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Miscellaneous Properties for a Class of Complex Functions Defined by Rodrigues Type Formula

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Abstract. In the present paper, we consider a class of holomorphic complex functions defined by Rodrigues type formula and we investigate some properties such as generating function and recurrence relations. Also, we discuss some applications of the given results.

1. Introduction

Orthogonal polynomials in two complex variables were studied by many researchers. Zernike introduced 2D analogue of Legendre polynomials which are orthogonal on $|z| \leq 1$ and which arise on optical problems [20, 21]. The more general disc or Zernike polynomials were given in [13, 17] (see also [4]).

Ito [12] defined the complex Hermite polynomials $H_{m,n}(z_1, z_2)$ generated by (similar polynomials with other names and notations were defined by [16])

$$\sum_{m,n=0}^{\infty} H_{m,n}(z_1, z_2) \frac{u^m}{m!} \frac{v^n}{n!} = e^{uz_1 + vz_2 - uv},$$

which are studied several mathematical physicists, for example, they are applied to quantum optics and quasi probabilities and they are seen in combinatorics [14, 18, 19]. The Rodrigues formula is as follows

$$H_{m,n}(z_1, z_2) = (-1)^{m+n} e^{z_1 z_2} \frac{\partial^{m+n}}{\partial z_1^n \partial z_2^m} \left\{ e^{-z_1 z_2} \right\}.$$

Also, q -analogues of the Ito's polynomials were introduced by Ismail and Zhang in [11]. Moreover, there are many research papers for complex Hermite polynomials, we refer to [3, 6–9]. A brief review of orthogonal polynomials in two complex variables was given by Ismail and Zhang [10]. For other complex orthogonal polynomials and q-analogues of them, we refer to [10]. Other references on the 2D-orthogonal polynomials are [5], [15], [16], etc.

In the previous works [1, 2], we have studied the family of real analytic functions defined by Rodrigues type formula and investigated their properties. Motivated by the class of real analytic functions, we consider

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a class of holomorphic complex functions. Our aim in the present paper is to discuss some properties of a class of holomorphic complex functions $Q_{m,n}(z_1,z_2)$ of Hermite type suggested by the Rodrigues formula

$$Q_{m,n}(z_1,z_2)=e^{-\varphi(z_1,z_2)}\frac{\partial^{m+n}}{\partial z_1^m\partial z_2^n}\left\{\phi(z_1,z_2)\,e^{\varphi(z_1,z_2)}\right\}\quad,\quad z_1,z_2\in\mathbb{C} \tag{1}$$

$(m,n=0,1,2,\ldots)$

which gives the complex Hermite polynomials $H_{m,n}(z_1,z_2)$ in the special case.

In Section 2, we first derive a generating function and then we obtain several recurrence relations by using the obtained generating function. Section 3 has some applications of the results obtained in the earlier section. The known properties of the complex Hermite polynomials $H_{m,n}(z_1,z_2)$ are given as examples of our results.

2. Miscellaneous Properties of the Function $Q_{m,n}(z_1,z_2)$

Let $f(z_1,z_2)$ be holomorphic on the closed polydisc $\overline{D}^2$, $D^2=\{z:=(z_1,z_2):|z_1|<1\,,\ |z_2|<1\}$. Then $f\in C^\infty\left(D^2\right)$ and for each $z\in D^2$,

$$\frac{\partial^{m+n}}{\partial z_1^m\partial z_2^n}f(z_1,z_2)=\frac{m!n!}{(2\pi i)^2}\oint\limits_{|\xi_1|=1}\oint\limits_{|\xi_2|=1}\frac{f(\xi_1,\xi_2)}{(\xi_1-z_1)^{m+1}(\xi_2-z_2)^{n+1}}d\xi_2d\xi_1.$$

By using Cauchy formula for the partial derivatives of f, we first give a generating function for the function $Q_{m,n}(z_1,z_2)$.

