Filomat 32:20 (2018), 6869–6877
https://doi.org/10.2298/FIL1820869S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Combinatorial Identities and Sums for Special Numbers and Polynomials

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Abstract. In this paper, by using some families of special numbers and polynomials with their generating functions and functional equations, we derive many new identities and relations related to these numbers and polynomials. These results are associated with well-known numbers and polynomials such as Euler numbers, Stirling numbers of the second kind, central factorial numbers and array polynomials. Furthermore, by using higher-order partial differential equations, we derive some combinatorial sums and identities. Finally, we give two computation algorithms for Euler numbers and central factorial numbers.

1. Introduction

The special numbers and their generating functions have many applications in combinatorics and probability theory. There are many advantages of the generating functions. By using generating functions for special numbers and polynomials, one can get not only various properties of these numbers and polynomials, but also enumerating arguments such as counting the number of subsets and the number of total ordering.

In order to give our results, we need to recall some well-known generating functions for the special combinatorial numbers such as Stirling numbers of the first kind, central factorial numbers, Euler numbers and polynomials, array polynomials and other special numbers as follows:

Apostol-Euler polynomials of the first kind of order k are defined by

$$F_{P1}(t,x;k,\lambda)=\left(\frac{2}{\lambda e^{t}+1}\right)^{k}e^{tx}=\sum_{n=0}^{\infty}E_{n}^{(k)}(x;\lambda)\frac{t^{n}}{n!}, \tag{1}$$

($|t|<\pi$ when $\lambda=1$ and $|t|<|\ln(-\lambda)|$ when $\lambda\neq 1$), $\lambda\in\mathbb{C}$, the set of complex numbers, $k\in\mathbb{N}$, the set of natural numbers. By (1), we easily see that

$$E_{n}^{(k)}(\lambda)=E_{n}^{(k)}(0;\lambda),$$

2010 *Mathematics Subject Classification*. 05A10, 05A15, 11B37, 11B39, 11B68, 11B83, 11M35, 26C05, 26C10, 33C05, 34A99, 35A99.
Keywords. Euler numbers, Central factorial numbers, Array polynomials, Stirling numbers, Generating functions, Binomial coefficients, Combinatorial sum.
Received: 10 April 2018; Accepted: 04 July 2018
Communicated by Miodrag Spalević
The paper was supported by the *Scientific Research Project Administration of Akdeniz University*. A part of this paper was presented in *the 14th International Conference on Permutation Patterns (PP 2016) hold at Howard University in Washington, DC, on June 27 July 1, 2016.*
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which denotes Apostol-Euler numbers of the first kind of order k. By substituting $k = \lambda = 1$ into (1), we have

$$E_n = E_n^{(1)}(1)$$

which denotes Euler numbers of the first kind (*cf.* [4]-[17], and the references cited therein).

Euler numbers E_n^* of the second kind of order $-k$ are defined by

$$F_{E2}(t,k) = \left(\frac{2}{e^t + e^{-t}}\right)^{-k} = \sum_{n=0}^{\infty} E_n^{*(-k)} \frac{t^n}{n!}, \tag{2}$$

where $|t| < \frac{\pi}{2}$ (*cf.* [13], [14], and the references cited therein).

Combining Remark 4.2 and Equation 12 in [13], we have the following explicit formula for the numbers $E_n^{*(-k)}$ as follows:

$$E_n^{*(-k)} = 2^{n-k} \sum_{j=0}^{k} \binom{k}{j} \left(j - \frac{k}{2}\right)^n. \tag{3}$$

We also note that there exists (presumably) different proofs for the above formula.

λ-Stirling numbers of the second kind $S_2(n, v; \lambda)$ defined by

$$F_S(t, v; \lambda) = \frac{(\lambda e^t - 1)^v}{v!} = \sum_{n=0}^{\infty} S_2(n, v; \lambda) \frac{t^n}{n!}, \tag{4}$$

where $v \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\lambda \in \mathbb{C}$ (*cf.* [8], [11], [16], and the references cited therein).

By using (4), we have

$$S_2(n, v; \lambda) = \frac{1}{v!} \sum_{j=0}^{v} \binom{v}{j} (-1)^{v-j} \lambda^j j^n$$

(*cf.* [8], [11], [16]).

Substituting $\lambda = 1$ into (4), we have Stirling numbers of the second kind $S_2(n, v)$ which denotes the number of ways to partition a set of n objects into v groups:

$$S_2(n, v) = S_2(n, v; 1).$$

(*cf.* [1]-[17]; see also the references cited therein).

