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Finite Difference Approximation for the 2D Heat Equation with Concentrated Capacity

Bratislav V. Sredojević[a], Dejan R. Bojović[b]

[a]*University of Kragujevac, Faculty of Mechanical and Civil Engineering in Kraljevo, Dositejeva 19, 36000 Kraljevo, Serbia*
[b]*University of Kragujevac, Faculty of Science, R. Domanovića 12, 34000 Kragujevac, Serbia*

Abstract. The convergence of difference scheme for two-dimensional initial-boundary value problem for the heat equation with concentrated capacity and time-dependent coefficients of the space derivatives, is considered. An estimate of the rate of convergence in a special discrete Sobolev norms , compatible with the smoothness of the coefficients and solution, is proved.

1. Introduction

The finite-difference method is one of the basic tools for the numerical solution of partial differential equations. In the case of problems with discontinuous coefficients and concentrated factors (Dirac delta functions, free boundaries, etc.) the solution has weak global regularity and it is impossible to establish convergence of finite difference schemes using the classical Taylor series expansion. Often, the Bramble-Hilbert lemma takes the role of the Taylor formula for functions from the Sobolev spaces [5], [6], [10].

Following Lazarov et al. [10], a convergence rate estimate of the form

$$\|u - v\|_{W^k_{2,h}} \leq Ch^{s-k}\|u\|_{W^s_2}, \quad s > k, \tag{1}$$

is called **compatible** with the smoothness (regularity) of the solution u of the boundary-value problem. Here v is the solution of the discrete problem, h is the spatial mesh step, W^s_2 and $W^k_{2,h}$ are Sobolev spaces of functions with continuous and discrete argument, respectively, C is a constant which doesn't depend on u and h. For the parabolic case typical estimates are of the form

$$\|u - v\|_{W^{k,k/2}_{2,h\tau}} \leq C(h + \sqrt{\tau})^{s-k}\|u\|_{W^{s,s/2}_2}, \quad s > k, \tag{2}$$

where τ is the time step. In the case of equations with variable coefficients the constant C in the error bounds depends on norms of the coefficients (see, for example, [6], [14], [1]).

One interesting class of parabolic problems model processes in heat-conducting media with concentrated capacity in which the heat capacity coefficient contains a Dirac delta function, or equivalently, the jump

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Email addresses: bratislavsredojevic9@gmail.com (Bratislav V. Sredojević), dbojovic68@gmail.com (Dejan R. Bojović)

of the heat flow in the singular point is proportional to the time-derivative of the temperature [11]. Such problems are nonstandard and the classical tools of the theory of finite difference schemes are difficult to apply to their convergence analysis.

In the present paper a finite-difference scheme, approximating the two-dimensonal initial-boundary value problem for the heat equation with concentrated capacity and time dependent coefficients is derived. Special Sobolev norms (corresponding to the norms $W_2^{1,1/2}$ and $W_2^{2,1}$ for a classical heat-conduction problem) is constructed. In this norm, a convergence rate estimate, compatible with the smoothness of the solution of the boundary value problem, is obtained.

Note that one-dimensional parabolic problem with weak solution is studied in [8] and [2]; 2D parabolic problem with concentrated capacity and variable (but not time-dependent) coefficients is considered in [9] and [3].

2. Differential problem and its approximation

Let us consider the 2D initial-boundary-value problem for the heat equation in the presence of a concentrated capacity at the line $x_2 = \xi$:

$$\begin{aligned} &(1 + K\delta(x_2 - \xi))\frac{\partial u}{\partial t} - \sum_{i}^{2} \frac{\partial}{\partial x_i}\left(a_i(x,t)\frac{\partial u}{\partial x_i}\right) + a(x,t)u = f\,, \ \text{ in } Q\,, \\ &u = 0\,, \ \text{ on } \partial\Omega \times (0,T)\,, \\ &u(x,0) = u_0(x)\,, \ \text{ on } \Omega\,, \end{aligned} \tag{3}$$

where $\delta(x)$ is the Dirac delta function, $K > 0$, $\Omega = (0,1)^2$, and $Q = \Omega \times (0,T)$.

Let $\bar{\omega}_h$– uniform mesh with step size h in $\bar{\Omega}$, $\omega_h = \bar{\omega}_h \cap \Omega$, $\omega_{1h} = \bar{\omega}_h \cap ([0,1) \times (0,1))$, $\omega_{2h} = \bar{\omega}_h \cap ((0,1) \times [0,1))$, $\sigma_h = \omega_h \cap \Sigma$, where $\Sigma = \{(x_1,\xi) | x_1 \in (0,1)\}$. Suppose that ξ is a rational number. Then one can choose step h so that $\sigma_h \neq \emptyset$. Let ω_τ be a uniform mesh on $(0,T)$ with the stepsize $\tau = T/m$, $\omega_\tau^- = \omega_\tau \cup \{0\}$, $\omega_\tau^+ = \omega_\tau \cup \{T\}$ and $\overline{\omega}_\tau = \omega_\tau \cup \{0,T\}$. Also we assume that the condition $c_1h^2 \le \tau \le c_2h^2$ is satisfied. Define finite differences in the usual way [13]:

$$v_{\bar{x}_i}(x,t) = \frac{v - v^{-i}}{h}\,, \ \ v_{x_i}(x,t) = \frac{v^{+i} - v}{h}\,, \ \ v_{\bar{t}}(x,t) = \frac{v(x,t) - v(x,t-\tau)}{\tau} = v_t(x,t-\tau)\,,$$

where $v^{\pm i}(x,t) = v(x \pm e_i h, t)$, $e_1 = (1,0)$, $e_2 = (0,1)$. The problem (3) can be approximated on the mesh $\overline{Q}_{h\tau} = \bar{\omega}_h \times \bar{\omega}_\tau$ by the following difference scheme with averaged right-hand side (see [7]):

$$\begin{aligned} &(1 + K\delta_h(x_2 - \xi))v_{\bar{t}} + L_h v = T_1^2 T_2^2 T_t^- f\,, \ \text{ in } Q_{h\tau}\,, \\ &v = 0\,, \ \text{ on } \gamma_h \times \omega_\tau^+\,, \quad v(x,0) = u_0(x)\,, \ \text{ on } \omega_h\,, \end{aligned} \tag{4}$$

where $L_h v = -\dfrac{1}{2}\displaystyle\sum_{i=1}^{2}((a_i v_{x_i})_{\bar{x}_i} + (a_i v_{\bar{x}_i})_{x_i}) + (T_1^2 T_2^2 T_t^- a)v,$

$$\delta_h(x_2 - \xi) = \begin{cases} 0\,, & x \notin \sigma_h \\ 1/h\,, & x \in \sigma_h \end{cases}$$

is the mesh Dirac function, and T_1^2, T_2^2, T_t^- are Steklov averaging operators defined as follows:

$$T_1 f(x_1,x_2) = T_1^{\pm} f(x_1 \mp h/2, x_2) = \frac{1}{h}\int_{x_1-h/2}^{x_1+h/2} f(x_1',x_2)dx_1' \,,$$

$$T_2 f(x_1,x_2) = T_2^{\pm} f(x_1, x_2 \mp h/2) = \frac{1}{h}\int_{x_2-h/2}^{x_2+h/2} f(x_1,x_2')dx_2' \,,$$

$$T_t^- f(x,t) = T_t^+ f(x,t-\tau) = \frac{1}{\tau}\int_{t-\tau}^{t} f(x,t')dt' \,.$$

Note that these operators are self-commutative and transforms the derivatives to divided differences, for example:

$$T_i^- \frac{\partial u}{\partial x_i} = u_{\bar{x}_i} \,, \quad T_i^+ \frac{\partial u}{\partial x_i} = u_{x_i} \,, \quad T_i^2 \frac{\partial^2 u}{\partial x_i^2} = u_{x_i\bar{x}_i} \,, \quad T_t^- \frac{\partial u}{\partial t} = u_{\bar{t}} \,.$$

We also define operators

$$T_2^{2-} f(x_1,x_2) = \frac{2}{h}\int_{x_2-h}^{x_2} \left(1 + \frac{x_2'-x_2}{h}\right) f(x_1,x_2')dx_2' \,, \quad T_2^{2+} f(x_1,x_2) = \frac{2}{h}\int_{x_2}^{x_2+h} \left(1 - \frac{x_2'-x_2}{h}\right) f(x_1,x_2')dx_2' \,.$$

We define the following inner products and norms:

$$(v,u)_{L_2(\omega_h)} = h^2 \sum_{x\in\omega_h} v(x)u(x) \,, \quad \|v\|_{L_2(\omega_h)} = (v,v)_{L_2(\omega_h)}^{1/2} \,,$$

$$(v,u)_{L_2(\omega_{ih})} = h^2 \sum_{x\in\omega_{ih}} v(x)u(x) \,, \quad \|v\|_{L_2(\omega_{ih})} = (v,v)_{L_2(\omega_{ih})}^{1/2} \,.$$

Further, we denote $B_h v = (1 + K\delta_h(x_2 - \xi))v$ and define the following norms:

$$\|v\|_{B_h}^2 = \|v\|_{L_2(\omega_h)}^2 + Kh \sum_{x\in\sigma_h} v^2(x) \,,$$

$$\|v\|_{B_h^{-1}}^2 = h^2 \sum_{x\in w_h \setminus \sigma_h} v^2(x) + \frac{h^3}{K+h} \sum_{x\in\sigma_h} v^2(x) \,,$$

$$\|v\|_{\widetilde{W}_{2,h}^2} = \sum_{i=1}^{2} \left(\|v_{x_i\bar{x}_i}\|_{B_h^{-1}}^2 + \|v_{x_i}\|_{L_2(\omega_{ih})}^2 \right) + \|v\|_{B_h}^2 \,,$$

$$\|v\|_{\widetilde{W}_2^{2,1}(Q_{h\tau})}^2 = \tau \sum_{t\in\bar{\omega}_\tau} \|v(\cdot,t)\|_{\widetilde{W}_{2,h}^2}^2 + \tau \sum_{t\in\omega_\tau^+} \|v_{\bar{t}}(\cdot,t)\|_{B_h}^2 \,.$$

