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Identities and Derivative Formulas for the Combinatorial and Apostol-Euler Type Numbers by Their Generating Functions

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Abstract. The first aim of this paper is to give identities and relations for a new family of the combinatorial numbers and the Apostol-Euler type numbers of the second kind, the Stirling numbers, the Apostol-Bernoulli type numbers, the Bell numbers and the numbers of the Lyndon words by using some techniques including generating functions, functional equations and inversion formulas. The second aim is to derive some derivative formulas and combinatorial sums by applying derivative operators including the Caputo fractional derivative operators. Moreover, we give a recurrence relation for the Apostol-Euler type numbers of the second kind. By using this recurrence relation, we construct a computation algorithm for these numbers. In addition, we derive some novel formulas including the Stirling numbers and other special numbers. Finally, we also some remarks, comments and observations related to our results.

1. Introduction

The special numbers and polynomials with their generating funstions, recently, have been studied by many authors in many different areas especially almost all branches of mathematics, probability and statistics and the other areas. On the other hand, combinatorial numbers and combinatorial sums provide significant tools to solve problems not only in number theory, but also in discrete probability theory. There are various kind of studies and applications related to generating functions for the special numbers and polynomials. For example, according to the work in [5, p. 55], we see that the generating functions techniques emerged from while studying the distributions of random variables by De moivre and Laplace in probability theory. It is well-known that interpreting probability and statistic problems by the combinatorial way is very important. For instance, special numbers especially combinatorial numbers are of many application in the theory of enumerative combinatorics, the theory of probability and algebraic combinatorics on words. In this paper, we give connection between the numbers counting the k-ary n-length Lyndon words and combinatorial numbers. The numbers of the k-ary n-length Lyndon words have many applications in algebraic combinatorics, lie algebra, analytic number theory, and also the theory of generating functions.

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Motivation of this paper is to combine special numbers and polynomials, combinatorial numbers and the numbers of Lyndon words (see [2], [8], [14], [15], [16], [17], [18]).

In order to achieve our goals of this paper, we consider a family of new combinatorial numbers and the Apostol-Euler type polynomials of the second kind, which is introduced and investigated in recent papers [19], [21] and [22]. By spirit of these numbers with their generating functions, we give many novel identities including of them.

It is time to recall some notations, definitions, identities and formulas related to some well-known numbers and polynomials with their generating functions.

Let $\mathbb{N} = \{1, 2, 3, \ldots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ corresponds the set of integers, the set of real numbers and the set of complex numbers, respectively. Furthermore, we assume that

$$0^n = \begin{cases} 1, & n = 0 \\ 0, & n \in \mathbb{N} \end{cases}.$$

Also, throughout this paper, we assume that

$$\binom{x}{0} = 1 \text{ and } \binom{x}{n} = \frac{(x)_n}{n!},$$

where $x \in \mathbb{C}$, $n \in \mathbb{N}_0$ and $(x)_n$ denotes the falling factorial defined as follows:

$$(x)_n = x(x-1)(x-2)\ldots(x-n+1) \qquad (x \in \mathbb{C}; n \in \mathbb{N})$$

and

$$(x)_0 = 1 \qquad (x \in \mathbb{C})$$

(*cf.* [1]-[24], and the references cited therein)

In this paper, our motivation is to study on the combinatorial numbers $y_1(n, k; \lambda)$ defined by means of the following generating function:

$$F_{y1}(t, k; \lambda) = \frac{1}{k!}\left(\lambda e^t + 1\right)^k = \sum_{n=0}^{\infty} y_1(n, k; \lambda)\frac{t^n}{n!} \tag{1}$$

(*cf.* [19], [21]).

The explicit formula for the combinatorial numbers $y_1(n, k; \lambda)$ is given by

$$y_1(n, k; \lambda) = \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j} j^n \lambda^j \tag{2}$$

(*cf.* [19], [21]).

One can easily see that for $\lambda = 1$, these combinatorial numbers are reduced to the following combinatorial sum:

$$B(n, k) = k! y_1(n, k; 1) = \sum_{j=0}^{k}\binom{k}{j} j^n \tag{3}$$

(*cf.* [19], [21]).