In [11], we defined λ-array polynomials $S_v^n(x; \lambda)$ by

$$F_A(t, x, v; \lambda) = \frac{(\lambda e^t - 1)^v}{v!} e^{tx} = \sum_{n=0}^{\infty} S_v^n(x; \lambda) \frac{t^n}{n!}, \tag{5}$$

where $v \in \mathbb{N}_0$ and $\lambda \in \mathbb{C}$ (*cf.* [5], [4], [11], [12], and the references cited therein).

Central factorial numbers $T(n, k)$ of the second kind are defined by

$$F_T(t, k) = \frac{1}{(2k)!} \left(e^t + e^{-t} - 2\right)^k = \sum_{n=0}^{\infty} T(n, k) \frac{t^{2n}}{(2n)!} \tag{6}$$

(*cf.* [2], [6], [12], [17], and the references cited therein).

Remark 1.1. *Central factorial numbers are used in combinatorial problems. That is, the number of ways to place k rooks on a size m triangle board in three dimensions is equal to*

$$T(m + 1, m + 1 - k),$$

where $0 \leq k \leq m$ (cf. [1]).

In [14], we defined the numbers $y_1(n,k;\lambda)$ by means of the following generating functions:

$$F_{y_1}(t,k;\lambda) = \frac{1}{k!}\left(\lambda e^t + 1\right)^k = \sum_{n=0}^{\infty} y_1(n,k;\lambda)\frac{t^n}{n!}, \tag{7}$$

where $k \in \mathbb{N}_0$ and $\lambda \in \mathbb{C}$. If we substitute $\lambda = -1$ into (7), then we have

$$S_2(n,k) = (-1)^k y_1(n,k;-1) \tag{8}$$

(*cf.* [14], [13]). The numbers $y_1(n,k;\lambda)$ is related to following combinatorial sum:

$$B(n,k) = k! y_1(n,k;1) = \sum_{j=0}^{k} \binom{k}{j} j^n = \frac{d^n}{dt^n}\left(e^t + 1\right)^k |_{t=0}\ , \tag{9}$$

where $n = 1, 2, \ldots$(cf. [7], [14]). In the work of Spivey [15, Identity 8-Identity 10], we see that

$$B(0,k) = 2^k, B(1,k) = k2^{k-1}, B(2,k) = k(k+1)2^{k-2},$$

and also

$$B(m,n) = \sum_{j=0}^{n} \binom{n}{j} j! 2^{n-j} S_2(m,j), \tag{10}$$

(*cf.* [3, p.4, Eq-(7)], [14]; see also the references cited therein). In [14], we gave a conjecture and two open questions associated with the numbers $B(n,k)$.

In [14], we defined the numbers $y_2(n,k;\lambda)$ by means of the following generating functions:

$$F_{y_2}(t,k;\lambda) = \frac{1}{(2k)!}\left(\lambda e^t + \lambda^{-1}e^{-t} + 2\right)^k = \sum_{n=0}^{\infty} y_2(n,k;\lambda)\frac{t^n}{n!}. \tag{11}$$

In [14], we gave some combinatorial interpretations for the numbers $y_1(n,k)$, $y_2(n,k)$ and $B(n,k)$ as well as the generalization of the central factorial numbers. We also see that these numbers were related to the rook numbers and polynomials.

We summarize our results as follows: In Section 2, by using functional equations of the generating fucntions, we derive various identities and relations related to the Stirling numbers, the Euler numbers, the central factorial numbers, the array polynomials, the numbers $y_1(n,k;\lambda)$ and the numbers $y_2(n,k;\lambda)$.

In Section 3, we give higher-order partial derivative for the generating functions. By using these functions, we give some combinatorial sums including the numbers $y_1(n,k;\lambda)$ and the numbers $y_2(n,k;\lambda)$ with their generating functions.

In Seciton 4, we give computation algorithms for the Euler numbers and the central factorial numbers.

2. Functional equations and related identities

By using generating functions for Stirling numbers, Euler numbers, central factorial numbers, array polynomials, the numbers $y_1(n,k;\lambda)$ and the numbers $y_2(n,k;\lambda)$ with their functional equations, we derive some identities and relations involving binomial coefficients and these numbers and polynomials. We also give computation formulas for Euler numbers and polynomials of the first kind and the second kind.