Also we define the following discrete norms and seminorms:

$$\|v\|^2_{L_2(Q_{h\tau})} = \tau \sum_{t\in\omega_\tau^+} \|v(\cdot,t)\|^2_{L_2(\omega_h)}, \quad \|v\|^2_{L_2(\sigma_h\times\omega_\tau)} = \tau \sum_{t\in\omega_\tau^+} \|v(\cdot,t)\|^2_{L_2(\sigma_h)}$$

$$|v|^2_{L_2(\omega_\tau;\, W_2^{1/2}(\sigma_h))} = \tau \sum_{t\in\omega_\tau^+} \|v(\cdot,t)\|^2_{W_2^{1/2}(\sigma_h)},$$

$$|v|^2_{W_2^{1/2}(\omega_\tau;\, L_2(\omega_h))} = \tau \sum_{t\in\overline{\omega}_\tau} \tau \sum_{t'\in\overline{\omega}_\tau,\, t'\neq t} \frac{\|v(\cdot,t)-v(\cdot,t')\|^2_{L_2(\omega_h)}}{|t-t'|^2},$$

$$|v|^2_{W_2^{1/2}(\omega_\tau;\, L_2(\sigma_h))} = \tau \sum_{t\in\overline{\omega}_\tau} \tau \sum_{t'\in\overline{\omega}_\tau,\, t'\neq t} \frac{\|v(\cdot,t)-v(\cdot,t')\|^2_{L_2(\sigma_h)}}{|t-t'|^2},$$

$$\|v\|^2_{\widetilde{W}_2^{1/2}(\omega_\tau;\, L_2(\omega_h))} = |v|^2_{W_2^{1/2}(\omega_\tau;\, L_2(\omega_h))} + \tau \sum_{t\in\overline{\omega}_\tau} \left(\frac{1}{t+\tau} + \frac{1}{T-t+\tau}\right) \|v(\cdot,t)\|^2_{L_2(\omega_h)},$$

$$\|v\|^2_{\widetilde{W}_2^{1/2}(\omega_\tau;\, L_2(\sigma_h))} = |v|^2_{W_2^{1/2}(\omega_\tau;\, L_2(\sigma_h))} + \tau \sum_{t\in\overline{\omega}_\tau} \left(\frac{1}{t+\tau} + \frac{1}{T-t+\tau}\right) \|v(\cdot,t)\|^2_{L_2(\sigma_h)},$$

$$\|v\|^2_{\widetilde{W}_2^{1,1/2}(Q_{h\tau})} = \tau \sum_{t\in\omega_\tau^+} \|v(\cdot,t)\|^2_{W_2^{1/2}(\omega_h)} + |v|^2_{W_2^{1/2}(\omega_\tau;\, L_2(\omega_h))} + |v|^2_{W_2^{1/2}(\omega_\tau;\, L_2(\sigma_h))}.$$

3. Convergence in $\widetilde{W}_2^{1,1/2}(Q_{h\tau})$

Now we shall prove the convergence of the difference scheme (4) in the $\widetilde{W}_2^{1,\,1/2}(Q_{h\tau})$ norm.
We shall assume that (see [7], [16]):

$$\begin{aligned} &a_i \in W_2^{2+\varepsilon,1+\varepsilon/2}(Q_1) \cap W_2^{2+\varepsilon,1+\varepsilon/2}(Q_2)\,,\; a \in W_2^{1+\varepsilon,(1+\varepsilon)/2}(Q_1) \cap W_2^{1+\varepsilon,(1+\varepsilon)/2}(Q_2)\,,\; a_i > 0\,,\; a > 0\,,\\ &f \in W_2^{1,1/2}(Q_1) \cap W_2^{1,1/2}(Q_2)\,, \quad u \in W_2^{3,3/2}(Q_1) \cap W_2^{3,3/2}(Q_2) \cap W_2^{3,3/2}(\Sigma\times(0,T))\,, \end{aligned} \tag{5}$$

where $\Omega_1 = (0,1)\times(0,\xi)$, $\Omega_2 = (0,1)\times(\xi,1)$, $Q_1 = \Omega_1\times(0,T)$, $Q_2 = \Omega_2\times(0,T)$, $\Sigma = \{(x_1,\xi) | x_1 \in (0,1)\}$ and $\varepsilon > 0$.