The Apostol-Euler type polynomials of the second kind of order k, $E_n^{*(k)}(x; \lambda)$ defined by means of the following generating functions:

$$F_P(t, x; k, \lambda) = \left(\frac{2}{\lambda e^t + \lambda^{-1}e^{-t}}\right)^k e^{tx} = \sum_{n=0}^{\infty} E_n^{*(k)}(x; \lambda)\frac{t^n}{n!} \tag{4}$$

and $E_n^{*(k)}(\lambda) = E_n^{*(k)}(0;\lambda)$ are the Apostol-Euler type numbers of the second kind of order k and their generating functions are given by

$$F_N(t;k,\lambda) = \left(\frac{2}{\lambda e^t + \lambda^{-1}e^{-t}}\right)^k = \sum_{n=0}^{\infty} E_n^{*(k)}(\lambda)\frac{t^n}{n!} \tag{5}$$

(*cf.* [19], [21]).

Moreover, for $\lambda = k = 1$, the $E_n^{*(k)}(\lambda)$ numbers are reduced to the Euler numbers of the second kind E_n^*:

$$E_n^* = E_n^{*(1)}(1)$$

which are given by the following generating function:

$$\frac{2}{e^t + e^{-t}} = \sum_{n=0}^{\infty} E_n^* \frac{t^n}{n!} \tag{6}$$

where $|t| < \frac{\pi}{2}$ (*cf.* [9], [21], [24]; and see also the references cited therein).

In [13], Kim *et al.* defined the λ-Bernoulli polynomials $B_n(\lambda;x)$ are given by the following generating function:

$$\frac{\log\lambda + t}{\lambda e^t - 1}e^{tx} = \sum_{n=0}^{\infty} B_n(\lambda;x)\frac{t^n}{n!} \tag{7}$$

For $x = 0$, these polynomials are reduced to the λ-Bernoulli numbers $B_n(\lambda)$:

$$B_n(\lambda) = B_n(\lambda;0)$$

which are given by the following generating function:

$$\frac{\log\lambda + t}{\lambda e^t - 1} = \sum_{n=0}^{\infty} B_n(\lambda)\frac{t^n}{n!} \tag{8}$$

(*cf.* [13], [11]; and see also the references cited therein).

In [13], Kim *et al.* considered the λ-Bernoulli numbers as follows:

$$\lambda(B(\lambda)+1)^n - B_n(\lambda) = \begin{cases} \log\lambda, & n = 0 \\ 1, & n = 1 \\ 0, & n > 1 \end{cases} \tag{9}$$

with the umbral calculus convention of replacing $B_n(\lambda)$ by $B^n(\lambda)$.

By using (9), some special values of $B_n(\lambda)$ are given as follows:

$$B_0(\lambda) = \frac{\log\lambda}{\lambda - 1}, B_1(\lambda) = \frac{\lambda - 1 - \lambda\log\lambda}{(\lambda-1)^2}, \ldots$$

(*cf.* [13], [11], [20]; and see also the references cited therein).

From equation (7) and equation (8), we have

$$B_n(\lambda;x) = \sum_{k=0}^{n} \binom{n}{k} B_k(\lambda)x^{n-k}$$

(*cf.* [13]).

Also, in [13, p. 9], Kim *et al.* derived the following formula including the sums powers of consecutive integers in terms of λ-Bernoulli numbers and polynomials:

$$B_l(\lambda;k)-\lambda^{-k}B_l(\lambda)=\lambda^{-k}l\sum_{n=0}^{k-1}\lambda^n n^{l-1}. \tag{10}$$

The Stirling numbers of the second kind $S_2(n,k)$ are defined by means of the following generating functions:

$$x^n=\sum_{k=0}^{n}S_2(n,k)(x)_k \tag{11}$$

and

$$\frac{1}{k!}(e^x-1)^k=\sum_{n=k}^{\infty}S_2(n,k)\frac{x^n}{n!}$$

where $k\in\mathbb{N}_0$ (*cf.* [3], [9], [4], [5], [6], [24]; and the references cited therein). The recurrence relation for the Stirling numbers of the second kind are given by

$$S_2(n,k)=S_2(n-1,k-1)+kS_2(n-1,k) \tag{12}$$

and $S_2(0,0)=1$, $S_2(n,k)=0$ if $k>n$; $S_2(n,0)=0$ if $n>0$. By using the above generating function, an explicit formula for the Stirling numbers of the second kind is given by

$$S_2(n,v)=\frac{1}{v!}\sum_{j=0}^{v}\binom{v}{j}(-1)^j(v-j)^n \tag{13}$$

(*cf.* [3], [4], [5], [6], [24]; and the references cited therein).