Theorem 2.1. *Let $\varphi(z_1,z_2)$ and $\phi(z_1,z_2)$ be holomorphic on the closed polydisc $\overline{D}^2$. For the function $Q_{m,n}(z_1,z_2)$, we have*

$$\sum_{m,n=0}^{\infty}Q_{m,n}(z_1,z_2)\frac{u^m}{m!}\frac{v^n}{n!}=G(z_1,z_2;u,v) \tag{2}$$

where

$$G(z_1,z_2;u,v)=\phi(z_1+u,z_2+v)\,e^{\varphi(z_1+u,z_2+v)-\varphi(z_1,z_2)}.$$

Proof. Using the definition of the function $Q_{m,n}(z_1,z_2)$ and Cauchy formula, one can write

$$\begin{aligned}&\sum_{m,n=0}^{\infty}Q_{m,n}(z_1,z_2)\frac{u^m}{m!}\frac{v^n}{n!}\\&=\sum_{m,n=0}^{\infty}e^{-\varphi(z_1,z_2)}\frac{\partial^{m+n}}{\partial z_1^m\partial z_2^n}\left(e^{\varphi(z_1,z_2)}\phi(z_1,z_2)\right)\frac{u^m}{m!}\frac{v^n}{n!}\\&=\frac{1}{(2\pi i)^2}\sum_{m,n=0}^{\infty}e^{-\varphi(z_1,z_2)}u^mv^n\oint\limits_{|\xi_1|=1}\oint\limits_{|\xi_2|=1}\frac{e^{\varphi(\xi_1,\xi_2)}\phi(\xi_1,\xi_2)}{(\xi_1-z_1)^{m+1}(\xi_2-z_2)^{n+1}}d\xi_2d\xi_1\\&=\frac{e^{-\varphi(z_1,z_2)}}{(2\pi i)^2}\oint\limits_{|\xi_1|=1}\oint\limits_{|\xi_2|=1}\frac{e^{\varphi(\xi_1,\xi_2)}\phi(\xi_1,\xi_2)}{(\xi_1-z_1)(\xi_2-z_2)}\sum_{m,n=0}^{\infty}\left(\frac{u}{\xi_1-z_1}\right)^m\left(\frac{v}{\xi_2-z_2}\right)^n d\xi_2d\xi_1\end{aligned}$$

for $\left|\frac{u}{\xi_1-z_1}\right|<1$ and $\left|\frac{v}{\xi_2-z_2}\right|<1$,

$$=\frac{e^{-\varphi(z_1,z_2)}}{(2\pi i)^2}\oint\limits_{|\xi_1|=1}\oint\limits_{|\xi_2|=1}\frac{e^{\varphi(\xi_1,\xi_2)}\phi(\xi_1,\xi_2)}{(\xi_1-z_1)(\xi_2-z_2)}\frac{1}{1-\frac{u}{\xi_1-z_1}}\frac{1}{1-\frac{v}{\xi_2-z_2}}d\xi_2 d\xi_1$$

$$=\frac{e^{-\varphi(z_1,z_2)}}{(2\pi i)^2}\oint\limits_{|\xi_1|=1}\oint\limits_{|\xi_2|=1}\frac{e^{\varphi(\xi_1,\xi_2)}\phi(\xi_1,\xi_2)}{(\xi_1-(z_1+u))(\xi_2-(z_2+v))}d\xi_2 d\xi_1$$

$$=e^{-\varphi(z_1,z_2)}\phi(z_1+u,z_2+v)\,e^{\varphi(z_1+u,z_2+v)},$$

which gives the desired. □

Remark 2.2. *In particular for $u=z_1$, $v=z_2$ and for $u=z_2$, $v=z_1$, we have*

$$\sum_{m,n=0}^{\infty}Q_{m,n}(z_1,z_2)\frac{z_1^m}{m!}\frac{z_2^n}{n!}=\phi(2z_1,2z_2)\,e^{\varphi(2z_1,2z_2)-\varphi(z_1,z_2)}$$

and

$$\sum_{m,n=0}^{\infty}Q_{m,n}(z_1,z_2)\frac{z_2^m}{m!}\frac{z_1^n}{n!}=\phi(z_1+z_2,z_1+z_2)\,e^{\varphi(z_1+z_2,z_1+z_2)-\varphi(z_1,z_2)}.$$