Let u be the solution of the boundary-value problem (3) and v the solution of the difference problem (4). The error $z = u - v$ satisfies the finite difference scheme

$$\begin{aligned} &(1 + K\delta_h(x_2-\xi))z_{\bar t} + L_h z = \sum_{i=1}^{2} \xi_{i,\bar x_i} + \chi_{\bar t} + \delta_{\sigma_h}\mu_{\bar t} + \eta\,, \;\text{ in } Q_{h\tau}\,,\\ &z = 0\,, \;\text{ on } \gamma_h\times\omega_\tau^+\,, \quad z(x,0) = 0\,, \;\text{ on } \omega_h\,, \end{aligned} \tag{6}$$

where

$$\xi_i = T_i^+ T_{3-i}^2 T_t^- \left(a_i \frac{\partial u}{\partial x_i}\right) - \frac{1}{2}(a_i + a_i^{+i}) u_{x_i}\,,$$

$$\chi = u - T_1^2 T_2^2 u\,,\;\; \mu = ku - T_1^2(ku)\,,\;\; \eta = (T_1^2 T_2^2 T_t^- a)u - T_1^2 T_2^2 T_t^-(au)\,.$$

Let us set $\xi_1 = \widetilde{\xi}_1 + \delta_{\sigma_h}\widehat{\xi}_1$, $\chi = \widetilde{\chi} + \delta_{\sigma_h}\widehat{\chi}$, $\eta = \widetilde{\eta} + \delta_{\sigma_h}\widehat{\eta}$, where

$$\widehat{\xi}_1 = \frac{h^2}{6} T_1^+ T_t^- \left(\left[a_1 \frac{\partial^2 u}{\partial x_1 \partial x_2} + \frac{\partial a_1}{\partial x_2}\frac{\partial u}{\partial x_1}\right]_\Sigma\right),$$

$$\widehat{\chi} = \frac{h^2}{6}\left[T_1^2 \frac{\partial u}{\partial x_2}\right]_\Sigma\,,$$

$$\widehat{\eta} = -\frac{h^2}{3}\left[(T_1^2 T_t^- a)\left(T_1^2 T_t^- \frac{\partial u}{\partial x_2}\right)\right]_\Sigma$$

and $[u]_\Sigma = u(x_1, \xi + 0, t) - u(x_1, \xi - 0, t)$.

The following a priori estimate for the solution of the difference scheme (6) is valid (see [9], [15]):

$$\|z\|_{\widetilde{W}_2^{1,1/2}(Q_{h\tau})} \le C\Big[\|\xi_2\|_{L_2(Q_{h\tau})} + \|\widetilde{\xi_1}\|_{L_2(Q_{h\tau})} + |\widehat{\xi_1}|_{L_2(\omega_\tau;\, W_2^{1/2}(\sigma_h))} + \|\widetilde{\eta}\|_{L_2(Q_{h\tau})} + \|\widehat{\eta}\|_{L_2(\sigma_h\times\omega_\tau)} + \|\widetilde{\chi}\|_{\widetilde{W}_2^{1/2}(\omega_\tau, L_2(\omega_h))} + \|\widehat{\chi}\|_{\widetilde{W}_2^{1/2}(\omega_\tau, L_2(\sigma_h))} + \|\mu\|_{\widetilde{W}_2^{1/2}(\omega_\tau, L_2(\sigma_h))}\Big]. \tag{7}$$

Therefore, in order to estimate the rate of convergence of the difference scheme (4), it is sufficient to estimate the right-hand side of the inequality (7).

The estimates of terms $\xi_2, \widetilde{\xi_1}, \widehat{\xi_1}$ are derived in [15]:

$$\|\xi_2\|_{L_2(Q_{h\tau})} \le Ch^2(\|a_2\|_{W_2^{2+\varepsilon,1+\varepsilon/2}(Q_1)}\|u\|_{W_2^{3,3/2}(Q_1)} + \|a_2\|_{W_2^{2+\varepsilon,1+\varepsilon/2}(Q_2)}\|u\|_{W_2^{3,3/2}(Q_2)}). \tag{8}$$

$$\|\widetilde{\xi_1}\|_{L_2(Q_{h\tau})} \le Ch^2(\|a_1\|_{W_2^{2+\varepsilon,1+\varepsilon/2}(Q_1)}\|u\|_{W_2^{3,3/2}(Q_1)} + \|a_1\|_{W_2^{2+\varepsilon,1+\varepsilon/2}(Q_2)}\|u\|_{W_2^{3,3/2}(Q_2)}). \tag{9}$$

$$|\widehat{\xi_1}|_{L_2(\omega_\tau, W_2^{1/2}(\sigma_h))} \le Ch^2(\|a_1\|_{W_2^{2+\varepsilon,1+\varepsilon/2}(Q_1)}\|u\|_{W_2^{3,3/2}(Q_1)} + \|a_1\|_{W_2^{2+\varepsilon,1+\varepsilon/2}(Q_2)}\|u\|_{W_2^{3,3/2}(Q_2)}). \tag{10}$$

Let us estimate the term $\widetilde{\eta}$. At the point $x \notin \sigma_h$ we have $\widetilde{\eta} = \eta$. We decompose term $\eta = \bar{\eta}_1 + \bar{\eta}_2$ where

$$\bar{\eta}_1 = (T_1^2T_2^2T_t^- a)(u - T_1^2T_2^2T_t^- u),$$
$$\bar{\eta}_2 = (T_1^2T_2^2T_t^- a)(T_1^2T_2^2T_t^- u) - T_1^2T_2^2T_t^-(au).$$