It may be worthy to note that the relation between the Stirling numbers of the second kind and the combinatorial numbers $y_1(n,k;\lambda)$ given by (*cf.* [19]):

$$S_2(n,k)=(-1)^k y_1(n,k;-1). \tag{14}$$

For $n\geq 1$, the Bell numbers Bl_n, enumerates all partitions of a set with n elements, are given by

$$Bl_n=\sum_{k=1}^{n}S_2(n,k) \tag{15}$$

and their generating functions are given by

$$e^{e^x-1}=\sum_{n=0}^{\infty}Bl_n\frac{x^n}{n!}$$

(*cf.* [3]; [6]; and also see the references cited therein).

In [10, Vol. 7, Eq-(2.29)], Gould gave the following identity:

$$\sum_{n=0}^{\infty}\frac{n^r x^n}{n!}=e^x\sum_{k=1}^{r}S_2(r,k)x^k. \tag{16}$$

The Möbius function μ, which is an arithmetical function, is given by (*cf.* [1]):

$$\mu(n)=\begin{cases}1 & \text{if } n=1,\\ (-1)^m & \text{if } n \text{ is a square-free integer with } m \text{ distinct prime factors},\\ 0 & \text{if } n \text{ has a squared prime factor.}\end{cases}$$

In order to give our identities, we need the Möbius inversion formula given by the following theorem:

Theorem 1.1. *Let f and g be arithmetical functions which satisfy*

$$f(n) = \sum_{d|n} g(d)$$

if and only if

$$g(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) f(d),$$

(cf. [1, p. 32]).

The numbers $L_k(n)$, counts the k-ary n length Lyndon words, are given as follows:

$$L_k(n) = \frac{1}{n} \sum_{d|n} \mu\left(\frac{n}{d}\right) k^d, \tag{17}$$

(*cf* [2], [8], [18], [16]).

In particular, in algebraic combinatorics, there exist various combinatorial interpretations related to the numbers $L_k(n)$. One of them is counting the numbers of the k-ary n-length Lyndon words from lexicographically ordered free semigroup composed of the k-letters. The k-ary n-length Lyndon words are representatives of the primitive necklaces and necklaces are formed by the ways of placing n colored beads circularly with set of k various colours. It is also known that the numbers $L_k(n)$ correspond dimension formulas in some works based on the theory of free lie algebra. The formula in (17) is also known as a Witt's formula. The numbers $L_k(n)$ are also used for the enumeration of monic irreducible polynomials of degree n over Galois field $GF(k)$ (*cf.* [2], [8], [14], [15], [16], [17], [18]).

The present paper consist of next four sections. We summarize our results as follows:

In Section 2, we construct not only recurrence relations for the Apostol-Euler type numbers of the second kind, but also a computation algorithm for the Apostol-Euler type numbers of the second kind using these recurrence relations.

In Section 3, by using some powerful techniques including generating functions, functional equations and inversion formulas for any two arithmetical functions, we derive some identities related to a family of new combinatorial numbers, the Apostol-Euler type numbers of the second kind, the Stirling numbers of the second kind, the Euler numbers of the second kind, the Apostol-Bernoulli type numbers, the Bell numbers and the numbers of special words.

In the last two sections, we derive some novel partial derivative formulas for a family of new combinatorial numbers by using derivative operator and the Caputo fractional derivative operator. These derivative formulas provides ways to derive many identities for the special numbers and polynomials.

2. Recurrence relations for the Apostol-Euler type numbers of the second kind

In this section, we give recurrence relations for the Apostol-Euler type numbers of the second kind $E_n^{*(k)}(\lambda)$. By using these relations, we compute few values of these numbers.