It is possible to obtain recurrence relations by using the generating function (2). For this purpose, we start with the following equations satisfied by $G(z_1,z_2;u,v)$:

$$\begin{aligned}\phi(z_1+u,z_2+v)\frac{\partial}{\partial u}G(z_1,z_2;u,v)&=G(z_1,z_2;u,v)\frac{\partial}{\partial u}\phi(z_1+u,z_2+v)\\&\quad+\phi(z_1+u,z_2+v)\,G(z_1,z_2;u,v)\frac{\partial}{\partial u}\varphi(z_1+u,z_2+v)\end{aligned}\tag{3}$$

and from the symmetry,

$$\begin{aligned}\phi(z_1+u,z_2+v)\frac{\partial}{\partial v}G(z_1,z_2;u,v)&=G(z_1,z_2;u,v)\frac{\partial}{\partial v}\phi(z_1+u,z_2+v)\\&\quad+\phi(z_1+u,z_2+v)\,G(z_1,z_2;u,v)\frac{\partial}{\partial v}\varphi(z_1+u,z_2+v).\end{aligned}\tag{4}$$

Theorem 2.3. *From (3) and (4), the following recurrence relations hold true*

$$\begin{aligned}&\sum_{i,j=0}^{m,n}\binom{m}{i}\binom{n}{j}Q_{m-i+1,n-j}(z_1,z_2)\frac{\partial^{i+j}}{\partial z_1^i\partial z_2^j}\phi(z_1,z_2)\\&=\sum_{i,j=0}^{m,n}\binom{m}{i}\binom{n}{j}Q_{m-i,n-j}(z_1,z_2)\frac{\partial^{i+j+1}}{\partial z_1^{i+1}\partial z_2^j}\phi(z_1,z_2)\\&+\sum_{i,j=0}^{m,n}\sum_{k,l=0}^{m-i,n-j}\frac{m!n!}{(m-i-k)!(n-j-l)!i!j!k!l!}Q_{m-i-k,n-j-l}(z_1,z_2)\frac{\partial^{i+j}}{\partial z_1^i\partial z_2^j}\phi(z_1,z_2)\frac{\partial^{k+l+1}}{\partial z_1^{k+1}\partial z_2^l}\varphi(z_1,z_2)\end{aligned}\tag{5}$$

and by symmetry, it follows

$$\begin{aligned}
&\sum_{i,j=0}^{m,n}\binom{m}{i}\binom{n}{j}Q_{m-i,n-j+1}(z_1,z_2)\frac{\partial^{i+j}}{\partial z_1^i\partial z_2^j}\phi(z_1,z_2)\\
&=\sum_{i,j=0}^{m,n}\binom{m}{i}\binom{n}{j}Q_{m-i,n-j}(z_1,z_2)\frac{\partial^{i+j+1}}{\partial z_1^i\partial z_2^{j+1}}\phi(z_1,z_2)\\
&+\sum_{i,j=0}^{m,n}\sum_{k,l=0}^{m-i,n-j}\frac{m!n!}{(m-i-k)!(n-j-l)!i!j!k!l!}Q_{m-i-k,n-j-l}(z_1,z_2)\frac{\partial^{i+j}}{\partial z_1^i\partial z_2^j}\phi(z_1,z_2)\frac{\partial^{k+l+1}}{\partial z_1^k\partial z_2^{l+1}}\varphi(z_1,z_2).
\end{aligned}\tag{6}$$