The term $\bar{\eta}_1$ is a bounded bilinear functional of the argument $(a, u) \in L_q(e) \times W_{2q/(q-2)}^{2,1}(e)$, $q > 2$, where $e = (x_1 - h, x_1 + h) \times (x_2 - h, x_2 + h) \times (t - \tau, t)$. Further, $\bar{\eta}_1 = 0$ whenever u is a polynomial of degree one in x_1 or x_2 or a constant in t. Applying Bramble-Hilbert lemma [5] we get:

$$|\bar{\eta}_1(x, t)| \le C|a|_{L_q(e)}|u|_{W_{2q/(q-2)}^{2,1}(e)}, \quad q > 2. \tag{11}$$

The term $\bar{\eta}_2$ is a bounded bilinear functional of the argument $(a, u) \in W_q^{1,1/2}(e)\times W_{2q/(q-2)}^{1,1/2}(e)$, $q > 2$. Further, $\bar{\eta}_2 = 0$ whenever is a constant or u is constant. Applying Bramble-Hilbert lemma we get:

$$|\bar{\eta}_2(x, t)| \le C|a|_{W_q^{1,1/2}(e)}|u|_{W_{2q/(q-2)}^{1,1/2}(e)}, \quad q > 2. \tag{12}$$

At the point $x \in \sigma_h$ we decompose term $\widetilde{\eta} = \eta_{(1)}^+ + \eta_{(1)}^- + \eta_{(2)}^+ + \eta_{(2)}^-$ where

$$\eta_{(1)}^\pm = (T_1^2T_2^{2\pm}T_t^- a)\Big[u - (T_1^2T_2^{2\pm}T_t^- u) \pm \frac{h}{3}(T_1^2T_t^- \frac{\partial u}{\partial x_2})\Big]\Big|_{x_2=\xi\pm0},$$
$$\eta_{(2)}^\pm = \Big[(T_1^2T_2^{2\pm}T_t^- a)(T_1^2T_2^{2\pm}T_t^- u) - T_1^2T_2^{2\pm}T_t^-(au) \pm \frac{h}{3}((T_1^2T_t^- a) - (T_1^2T_2^{2\pm}T_t^- a)(T_1^2T_t^- \frac{\partial u}{\partial x_2})\Big]\Big|_{x_2=\xi\pm0}$$

The term $\eta_{(1)}^\pm$ is a bounded bilinear functional of the argument $(a, u) \in L_q(e_1^\pm) \times W_{2q/(q-2)}^{2,1}(e_1^\pm)$, $q > 2$, where $e_1^+ = (x_1 - h, x_1 + h)\times(\xi, \xi + h)\times(t - \tau, t)$, $e_1^- = (x_1 - h, x_1 + h)\times(\xi - h, \xi)\times(t - \tau, t)$. Further, $\eta_{(1)}^\pm = 0$ whenever u is a polynomial of degree one in x_1 or x_2 or constant. Applying Bramble-Hilbert lemma we get the following estimate:

$$|\eta_{(1)}^\pm(x, t)| \le C|a|_{L_q(e_1^\pm)}|u|_{W_{2q/(q-2)}^{2,1}(e_1^\pm)}, \quad q > 2. \tag{13}$$

The term $\eta_{(2)}^\pm$ is a bounded bilinear functional of the argument $(a, u) \in W_q^{1,1/2}(e_1^\pm) \times W_{2q/(q-2)}^{1,1/2}(e_1^\pm)$. Further, $\eta_{(2)}^\pm = 0$ whenever a is a constant or u is constant. Applying Bramble-Hilbert lemma we get the following estimate:

$$|\eta_{(2)}^\pm(x, t)| \le C|a|_{W_q^{1,1/2}(e_1^\pm)}|u|_{W_{2q/(q-2)}^{1,1/2}(e_1^\pm)}, \quad q > 2. \tag{14}$$

From estimates (11)-(12) and (13)-(14), after summation and using imbeddings $W_2^{1+\varepsilon,(1+\varepsilon)/2}(Q_k) \subset W_q^{1,1/2}(Q_k)$, $W_2^{1+\varepsilon,(1+\varepsilon)/2}(Q_k) \subset L_q(Q_k)$, $W_2^{3,3/2}(Q_k) \subset W_{2q/(q-2)}^{2,1}(Q_k)$, $W_2^{3,3/2}(Q_k) \subset W_{2q/(q-2)}^{1,1/2}(Q_k)$, for $q = 4/(2-\varepsilon)$, $\varepsilon > 0$, $k = 1, 2$, we have

$$\|\widetilde{\eta}\|_{L_2(Q_{h\tau})} \le Ch^2(\|a\|_{W_2^{1+\varepsilon,(1+\varepsilon)/2}(Q_1)}\|u\|_{W_2^{3,3/2}(Q_1)} + \|a\|_{W_2^{1+\varepsilon,(1+\varepsilon)/2}(Q_2)}\|u\|_{W_2^{3,3/2}(Q_2)}). \tag{15}$$