By using Equation (5), we obtain a recurrence relation for the Apostol-Euler type numbers of the second kind $E_n^{*(k)}(\lambda)$ by the following theorem:

Theorem 2.1. *Let $n \in \mathbb{N}_0$. Starting with*

$$E_n^{*(0)}(\lambda) = \begin{cases} 1, & n = 0 \\ 0, & n > 0 \end{cases}$$

and

$$E_0^{*(1)}(\lambda) = \frac{2\lambda}{\lambda^2 + 1}.$$

If $n>0$, then we have

$$\sum_{j=0}^{n}\binom{n}{j}\left(\lambda+(-1)^{n-j}\lambda^{-1}\right)E_j^{*(1)}(\lambda)=0. \tag{18}$$

By using the above relations, we compute the values of some $E_n^{*(1)}(\lambda)$ numbers as follows:

$$E_1^{*(1)}(\lambda)=\frac{2\lambda\left(1-\lambda^2\right)}{\left(\lambda^2+1\right)^2},\quad E_2^{*(1)}(\lambda)=\frac{2\lambda\left(\lambda^2-2\lambda-1\right)\left(\lambda^2+2\lambda-1\right)}{\left(\lambda^2+1\right)^3},\dots$$

It is time to compute the Apostol-Euler type numbers of the second kind of higher-order, $E_n^{*(k)}(\lambda)$ by the following theorem:

Theorem 2.2.

$$E_n^{*(k_1+k_2)}(\lambda)=\sum_{j=0}^{n}\binom{n}{j}E_j^{*(k_1)}(\lambda)E_{n-j}^{*(k_2)}(\lambda). \tag{19}$$

Proof. We set the following functional equation:

$$F_N(t;k_1+k_2,\lambda)=F_N(t;k_1,\lambda)F_N(t;k_2,\lambda).$$

Combining the above equation with (5), we get

$$\sum_{n=0}^{\infty}E_n^{*(k_1+k_2)}(\lambda)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}E_j^{*(k_1)}(\lambda)E_{n-j}^{*(k_2)}(\lambda)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. □

By using the above relations, we compute the values of some $E_n^{*(k)}(\lambda)$ numbers as follows:

$$E_0^{*(2)}(\lambda)=\frac{4\lambda^2}{\left(\lambda^2+1\right)^2},\quad E_1^{*(2)}(\lambda)=\frac{8\lambda^2\left(1-\lambda^2\right)}{\left(\lambda^2+1\right)^2},\dots$$

2.1. Computation algorithm for the Apostol-Euler type numbers of the second kind $E_n^{*(k)}(\lambda)$

Here, we give a computation algorithm for the Apostol-Euler type numbers of the second kind $E_n^{*(k)}(\lambda)$ by using Theorem 2.1 and Theorem 2.2.

Algorithm 1 Let $n, k \in \mathbb{N}_0$. This algorithm will return the Apostol-Euler type numbers of the second kind, $E_n^{*(k)}(\lambda)$ by using Theorem 2.1 and Theorem 2.2., recursively.

```
procedure Second_Kind_Apostol_Euler_Type_numbers(n: nonnegative integer, k: nonnegative integer, λ)
    Begin
    Lobal variables:
    j : integer
    if k = 0 then
        if n = 0 then
            return 1
        else
            if n > 0 then
                return 0
            end if
        end if
    else
        if k = 1 then
            if n = 0 then
                return 2λ/(λ^2 + 1)
            else
                if n > 0 then
                    return −(1/(λ^2 + 1)) ∗ sum( Binomial_Coef(n, j)∗(λ^2 + Power(−1, n − j))
                        ↪ ∗SECOND_KIND_APOSTOL_EULER_TYPE_NUMBERS(n − j, k, λ), j, 1, n)
                end if
            end if
        else
            if k > 1 then
                return sum(Binomial_Coef(n, j) ∗SECOND_KIND_APOSTOL_EULER_TYPE_NUMBERS(j, λ, k − 1)
                        ↪ ∗SECOND_KIND_APOSTOL_EULER_TYPE_NUMBERS(n − j, λ, 1), j, 0, n)
            end if
        end if
    end if
end procedure
```

3. Identities and relations

In this section, we investigate two kind of identities and relations. One of them is related to the combinatorial numbers and the Apostol-Euler type numbers of the second kind. For these identities, our methods are depended on generating functions and their functional equations for these numbers. In order to prove the other identities for the Möbius function and the Lyndon words, we use an arithmetical function and Mobius inversion formula method.

3.1. *Identities and relations for the combinatorial numbers and the Apostol-Euler type numbers of the second kind*

Here, we give some functional equations for the generating functions of the combinatorial numbers and the Apostol-Euler type numbers of the second kind. By using these equations, we derive many identities and relations.