Theorem 2.4. *Other relations for $Q_{m,n}(z_1,z_2)$ are given as follows*

$$\begin{aligned}
&\sum_{i,j=0}^{m,n}\binom{m}{i}\binom{n}{j}\frac{\partial}{\partial z_1}Q_{m-i,n-j}(z_1,z_2)\frac{\partial^{i+j}}{\partial z_1^i\partial z_2^j}\phi(z_1,z_2)\\
&=\sum_{i,j=0}^{m,n}\binom{m}{i}\binom{n}{j}Q_{m-i,n-j}(z_1,z_2)\frac{\partial^{i+j+1}}{\partial z_1^{i+1}\partial z_2^{j}}\phi(z_1,z_2)\\
&-\frac{\partial}{\partial z_1}\varphi(z_1,z_2)\sum_{i,j=0}^{m,n}\binom{m}{i}\binom{n}{j}Q_{m-i,n-j}(z_1,z_2)\frac{\partial^{i+j}}{\partial z_1^i\partial z_2^j}\phi(z_1,z_2)\\
&+\sum_{i,j=0}^{m,n}\sum_{k,l=0}^{m-i,n-j}\frac{m!n!}{(n-j-l)!(m-i-k)!i!j!k!l!}Q_{m-i-k,n-j-l}(z_1,z_2)\frac{\partial^{i+j}}{\partial z_1^i\partial z_2^j}\phi(z_1,z_2)\frac{\partial^{k+l+1}}{\partial z_1^{k+1}\partial z_2^{l}}\varphi(z_1,z_2)
\end{aligned}\tag{7}$$

and

$$\begin{aligned}
&\sum_{i,j=0}^{m,n}\binom{m}{i}\binom{n}{j}\frac{\partial}{\partial z_2}Q_{m-i,n-j}(z_1,z_2)\frac{\partial^{i+j}}{\partial z_1^i\partial z_2^j}\phi(z_1,z_2)\\
&=\sum_{i,j=0}^{m,n}\binom{m}{i}\binom{n}{j}Q_{m-i,n-j}(z_1,z_2)\frac{\partial^{i+j+1}}{\partial z_1^{i}\partial z_2^{j+1}}\phi(z_1,z_2)\\
&-\frac{\partial}{\partial z_2}\varphi(z_1,z_2)\sum_{i,j=0}^{m,n}\binom{m}{i}\binom{n}{j}Q_{m-i,n-j}(z_1,z_2)\frac{\partial^{i+j}}{\partial z_1^i\partial z_2^j}\phi(z_1,z_2)\\
&+\sum_{i,j=0}^{m,n}\sum_{k,l=0}^{m-i,n-j}\frac{m!n!}{(n-j-l)!(m-i-k)!i!j!k!l!}Q_{m-i-k,n-j-l}(z_1,z_2)\frac{\partial^{i+j}}{\partial z_1^i\partial z_2^j}\phi(z_1,z_2)\frac{\partial^{k+l+1}}{\partial z_1^{k}\partial z_2^{l+1}}\varphi(z_1,z_2).
\end{aligned}\tag{8}$$

Proof. It is seen that the following relations are satisfied by the function $G(z_1,z_2;u,v)$

$$\begin{aligned}
&\phi(z_1+u,z_2+v)\frac{\partial}{\partial z_1}G(z_1,z_2;u,v)=G(z_1,z_2;u,v)\frac{\partial}{\partial z_1}\phi(z_1+u,z_2+v)\\
&+\phi(z_1+u,z_2+v)G(z_1,z_2;u,v)\left(\frac{\partial}{\partial z_1}\varphi(z_1+u,z_2+v)-\frac{\partial}{\partial z_1}\varphi(z_1,z_2)\right)
\end{aligned}$$

and

$$\begin{aligned}
&\phi(z_1+u,z_2+v)\frac{\partial}{\partial z_2}G(z_1,z_2;u,v)=G(z_1,z_2;u,v)\frac{\partial}{\partial z_2}\phi(z_1+u,z_2+v)\\
&+\phi(z_1+u,z_2+v)G(z_1,z_2;u,v)\left(\frac{\partial}{\partial z_2}\varphi(z_1+u,z_2+v)-\frac{\partial}{\partial z_2}\varphi(z_1,z_2)\right).
\end{aligned}\tag{9}$$