By using (7) and (4), we obtain the following functional equation:

$$F_{y_1}(2t,k;-\lambda^2) = (-1)^k k! F_{y_1}(t,k;\lambda) F_S(t,k;\lambda).$$

By using the above equation, we get

$$\sum_{n=0}^{\infty} 2^n y_1\left(n,k;-\lambda^2\right)\frac{t^n}{n!} = (-1)^k k! \sum_{n=0}^{\infty} y_1(n,k;\lambda)\frac{t^n}{n!}\sum_{n=0}^{\infty} S_2\left(n,k;\lambda\right)\frac{t^n}{n!}.$$

By using the Cauchy product in the above equation, we obtain

$$\sum_{n=0}^{\infty} 2^n y_1\left(n,k;-\lambda^2\right)\frac{t^n}{n!} = (-1)^k k! \sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l} S_2(l,k;\lambda) y_1(n-l,k;\lambda)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive the following theorem:

Theorem 2.1.

$$y_1(n,k;-\lambda^2) = (-1)^k k! 2^{-n} \sum_{l=0}^{n}\binom{n}{l} S_2(l,k;\lambda) y_1(n-l,k;\lambda). \tag{12}$$

By substituting $\lambda = 1$ into (12) and combining (8) and (10), we arrive at the following corollary:

Corollary 2.2.

$$S_2\left(n,k\right) = \sum_{l=0}^{n}\sum_{j=0}^{k}\binom{n}{l}\frac{2^{k-j-n}}{(k-j)!} S_2\left(l,k\right) S_2\left(n-l,j\right).$$

By combining (5) with (6) and (11), we obtain the following functional equation:

$$F_A(2t,-k,2k;1) = (2k)! F_T(t,k) F_{y_2}(t,k;1).$$

Using the above equation, we get

$$\sum_{n=0}^{\infty} 2^n S_{2k}^n(-k)\frac{t^n}{n!} = (2k)! \sum_{n=0}^{\infty} T(n,k)\frac{t^{2n}}{(2n)!}\sum_{n=0}^{\infty} y_2\left(n,k;1\right)\frac{t^{2n}}{(2n)!}.$$

Therefore

$$\sum_{n=0}^{\infty} 2^n S_{2k}^n(-k)\frac{t^n}{n!} = (2k)! \sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l} T(j,k) y_2\left(n-l,k;1\right)\frac{t^{2n}}{(2n)!}.$$

By using the above equation, we arrive at the following theorem:

Theorem 2.3.

$$S_{2k}^{2n}(-k) = (2k)! 2^{-2n} \sum_{l=0}^{n}\binom{n}{l} T(l,k) y_2\left(n-l,k;1\right).$$

Lemma 2.4. *([9, Lemma 11, Eq-(7)])*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(n,k) = \sum_{n=0}^{\infty}\sum_{k=0}^{\left[\frac{n}{2}\right]} A(n,n-2k),$$

where $[x]$ *denotes the greatest integer function.*

By combining (5) and (6) with (7), we get the following functional equation:

$$F_T(t,k) = \frac{k!}{(2k)!}\sum_{l=0}^{k}\frac{(2l)!}{l!}F_T\left(\frac{t}{2},l\right)F_A\left(-\frac{t}{2},\frac{l}{2},k-l;1\right).$$

By using the above equation, we obtain

$$\sum_{n=0}^{\infty}T(n,k)\frac{t^{2n}}{(2n)!} = \frac{k!}{(2k)!}\sum_{l=0}^{k}\frac{(2l)!}{l!}\sum_{n=0}^{\infty}2^{-2n}T(n,l)\frac{t^{2n}}{(2n)!}\sum_{n=0}^{\infty}S_{k-l}^{n}\left(\frac{l}{2},1\right)\frac{t^n}{n!}.$$

By using Lemma 2.4, we get

$$\sum_{n=0}^{\infty}T(n,k)\frac{t^{2n}}{(2n)!} = \frac{k!}{(2k)!}\sum_{l=0}^{k}\frac{(2l)!}{l!}\sum_{n=0}^{\infty}\sum_{j=0}^{\left[\frac{n}{2}\right]}T(j,l)\,S_{k-l}^{n-2j}\left(\frac{l}{2},1\right)\frac{2^{-2j}}{(2j)!}\frac{t^n}{(n-2j)!}.$$

Comparing the coefficients on both sides of the above equation, we arrive the following theorem:

Theorem 2.5. *If n is an even integer, we have*

$$T(n,k) = \frac{(2n)!k!}{(2k)!n!}\sum_{l=0}^{k}\sum_{j=0}^{\left[\frac{n}{2}\right]}\binom{n}{2j}\frac{(2l)!}{2^{2j}l!}T(j,l)\,S_{k-l}^{n-2j}\left(\frac{l}{2},1\right)$$

and if n is an odd integer, we have

$$\sum_{l=0}^{k}\sum_{j=0}^{\left[\frac{n}{2}\right]}\binom{n}{2j}\frac{(2l)!}{2^{2j}l!}T(j,l)\,S_{k-l}^{n-2j}\left(\frac{l}{2},1\right) = 0.$$