By using (1), we give the following functional equation:

$$\lambda e^t \sum_{j=1}^{k-1} j! F_{y1}(t, j; \lambda) = k! F_{y1}(t, k; \lambda) - F_{y1}(t, 1; \lambda).$$

By using the above equation, we obtain

$$\lambda \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{j=1}^{k-1} j! \sum_{n=0}^{\infty} y_1(n, j; \lambda) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left(k! y_1(n, k; \lambda) - y_1(n, 1; \lambda)\right) \frac{t^n}{n!}.$$

By using the Cauchy product, we get

$$\sum_{n=0}^{\infty} \left(\lambda \sum_{j=1}^{k-1} \sum_{m=0}^{n} \binom{n}{m} j! y_1(m, j; \lambda) \right) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left(k! y_1(n, k; \lambda) - y_1(n, 1; \lambda)\right) \frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the following theorem:

Theorem 3.1.

$$k! y_1(n, k; \lambda) = y_1(n, 1; \lambda) + \lambda \sum_{j=1}^{k-1} \sum_{m=0}^{n} \binom{n}{m} j! y_1(m, j; \lambda).$$

From (1) and (5), we have the following functional equation:

$$F_{y1}\left(2t, k; \lambda^2\right) F_N(t; k, \lambda) = \frac{(2\lambda)^k}{k!} e^{tk}.$$

From the above equation, we get

$$\sum_{n=0}^{\infty} 2^n y_1\left(n, k; \lambda^2\right) \frac{t^n}{n!} \sum_{n=0}^{\infty} E_n^{*(k)}(\lambda) \frac{t^n}{n!} = \frac{(2\lambda)^k}{k!} \sum_{n=0}^{\infty} k^n \frac{t^n}{n!}.$$

Therefore, using the Cauchy product in the above equation, we obtain

$$\sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \binom{n}{m} 2^m y_1\left(m, k; \lambda^2\right) E_{n-m}^{*(k)}(\lambda) \right) \frac{t^n}{n!} = \frac{(2\lambda)^k}{k!} \sum_{n=0}^{\infty} k^n \frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the following theorem:

Theorem 3.2.

$$\sum_{m=0}^{n} \binom{n}{m} 2^m y_1\left(m, k; \lambda^2\right) E_{n-m}^{*(k)}(\lambda) = \frac{(2\lambda)^k}{k!} k^n. \tag{20}$$

Firstly, from equation (20), we have

$$\frac{k!}{2^k} \sum_{m=0}^{n} \binom{n}{m} 2^m y_1\left(m, k; \lambda^2\right) E_{n-m}^{*(k)}(\lambda) = \lambda^k k^n.$$

Summing the above equation over all $0 \leq k \leq v$, we obtain the following combinatorial sums:

$$\sum_{k=0}^{v} \sum_{m=0}^{n} \binom{n}{m} k! 2^{m-k} y_1\left(m, k; \lambda^2\right) E_{n-m}^{*(k)}(\lambda) = \sum_{k=0}^{v} \lambda^k k^n.$$

Therefore, by combining the above equation with (10), we get the following theorem:

Theorem 3.3.

$$\sum_{k=0}^{v}\sum_{m=0}^{n}\binom{n}{m}k!2^{m-k}y_1\left(m,k;\lambda^2\right)E_{n-m}^{*(k)}(\lambda)=\frac{\lambda^{v+1}B_{n+1}(\lambda;v+1)-B_{n+1}(\lambda)}{n+1}.$$

Secondly, summing both side of the equation (20) from 0 to infinity over k and using (16), we arrive at the following result:

Corollary 3.4. *Let $n\in\mathbb{N}$. Then we have*

$$\sum_{k=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}2^m y_1\left(m,k;\lambda^2\right)E_{n-m}^{*(k)}(\lambda)=e^{2\lambda}\sum_{j=1}^{n}(2\lambda)^j S_2(n,j). \tag{21}$$

Substituting $\lambda=\frac{1}{2}$ into (21), we get sum of infinite series by the following corollary:

Corollary 3.5. *Let $n\in\mathbb{N}$. Then we have*

$$\sum_{k=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}2^m y_1\left(m,k;\frac{1}{4}\right)E_{n-m}^{*(k)}\left(\frac{1}{2}\right)=e\sum_{j=1}^{n}S_2(n,j).$$

