathbb{A}}$ is N-compact.*

Proof. Let $\mathbb{F}$ be a clopen ultrafilter in X having the CIP. We have to prove that $\mathbb{F}$ is fixed.

Since $\mathbb{F}_{\mathbb{A}} = \{A \in \mathbb{A}; A \in \mathbb{F}\}$ is a maximal filter in $\mathbb{A}$, there exists $x \in X$ such that $\mathbb{F}_{\mathbb{A}} = \{A \in \mathbb{A}; x \in A\}$. It suffices to prove that $x \in F$ for all $F \in \mathbb{F}$ Now, indirectly, suppose that there exists $F \in \mathbb{F}$ such that $x \in (X \setminus F) \in$ Clopen($\mathbb{A}$). Since $\mathbb{A}$ is a base, there exists $A \in \mathbb{A}$ such that $x \in A \subset (X \setminus F)$. Then $F \subset (X \setminus A)$ and $(X \setminus A) \in F_{\mathbb{A}}$, a contradiction.

For $i = 1, 2$, let $\mathbb{A}_i$ be a reduced field of subsets of X_i, let $\tau_{\mathbb{A}_i}$ be the corresponding 0-dimensional topology on X_i, and let Clopen($\mathbb{A}_i$) be the field of clopen sets. Let f be a map of X_1 into X_2.

The proof of the next lemma is omitted.

Lemma 3.2. *The following are equivalent:*

(i) f is a measurable map of $(X_1, Clopen(\mathbb{A}_1))$ into $(X_2, Clopen(\mathbb{A}_2))$.

(ii) f is a continuous map of $(X_1, \tau_{\mathbb{A}_1})$ into $(X_2, \tau_{\mathbb{A}_2})$.

Since in a 0-dimensional space the topology and the field of clopen sets are equivalent, Lemma 3.2 enables us to look at Proposition 1.3 from the topological viewpoint. Denote by $CNSPFS$ the subcategory of FS consisting of all s-perfect fields $\mathbb{A}$ such that $\mathbb{A}$=Clopen($\mathbb{A}$). Denote by NC the category of all N-compact topological spaces (i.e. Hausdorff 0-dimensional spaces in which each clopen ultrafilter having the CIP is fixed).

Proposition 3.3. *The dual category $CNSPFS^{op}$ of the category $CNSPFS$ and the category NC are isomorphic.*

Proof. For $i = 1, 2$, assume that $\mathbb{A}_i$ is s-perfect and $\mathbb{A}_i$=Clopen($\mathbb{A}_i$). Let φ be a sequentially continuous homomorphism of $\mathbb{A}_2$ into $\mathbb{A}_1$. Then (cf. Proposition 2.6 in FRIČ [19∞]) there is a unique measurable map f of $(X_1, \mathbb{A}_1)$ into $(X_2, \mathbb{A}_2)$ such that $\varphi = f^{\leftarrow}$. According to Lemma 3.2, f is a continuous map of $(X_1, \tau_{\mathbb{A}_1})$ into $(X_2, \tau_{\mathbb{A}_2})$. But from Lemma 3.2 it follows that this yields a one-to-one correspondence between morphisms in $CNSPFS$ and NC and, clearly, an isomorphism between $CNSPFS^{op}$ and NC.

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