Let us estimate the term $\widehat{\eta}$. At the point $(x,t) \in \sigma_h \times \omega_\tau$, using Hölder inequality, we obtain the following estimate

$$\|\hat{\eta}(x,t)\| \le Ch^{1/2}\left(\|a\|_{L_q(\sigma_h^+)}\left\|\frac{\partial u}{\partial x_2}\right\|_{L_{2q/(q-2)}(\sigma_h^+)} + \|a\|_{L_q(\sigma_h^-)}\left\|\frac{\partial u}{\partial x_2}\right\|_{L_{2q/(q-2)}(\sigma_h^-)}\right)$$

where $\|a\|_{L_q(\sigma_h^\pm)} = \|a(\cdot,\xi\pm 0,\cdot)\|_{L_q((x_1-h,x_1+h)\times(t-\tau,t))}$. After summation over the mesh $\sigma_h \times \omega_\tau$ we have

$$\|\hat{\eta}\|_{L_2(\sigma_h\times\omega_\tau)} \le Ch^2\left(\|a\|_{L_q(\Sigma^+)}\|u\|_{W_{2q/(q-2)}^{1,1/2}(\Sigma^+)} + \|a\|_{L_q(\Sigma^-)}\|u\|_{W_{2q/(q-2)}^{1,1/2}(\Sigma^-)}\right)$$

where $\|a\|_{L_q(\Sigma^\pm)} = \|a(\cdot,\xi\pm 0,\cdot)\|_{L_q(Q_\xi)}$, $Q_\xi = (0,1)\times(0,T)$. Further, using imbeddings

$$W_2^{1/2,1/4}(Q_\xi) \subset L_q(Q_\xi)\,,\ W_2^{5/2,5/4}(Q_\xi) \subset W_{2q/(q-2)}^{1,1/2}(Q_\xi)\,,\ 2 < q \le 3\,,$$

we have

$$\|\hat{\eta}\|_{L_2(\sigma_h\times\omega_\tau)} \le Ch^2\Big(\|a\|_{W_2^{1/2,1/4}(\Sigma^+)}\|u\|_{W_2^{5/2,5/4}(\Sigma^+)} + \|a\|_{W_2^{1/2,1/4}(\Sigma^-)}\|u\|_{W_2^{5/2,5/4}(\Sigma^-)}\Big).$$

Further, the following result is valid [9]:

$$\|u(\cdot,\xi,\cdot)\|_{W_2^{s-1/2,s/2-1/4}(Q_\xi)} \le C\|u\|_{W_2^{s,s/2}(Q)}.$$

Using previous result to all the norms in the previous estimate simply follows

$$\|\hat{\eta}\|_{L_2(\sigma_h\times\omega_\tau)} \le Ch^2\Big(\|a\|_{W_2^{1+\varepsilon,(1+\varepsilon)/2}(Q_1)}\|u\|_{W_2^{3,3/2}(Q_1)} + \|a\|_{W_2^{1+\varepsilon,(1+\varepsilon)/2}(Q_2)}\|u\|_{W_2^{3,3/2}(Q_2)}\Big) \tag{16}$$

The estimates of terms $\widetilde{\chi}, \mu, \widehat{\chi}$ are derived in [9]:

$$\|\widetilde{\chi}\|_{\widetilde{W}_2^{1/2}(\omega_\tau, L_2(\omega_h))} \le Ch^2\sqrt{\log\frac{1}{h}}(\|u\|_{W_2^{3,3/2}(Q_1)} + \|u\|_{W_2^{3,3/2}(Q_2)}), \tag{17}$$

$$\|\mu\|_{\widetilde{W}_2^{1/2}(\omega_\tau, L_2(\sigma_h))} \le Ch^2\sqrt{\log\frac{1}{h}}\|u\|_{W_2^{3,3/2}(\Sigma\times(0,T))}, \tag{18}$$

$$\|\widehat{\chi}\|_{\widetilde{W}_2^{1/2}(\omega_\tau, L_2(\sigma_h))} \le Ch^2\sqrt{\log\frac{1}{h}}\|u\|_{W_2^{2,1}(\Sigma\times(0,T))}. \tag{19}$$

Finally from (7)-(10) and (15)-(19) we obtain the following result.

Theorem 3.1. *Suppose that solution and coefficients of the differential problem (3) satisfy conditions (5). Then, the solution of the difference scheme (4) converges in* $\widetilde{W}_2^{1,1/2}(Q_{h\tau})$ *to the solution of the differential problem (3) and, assuming that* $\tau \asymp h^2$, *the following estimate is valid:*

$$\|u-v\|_{\widetilde{W}_2^{1,1/2}(Q_{h\tau})} \le Ch^2\Big(\max_i\|a_i\|_{W_2^{2+\varepsilon,1+\varepsilon/2}(Q_1)} + \max_i\|a_i\|_{W_2^{2+\varepsilon,1+\varepsilon/2}(Q_2)} + \|a\|_{W_2^{1+\varepsilon,(1+\varepsilon)/2}(Q_1)} + \|a\|_{W_2^{1+\varepsilon,(1+\varepsilon)/2}(Q_2)} + l(h)\Big)$$
$$\times\Big(\|u\|_{W_2^{3,3/2}(Q_1)} + \|u\|_{W_2^{3,3/2}(Q_2)} + \|u\|_{W_2^{3,3/2}(\Sigma\times(0,T))}\Big)\,,$$

where $l(h) = \sqrt{\log 1/h}$.