If we substitute Taylor series expressions of the functions $\phi(z_1+u, z_2+v)$ and $\varphi(z_1+u, z_2+v)$ about $(u,v) = (0,0)$ and the expression (2) of $G(z_1, z_2; u, v)$ in the first relation and then make necessary calculations, we obtain

$$\begin{aligned}
&\sum_{i,j=0}^{m,n}\binom{m}{i}\binom{n}{j}\frac{\partial}{\partial z_1}Q_{m-i,n-j}(z_1,z_2)\frac{\partial^{i+j}}{\partial z_1^i \partial z_2^j}\phi(z_1,z_2)\\
&=\sum_{i,j=0}^{m,n}\binom{m}{i}\binom{n}{j}Q_{m-i,n-j}(z_1,z_2)\frac{\partial^{i+j+1}}{\partial z_1^{i+1}\partial z_2^j}\phi(z_1,z_2)\\
&-\frac{\partial}{\partial z_1}\varphi(z_1,z_2)\sum_{i,j=0}^{m,n}\binom{m}{i}\binom{n}{j}Q_{m-i,n-j}(z_1,z_2)\frac{\partial^{i+j}}{\partial z_1^i\partial z_2^j}\phi(z_1,z_2)\\
&+\sum_{i,j=0}^{m,n}\sum_{k,l=0}^{m-i,n-j}\frac{m!n!}{(n-j-l)!(m-i-k)!i!j!k!l!}Q_{m-i-k,n-j-l}(z_1,z_2)\frac{\partial^{i+j}}{\partial z_1^i\partial z_2^j}\phi(z_1,z_2)\frac{\partial^{k+l+1}}{\partial z_1^{k+1}\partial z_2^l}\varphi(z_1,z_2).
\end{aligned}$$

From symmetry, the second relation holds true in view of (9). □

Corollary 2.5. *Considering the relations (5) and (7) together, we have the next result*

$$\frac{\partial}{\partial z_1}Q_{m,n}(z_1,z_2) = Q_{m+1,n}(z_1,z_2) - Q_{m,n}(z_1,z_2)\frac{\partial}{\partial z_1}\varphi(z_1,z_2). \tag{10}$$

Similarly, as a result of (6) and (8), it follows

$$\frac{\partial}{\partial z_2}Q_{m,n}(z_1,z_2) = Q_{m,n+1}(z_1,z_2) - Q_{m,n}(z_1,z_2)\frac{\partial}{\partial z_2}\varphi(z_1,z_2). \tag{11}$$

Theorem 2.6. *For $k = 1, 2, ...,$ we have*

$$D^kQ_{m,n}(z_1,z_2) = Q_{m+k,n+k}(z_1,z_2)$$

where the operator D is given by

$$D = \frac{\partial^2}{\partial z_1\partial z_2} + \varphi_{z_1}(z_1,z_2)\frac{\partial}{\partial z_2} + \varphi_{z_2}(z_1,z_2)\frac{\partial}{\partial z_1} + \varphi_{z_1}(z_1,z_2)\varphi_{z_2}(z_1,z_2) + \varphi_{z_1z_2}(z_1,z_2).$$

Proof. If we differentiate the relation (10) with respect to z_2 and use the relation (11), we get

$$DQ_{m,n}(z_1,z_2) = Q_{m+1,n+1}(z_1,z_2),$$

by repeating k times successively, it yields

$$D^kQ_{m,n}(z_1,z_2) = Q_{m+k,n+k}(z_1,z_2)$$

for $k = 1, 2, ...$ □

3. Some Special Cases of the Function $Q_{m,n}(z_1, z_2)$

In this section, we consider the special case of the function $Q_{m,n}(z_1,z_2)$, which is the complex Hermite polynomials. For these polynomials, we give the results obtained previous section.