Substituting (15) into Corollary 3.5, we also get the following combinatorial sum:

$$\sum_{k=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}2^m y_1\left(m,k;\frac{1}{4}\right)E_{n-m}^{*(k)}\left(\frac{1}{2}\right)=eBl_n.$$

Moreover, substituting $\lambda=1$ into the equation (20) and combining with (3), we arrive at the following corollary:

Corollary 3.6.

$$\sum_{m=0}^{n}\binom{n}{m}2^m B(m,k)E_{n-m}^{*(k)}(1)=2^k k^n.$$

By substituting $k=1$ into the above equation and using (3) and (6), we also get a (presumably new) formula for the Euler numbers of the second kind E_n^* by the following corollary:

Corollary 3.7.

$$\sum_{m=1}^{n}\binom{n}{m}2^{m-1}E_{n-m}^{*}=1-E_n^{*}.$$

3.2. Identities related to the Möbius function and the Lyndon words

Here, we derive two identities associated with the Möbius function and the Lyndon words by using an arithmetical function and Mobius inversion formula.

By combining (2) with (17), we arrive at the following theorem:

Theorem 3.8.

$$\sum_{d|n}\mu\left(\frac{n}{d}\right)y_1(d,k;\lambda)=\frac{n}{k!}\sum_{j=0}^{k}\binom{k}{j}\lambda^j L_j(n). \tag{22}$$

By applying the Möbius inversion formula to equation (22), we get the following novel identity related to the numbers $L_k(n)$ and $y_1(n,k;\lambda)$:

Theorem 3.9.

$$y_1(n,k;\lambda)=\sum_{d|n}\sum_{j=0}^{k}\binom{k}{j}\frac{d}{k!}\lambda^j L_j(d).$$

4. Partial derivative equations for the functions $F_{y1}(t,k;\lambda)$ and their applications

In this section, we give partial derivative equations including the functions $F_{y1}(t,k;\lambda)$. By using these equations, we derive recurrence relations for the combinatorial numbers $y_1(n,k;\lambda)$.

Differentiating both side of (1) with respect to t, we get the following partial differential equation:

$$\frac{\partial}{\partial t}\left\{F_{y1}(t,k;\lambda)\right\} = \lambda e^t F_{y1}(t,k-1;\lambda). \tag{23}$$

and

$$\frac{\partial^2}{\partial t^2}\left\{F_{y1}(t,k;\lambda)\right\} = \lambda^2 e^{2t} F_{y1}(t,k-2;\lambda) + \lambda e^t F_{y1}(t,k-1;\lambda).$$

By iterating the above derivation for the variable t and using induction method, we arrive at the following higher order differential equation for the functions $F_{y1}(t,k;\lambda)$ by the following theorem:

Theorem 4.1. *Let $m \in \mathbb{N}$. Then*

$$\frac{\partial^m}{\partial t^m}\left\{F_{y1}(t,k;\lambda)\right\} = \sum_{j=1}^{m} \lambda^j e^{jt} S_2(m,j) F_{y1}(t,k-j;\lambda). \tag{24}$$

Remark 4.2. *Replacing k by $2k$ in* (24), *we arrive at another differantial equation which was given by Simsek in* [23].

By using (1), we have

$$\frac{\partial^m}{\partial t^m}\left\{F_{y1}(t,k;\lambda)\right\} = \sum_{n=0}^{\infty} y_1(n+m,k;\lambda)\frac{t^n}{n!}.$$

Substituting the above equation and (1) into (24), we get

$$\sum_{n=0}^{\infty} y_1(n+m,k;\lambda)\frac{t^n}{n!} = \sum_{j=1}^{m} \lambda^j S_2(m,j) \sum_{n=0}^{\infty} j^n \frac{t^n}{n!} \sum_{n=0}^{\infty} y_1(n,k-j;\lambda)\frac{t^n}{n!}$$

and using the Cauchy product, we obtain

$$\sum_{n=0}^{\infty} y_1(n+m,k;\lambda)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left(\sum_{j=1}^{m}\sum_{l=0}^{n}\binom{n}{l}\lambda^j j^{n-l} S_2(m,j)\, y_1(l,k-j;\lambda)\right)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the following theorem:

Theorem 4.3. *Let $m \in \mathbb{N}$ and $n \in \mathbb{N}_0$. Then*

$$y_1(n+m,k;\lambda) = \sum_{j=1}^{m}\sum_{l=0}^{n}\binom{n}{l}\lambda^j j^{n-l} S_2(m,j)\, y_1(l,k-j;\lambda). \tag{25}$$

It is time to give some applications of (25). By substituting some special value of λ, we obtain some (presumably new) identities.