4. Convergence in $\widetilde{W}_2^{2,1}(Q_{h\tau})$

In this section we shall prove the convergence of the difference scheme (4) in the $\widetilde{W}_2^{2,1}(Q_{h\tau})$ norm. We shall assume that

$$a_i \in W_2^{3,3/2}(Q_1) \cap W_2^{3,3/2}(Q_2)\,,\;\; a \in W_2^{2,1}(Q_1) \cap W_2^{2,1}(Q_2)\,,\;\; a_i > 0\,,\; a > 0\,,$$
$$f \in W_2^{2,1}(Q_1) \cap W_2^{2,1}(Q_2)\,,\quad u \in W_2^{4,2}(Q_1) \cap W_2^{4,2}(Q_2) \cap W_2^{4,2}(\Sigma \times (0,T))\,. \tag{20}$$

We also assume that the coefficients $a_i(x,t)$ are decreasing functions in the variable t.
The following a priori estimate for the solution of the difference scheme (6) is valid (see [4]):

$$\|z\|_{\widetilde{W}_2^{2,1}(Q_{h\tau})} \le C\left(\tau \sum_{t\in\omega_\tau^+} (\|\xi_{1,\bar{x}_1} + \xi_{2,\bar{x}_2}\|^2_{B_h^{-1}} + \|\chi_{\bar{t}} + \delta_{\sigma_h}\mu_{\bar{t}}\|^2_{B_h^{-1}} + \|\eta\|^2_{B_h^{-1}})\right)^{1/2}. \tag{21}$$

The following estimates are obtained in [3] and [4]:

$$\left(\tau \sum_{\tau\in\omega_\tau^+} \|\chi_{\bar{t}} + \delta_{\sigma_h}\mu_{\bar{t}}\|^2_{B_h^{-1}}\right)^{1/2} \le Ch^2(\|u\|_{W_2^{4,2}(Q_1)} + \|u\|_{W_2^{4,2}(Q_2)} + \|u\|^2_{W_2^{4,2}(\Sigma\times(0,T))})\,, \tag{22}$$

$$\left(\tau \sum_{\tau\in\omega_\tau^+} \|\xi_{i,\bar{x}_i}\|^2_{B_h^{-1}}\right)^{1/2} \le Ch^2(\|a_i\|_{W_2^{3,3/2}(Q_1)}\|u\|_{W_2^{4,2}(Q_1)} + \|a_i\|_{W_2^{3,3/2}(Q_2)}\|u\|_{W_2^{4,2}(Q_2)})\,,\; i = 1,2\,. \tag{23}$$

Let us estimate the term η. From (11) and (12), after summation and using imbeddings $W_2^{2,1}(Q_k) \subset L_q(Q_k)$, $W_2^{2,1}(Q_k) \subset W_q^{1,1/2}(Q_k)$, $W_2^{4,2}(Q_k) \subset W_{2q/(q-2)}^{1,1/2}(Q_k)$, $W_2^{4,2}(Q_k) \subset W_{2q/(q-2)}^{2,1}(Q_k)$, $2 < q \le 4$, $k = 1,2$, we have

$$\left(h^2\tau \sum_{t\in\omega_\tau^+} \sum_{x\in\omega_h\setminus\sigma_h} |\eta(x,t)|^2\right)^{1/2} \le Ch^2(\|a\|_{W_2^{2,1}(Q_1)}\|u\|_{W_2^{4,2}(Q_1)} + \|a\|_{W_2^{2,1}(Q_2)}\|u\|_{W_2^{4,2}(Q_2)}). \tag{24}$$

At the point $x \in \sigma_h$ we decompose term $\eta = \eta^+ + \eta^-$, $\eta^\pm = \sum_{k=1}^2 \eta_k^\pm$, where

$$\eta_1^\pm = (T_1^2 T_2^{2\pm} T_t^- a)(u - T_1^2 T_2^{2\pm} T_t^- u)\,,$$
$$\eta_2^\pm = (T_1^2 T_2^{2\pm} T_t^- a)(T_1^2 T_2^{2\pm} T_t^- u) - T_1^2 T_2^{2\pm} T_t^-(au).$$

The term η_1^+ is a bounded bilinear functional of the argument $(a,u) \in L_q(e_1) \times W_{2q/(q-2)}^{1,1/2}(e_1)$, where $e_1 = (x_1 - h, x_1 + h) \times (x_2, x_2 + h) \times (t - \tau, t)$.Further , $\eta_1^+ = 0$ whenever u is constant . Applying Bramble-Hilbert lemma we get:

$$|\eta_1^+(x,t)| \le \frac{C}{h}|a|_{L_q(e_1)}|u|_{W_{2q/(q-2)}^{1,1/2}(e_1)}\,,\; q > 2\,.$$