Let $\phi(z_1,z_2) = 1$ and $\varphi(z_1,z_2) = -z_1z_2$. The function $Q_{m,n}(z_1,z_2)$ reduces to the polynomials

$$Q_{m,n}(z_1,z_2) = (-1)^{m+n}H_{n,m}(z_1,z_2) \tag{12}$$

where $H_{m,n}(z_1,z_2)$ denotes Ito's 2D-Hermite polynomials. From (2), it is generated by

$$\sum_{m,n=0}^{\infty}(-1)^{m+n}H_{n,m}(z_1,z_2)\frac{u^m}{m!}\frac{v^n}{n!}=e^{-vz_1-uz_2-uv},$$

by taking $u\to -v$, $v\to -u$ and $m\leftrightarrow n$, it yields

$$\sum_{m,n=0}^{\infty}H_{m,n}(z_1,z_2)\frac{u^m}{m!}\frac{v^n}{n!}=e^{uz_1+vz_2-uv},$$

which is the known result [10, 12]. From Remark 2.2,

$$\sum_{m,n=0}^{\infty}H_{m,n}(z_1,z_2)\frac{z_1^m}{m!}\frac{z_2^n}{n!}=e^{z_1^2+z_2^2-z_1z_2}\quad\text{and}\quad\sum_{m,n=0}^{\infty}H_{m,n}(z_1,z_2)\frac{z_1^n}{n!}\frac{z_2^m}{m!}=e^{z_1z_2}$$

holds (see [16]).

Corollary 3.1. *Under the choice of $\phi(z_1,z_2)=1$ and $\varphi(z_1,z_2)=-z_1z_2$, the relations (5) and (6) in Theorem 2.3 reduce to the relations, respectively*

$$Q_{m+1,n}(z_1,z_2)=-z_2Q_{m,n}(z_1,z_2)-nQ_{m,n-1}(z_1,z_2)$$

and

$$Q_{m,n+1}(z_1,z_2)=-z_1Q_{m,n}(z_1,z_2)-mQ_{m-1,n}(z_1,z_2).$$

In view of (12), we obtain the known results for the complex Hermite polynomials $H_{m,n}(z_1,z_2)$ [7, 8, 10, 16]

$$H_{m,n+1}(z_1,z_2)=z_2H_{m,n}(z_1,z_2)-mH_{m-1,n}(z_1,z_2)$$

and

$$H_{m+1,n}(z_1,z_2)=z_1H_{m,n}(z_1,z_2)-nH_{m,n-1}(z_1,z_2).$$

Corollary 3.2. *From the relations in Theorem 2.4, it follows*

$$\frac{\partial}{\partial z_1}Q_{m,n}(z_1,z_2)=-nQ_{m,n-1}(z_1,z_2)$$

and

$$\frac{\partial}{\partial z_2}Q_{m,n}(z_1,z_2)=-mQ_{m-1,n}(z_1,z_2),$$

by taking into account (12), which conclude the known results [3, 7, 16]

$$\frac{\partial}{\partial z_1}H_{m,n}(z_1,z_2)=mH_{m-1,n}(z_1,z_2)$$

and

$$\frac{\partial}{\partial z_2}H_{m,n}(z_1,z_2)=nH_{m,n-1}(z_1,z_2).$$

As a result of Corollary 2.5, the next known results are satisfied.

Corollary 3.3. *For $H_{m,n}(z_1,z_2)$, the known relations [3, 7, 16] hold*

$$\frac{\partial}{\partial z_1}H_{m,n}(z_1,z_2) = -H_{m,n+1}(z_1,z_2) + z_2 H_{m,n}(z_1,z_2)$$

and

$$\frac{\partial}{\partial z_2}H_{m,n}(z_1,z_2) = -H_{m+1,n}(z_1,z_2) + z_1 H_{m,n}(z_1,z_2).$$

Corollary 3.4. *As a conclusion of Theorem 2.6, it is satisfied that*

$$D^k H_{m,n}(z_1,z_2) = H_{m+k,n+k}(z_1,z_2)$$

for $k = 1, 2, ...$ where the operator D is given by

$$D = \frac{\partial^2}{\partial z_1 \partial z_2} - z_2\frac{\partial}{\partial z_2} - z_1\frac{\partial}{\partial z_1} + z_1 z_2 - 1.$$

References

[1] R. Aktaş, A. Altın, F. Taşdelen, A new family of analytic functions defined by means of Rodrigues type formula, Mathematica Slovaca 68(3) (2018) 607-616.