If we substitute $\lambda = -1$ into (25) and using (14), we arrive at the following theorem:

Theorem 4.4. *Let $m \in \mathbb{N}$ and $n \in \mathbb{N}_0$. Then*

$$S_2(n+m,k) = \sum_{j=1}^{m}\sum_{l=0}^{n}\binom{n}{l} j^{n-l} S_2(m,j)\, S_2(l,k-j). \tag{26}$$

Remark 4.5. *In* [23], *Simsek also gave a similar formula to (26) by using functional equation method. We use partial differential equation in our proof which is different from that of* [23].

Substituting $\lambda = 1$ into (25) and using (3), we also arrive at the following corollary:

Corollary 4.6. *Let $m \in \mathbb{N}$ and $n \in \mathbb{N}_0$. Then*

$$\frac{1}{k!}B(n+m,k) = \sum_{j=1}^{m}\sum_{l=0}^{n}\binom{n}{l}\frac{j^{n-l}S_2(m,j)\,B(l,k-j)}{(k-j)!}.$$

Therefore, by using the above equation with the definition of $B(n,k)$ in (3), we arrive at a combinatorial sum given by the following corollary:

Corollary 4.7.

$$\sum_{j=0}^{k}\binom{k}{j}j^{n+m} = \sum_{j=1}^{m}\sum_{l=0}^{n}\sum_{v=0}^{k-j}\binom{n}{l}\binom{k}{j}\binom{k-j}{v}v^l j^{n-l} j! S_2(m,j).$$

Differentiating both side of (1) m times with respect to λ, we get the following partial differential equation:

$$\frac{\partial^m}{\partial\lambda^m}\left\{F_{y1}(t,k;\lambda)\right\} = e^{mt}F_{y1}(t,k-m;\lambda). \tag{27}$$

Remark 4.8. *The special case of* (27) *when $m = 1$ is reduced to the Eq.* (15) *given by Simsek in* [21].

By using (1), we have

$$\frac{\partial^m}{\partial\lambda^m}\left\{F_{y1}(t,k;\lambda)\right\} = \sum_{n=0}^{\infty}\frac{\partial^m}{\partial\lambda^m}\left\{y_1(n,k;\lambda)\right\}\frac{t^n}{n!}.$$

Substituting the above equation and (1) into (27), we get

$$\sum_{n=0}^{\infty}\frac{\partial^m}{\partial\lambda^m}\left\{y_1(n,k;\lambda)\right\}\frac{t^n}{n!} = \sum_{n=0}^{\infty}m^n\frac{t^n}{n!}\sum_{n=0}^{\infty}y_1(n,k-m;\lambda)\frac{t^n}{n!}$$

and using the Cauchy product, we obtain

$$\sum_{n=0}^{\infty}\frac{\partial^m}{\partial\lambda^m}\left\{y_1(n,k;\lambda)\right\}\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\binom{n}{j}m^j y_1(n-j,k-m;\lambda)\right)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the following theorem:

Theorem 4.9.

$$\frac{\partial^m}{\partial\lambda^m}\left\{y_1(n,k;\lambda)\right\} = \sum_{j=0}^{n}\binom{n}{j}m^j y_1(n-j,k-m;\lambda). \tag{28}$$

Differentiating both side of (23) with respect to λ, we get the following partial differential equation:

$$\frac{\partial^2}{\partial t\partial\lambda}\left\{F_{y1}(t,k;\lambda)\right\} = \lambda e^{2t}F_{y1}(t,k-2;\lambda) + e^t F_{y1}(t,k-1;\lambda). \tag{29}$$

By using (1), we have

$$\frac{\partial^2}{\partial t \partial \lambda}\left\{F_{y1}(t,k;\lambda)\right\} = \sum_{n=0}^{\infty} \frac{\partial}{\partial \lambda}\left\{y_1(n+1,k;\lambda)\right\}\frac{t^n}{n!}.$$