Summing over the meshes ω_τ^+ and σ_h and using imbeddings $W_2^{2,1}(Q_2) \subset W_q^{1,1/2}(Q_2)$, $W_2^{4,2}(Q_2) \subset W_{2q/(q-2)}^{2,1}(Q_2)$, $2 < q \le 4$, we have (see [12]):

$$\left(\frac{\tau h^3}{k+h} \sum_{t\in\omega_\tau^+} \sum_{x\in\sigma_h} |\eta_1^+(x,t)|^2\right)^{1/2} \le Ch^{3/2}\|a\|_{L_q(Q_2^h)}\|u\|_{W_{2q/(q-2)}^{1,1/2}(Q_2^h)}$$
$$\le Ch^2\|a\|_{W_q^{1,1/2}(Q_2)}\|u\|_{W_{2q/(q-2)}^{2,1}(Q_2)} \le Ch^2\|a\|_{W_2^{2,1}(Q_2)}\|u\|_{W_2^{4,2}(Q_2)}\,, \tag{25}$$

where $Q_2^h = (0,1)\times(\xi,\xi+h)\times(0,T)$. The η_2^+ is a bounded bilinear functional of the argument $(a,u)\in W_q^{1,1/2}(e_1)\times W_{2q/(q-2)}^{1,1/2}(e_1)$, $q>2$. Further $\eta_2^+=0$ whenever a is a constant or u is constant. Applying Bramble-Hilbert lemma we get the following estimate:

$$|\eta_2^+(x,t)| \le C|a|_{W_q^{1,1/2}(e_1)}|u|_{W_{2q/(q-2)}^{1,1/2}(e_1)}\,,\ q>2\,.$$

Summing over the meshes ω_τ^+ and σ_h and using imbeddings $W_2^{2,1}(Q_2)\subset W_q^{1,1/2}(Q_2)$, $W_2^{4,2}(Q_2)\subset W_{2q/(q-2)}^{1,1/2}(Q_2)$, $q>2$, we have:

$$\left(\frac{\tau h^3}{k+h}\sum_{t\in\omega_\tau^+}\sum_{x\in\sigma_h}|\eta_2^+(x,t)|^2\right)^{1/2} \le Ch^2\|a\|_{W_2^{2,1}(Q_2)}\|u\|_{W_2^{4,2}(Q_2)}. \tag{26}$$

From estimates (25) and (26) we have

$$\left(\frac{\tau h^3}{k+h}\sum_{t\in\omega_\tau^+}\sum_{x\in\sigma_h}|\eta^+(x,t)|^2\right)^{1/2} \le Ch^2\|a\|_{W_2^{2,1}(Q_2)}\|u\|_{W_2^{4,2}(Q_2)}. \tag{27}$$

Analogous estimate holds for term η^-:

$$\left(\frac{\tau h^3}{k+h}\sum_{t\in\omega_\tau^+}\sum_{x\in\sigma_h}|\eta^-(x,t)|^2\right)^{1/2} \le Ch^2\|a\|_{W_2^{2,1}(Q_1)}\|u\|_{W_2^{4,2}(Q_1)}. \tag{28}$$

From (24), (27) and (28) we obtain

$$\left(\tau\sum_{t\in\omega_\tau^+}\|\eta\|^2_{B_h^{-1}}\right)^{1/2} \le Ch^2(\|a\|_{W_2^{2,1}(Q_1)}\|u\|_{W_2^{4,2}(Q_1)} + \|a\|_{W_2^{2,1}(Q_2)}\|u\|_{W_2^{4,2}(Q_2)}). \tag{29}$$

Finally, from (21), (22), (23) and (29) we get:

Theorem 4.1. *Suppose that solution and coefficients of the differential problem (3) satisfy conditions (20). Then, the solution of the difference scheme (4) converges in $\widetilde{W}_2^{2,1}(Q_{h\tau})$ to the solution of the differential problem (3) and, assuming that $\tau\asymp h^2$, the following estimate is valid:*

$$\begin{aligned}\|u-v\|_{\widetilde{W}_2^{2,1}(Q_{h\tau})} \le Ch^2\Big(&\max_i\|a_i\|_{W_2^{3,3/2}(Q_1)} + \max_i\|a_i\|_{W_2^{3,3/2}(Q_2)} + \|a\|_{W_2^{2,1}(Q_1)} + \|a\|_{W_2^{2,1}(Q_2)} + 1\Big)\\ &\times\left(\|u\|_{W_2^{4,2}(Q_1)} + \|u\|_{W_2^{4,2}(Q_2)} + \|u\|_{W_2^{4,2}(\Sigma\times(0,T))}\right).\end{aligned}$$

This estimate is compatible with the smoothness of the coefficients and solution of the differential problem (3).

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