[2] A. Altın, R. Aktaş, A class of polynomials in two variables, Mathematica Moravica 14(1) (2010) 1-14.

[3] N. Cotfas, J.-P. Gazeau, K. Gorska, Complex and real Hermite polynomials and related quantizations, J. Phys. A: Math. Theor. 43 (2010) 305304 (14 pp)

[4] C. Dunkl, Y. Xu, Orthogonal Polynomials of Several Variables, 2nd ed., Cambridge University Press, Cambridge, 2014.

[5] A. Ghanmi, A class of generalized complex Hermite polynomials, J. Math. Anal. Appl. 340(2008) 1395–1406.

[6] A. Ghanmi, Operational formulae for the complex Hermite polynomials $H_{p,q}(z,\overline{z})$, Integral Trans. Spec. Func. 24(11) (2013) 884-895.

[7] K. Gorska, A. Horzela, F. H. Szafraniec, Holomorphic Hermite polynomials in two variables, J. Math. Anal. Appl. 470(2) (2019) 750-769.

[8] M. E. H. Ismail, Analytic properties of complex Hermite polynomials, Trans. Amer. Math. Soc. 368(2) (2016) 1189–1210.

[9] M. E. H. Ismail, P. Simeonov, Complex Hermite polynomials: Their Combinatorics and integral operators , Proc. Amer. Math. Soc. 143(4) (2015) 1397–1410.

[10] M.E.H. Ismail, R. Zhang, A review of multivariate orthogonal polynomials, Journal of the Egyptian Mathematical Society 25 (2017) 91–110.

[11] M.E.H. Ismail, R. Zhang, On some 2D orthogonal q -polynomials, Trans. Amer. Math. Soc. 369 (2017) 6779-6821.

[12] K. Ito, Complex multiple Wiener integral, Japan J. Math. 22 (1952) 63–86.

[13] T.H. Koornwinder, Two-variable analogues of the classical orthogonal polynomials, in: R. Askey (Ed.), Theory and Application of Special Functions, Academic Press, New York, 1975, pp. 435–495.

[14] I. Novik, A. Postnikov, B. Sturmfels, Syzygies of oriented matroids, Duke J. Math. 111 (2002) 287–317.

[15] N. Saad, K. Thirulogasanthar, Generalized 2D Laguerre polynomials and their quaternionic extensions, J. Comp. App. Math. 308 (2016) 301-317.

[16] A. Wünsche, Generating functions for products of special Laguerre 2D and Hermite 2D polynomials, Applied Mathematics 6 (12) (2015) 2142-2168.

[17] A. Wünsche, Generalized zernike or disc polynomials, J. Comp. App. Math. 174 (2005) 135–163.

[18] A. Wünsche, Laguerre 2D -functions and their applications in quantum optics, J. Phys. A 31 (1998) 8267–8287.

[19] A. Wünsche, Transformations of Laguerre 2D -polynomials and their applications to quasiprobabilities, J. Phys. A 21 (1999) 3179–3199.

[20] F. Zernike, Beugungstheorie des schneidenverfahrens und seiner verbesserten form, der phasenkontrastmethode, Physica 1 (1934) 689–701.

[21] F. Zernike and H.C. Brinkman, Hypersphdrische funktionen und die in spharischen bereischen orthogonalen polynome, Nederl. Akad. Wetensch. Indag. Math. 38 (1935) 161–171.