Substituting the above equation and (1) into (29), we get

$$\sum_{n=0}^{\infty} \frac{\partial}{\partial \lambda}\left\{y_1(n+1,k;\lambda)\right\}\frac{t^n}{n!} = \lambda \sum_{n=0}^{\infty} 2^n \frac{t^n}{n!}\sum_{n=0}^{\infty} y_1(n,k-2;\lambda)\frac{t^n}{n!} + \sum_{n=0}^{\infty}\frac{t^n}{n!}\sum_{n=0}^{\infty} y_1(n,k-1;\lambda)\frac{t^n}{n!}$$

and using the Cauchy product, we obtain

$$\sum_{n=0}^{\infty} \frac{\partial}{\partial \lambda}\left\{y_1(n+1,k;\lambda)\right\}\frac{t^n}{n!} = \sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}\left(\lambda 2^{n-j} y_1(j,k-2;\lambda) + y_1(j,k-1;\lambda)\right)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the following theorem:

Theorem 4.10.

$$\frac{\partial}{\partial \lambda}\left\{y_1(n+1,k;\lambda)\right\} = \sum_{j=0}^{n}\binom{n}{j}\left(\lambda 2^{n-j} y_1(j,k-2;\lambda) + y_1(j,k-1;\lambda)\right).$$

5. Applications of the Caputo fractional derivative to $y_1(n,k;\lambda)$

In this section, firstly we briefly give definition of the caputo fractional derivative, gamma and beta functions. Secondly, by applying the caputo fractional derivative operator to $y_1(n,k;\lambda)$, which is an analytic function of λ variable, with beta integral, we derive a novel derivative formula for the $y_1(n,k;\lambda)$.

Let $\alpha \in (0,\infty)$ and n be the nearest positive integer greater than α. The Caputo fractional derivative of order α of a function f is defined as follows:

$$D_*^{\alpha}\{f(t)\} = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-u)^{n-\alpha-1}\frac{d^n}{du^n}\{f(u)\}\,du,$$

(*cf.* [7], [12]).

Lemma 5.1.

$$\int_a^b (x-a)^{\alpha-1}(b-x)^{\beta-1}\,dx = (b-a)^{\alpha+\beta-1}B(\alpha,\beta). \tag{30}$$

where $a > b$, $Re(\alpha) > 0$ and $Re(\beta) > 0$ and

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

(cf. [25, p. 10, Eq-(69)]).

Applying the operator D_*^{α} to $y_1(n,k;\lambda)$, we get

$$D_*^{\alpha}\{y_1(n,k;\lambda)\} = \frac{1}{\Gamma(m-\alpha)}\int_0^{\lambda}(\lambda-u)^{m-\alpha-1}\frac{d^m}{du^m}\{y_1(n,k;u)\}\,du$$

where m is the nearest positive integer greater than α; that is, m depends on α.

By combining the above equation with (28) and (2), we get

$$D_{*}^{\alpha}\left\{y_1(n,k;\lambda)\right\} = \frac{1}{\Gamma(m-\alpha)}\sum_{j=0}^{n}\binom{n}{j}\frac{m^j}{(k-m)!}\sum_{l=0}^{k-m}\binom{k-m}{l}l^{n-j}\int_{0}^{\lambda}(\lambda-u)^{m-\alpha-1}u^l du.$$

Combining the above equation with (30), we arrive at the following theorem:

Theorem 5.2. *Let $k \geq m$. Then we have*

$$D_{*}^{\alpha}\left\{y_1(n,k;\lambda)\right\} = \sum_{j=0}^{n}\sum_{l=0}^{k-m}\binom{n}{j}\frac{m^j l^{n-j}\lambda^{m-\alpha+l}}{\Gamma(m-\alpha+l+1)(k-m-l)!}. \tag{31}$$

Substituting $\alpha = m \in \mathbb{N}$ into (31), we arrive at the following result:

Corollary 5.3. *Let $k \geq m$. Then we have*

$$D_{*}^{m}\left\{y_1(n,k;\lambda)\right\} = \frac{\partial^m}{\partial\lambda^m}\left\{y_1(n,k;\lambda)\right\}.$$

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