By combining (5) with (2), we obtain the following functional equation:

$$F_T(2t,k) = \frac{2^{2k}}{(2k)!}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}F_{E2}(t,-2j).$$

By using the above functional equation, we get

$$\sum_{n=0}^{\infty}2^nT(n,k)\frac{t^{2n}}{(2n)!} = \frac{2^{2k}}{(2k)!}\sum_{n=0}^{\infty}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}E_n^{*(-2j)}\frac{t^{2n}}{(2n)!}.$$

Comparing the coefficients of $\frac{t^{2n}}{(2n)!}$ on both sides of the above equation, we arrive at the following theorem:

Theorem 2.6.

$$T(n,k) = \frac{2^{2k-n}}{(2k)!}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}E_n^{*(-2j)}. \tag{13}$$

By using (11) and (4), we get the following functional equation:

$$F_{y_2}(t,k;-\lambda) = \frac{k!}{(2k)!}\sum_{j=0}^{k}(-1)^kF_S(t,j;\lambda)F_S(-t,k-j;\lambda^{-1}).$$

By using the above functional equation, we obtain

$$\sum_{n=0}^{\infty} y_2(n,k;\lambda)\frac{t^n}{n!} = \frac{k!}{(2k)!}\sum_{j=0}^{k}(-1)^k\left(\sum_{n=0}^{\infty} S_2(n,j;\lambda)\frac{t^n}{n!}\sum_{n=0}^{\infty} S_2\left(n,k-j;\lambda^{-1}\right)\frac{(-t)^n}{n!}\right).$$

By using the Cauchy product in the right-hand side of the above equation, we obtain

$$\sum_{n=0}^{\infty} y_2(n,k;\lambda)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\frac{k!}{(2k)!}\sum_{j=0}^{k}\sum_{d=0}^{n}(-1)^{k+n-d}\binom{n}{d}S_2(d,j;\lambda)S_2\left(n-d,k-j;\lambda^{-1}\right)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the following theorem:

Theorem 2.7.

$$y_2(n,k;\lambda) = \frac{k!}{(2k)!}\sum_{j=0}^{k}\sum_{d=0}^{n}(-1)^{k+n-d}\binom{n}{d}S_2(d,j;\lambda)S_2\left(n-d,k-j;\lambda^{-1}\right).$$

By using (11) and (1), we obtain the following functional equation:

$$F_{y_2}(t,k;\lambda) = \frac{\lambda^{-k}}{(2k)!}\sum_{j=0}^{k}\binom{k}{j}F_{P1}\left(2t,\frac{k}{2};j,\lambda^2\right).$$

By using the above equation, we get

$$\sum_{n=0}^{\infty} y_2(n,k;\lambda)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\frac{2^n\lambda^{-k}}{(2k)!}\sum_{j=0}^{k}\binom{k}{j}E_n^{(-k)}\left(\frac{k}{2};\lambda^2\right)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the following theorem:

Theorem 2.8.

$$y_2(n,k;\lambda) = \frac{2^n\lambda^{-k}}{(2k)!}\sum_{j=0}^{k}\binom{k}{j}E_n^{(-k)}\left(\frac{k}{2};\lambda^2\right).$$

By using (7), (11) and (4), we obtain the following functional equations:

$$F_{y_2}(t,k;\lambda) = F_{y_1}(t,2k;\lambda)\sum_{j=0}^{k}\binom{k}{j}j!F_S\left(-t,j;\lambda^{-1}\right) \tag{14}$$

and

$$F_{y_2}(t,k;\lambda) = F_{y_1}(t,2k;\lambda)\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j!F_{y_1}\left(-t,j;\lambda^{-1}\right). \tag{15}$$

By using (14), we get

$$\sum_{n=0}^{\infty} y_2(n,k;\lambda)\frac{t^n}{n!} = \sum_{j=0}^{k}\binom{k}{j}j!\sum_{n=0}^{\infty} y_1(n,2k;\lambda)\frac{t^n}{n!}\sum_{n=0}^{\infty} S_2(n,j;\lambda^{-1})\frac{(-t)^n}{n!}.$$

Therefore

$$\sum_{n=0}^{\infty} y_2(n,k;\lambda)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\sum_{l=0}^{n}\sum_{j=0}^{k}(-1)^{n-l}\binom{k}{j}\binom{n}{l}j!y_1(l,2k;\lambda)S_2(n-l,j;\lambda^{-1})\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the following theorem:

Theorem 2.9.

$$y_2(n,k;\lambda) = \sum_{l=0}^{n}\sum_{j=0}^{k}(-1)^{n-l}\binom{k}{j}\binom{n}{l} j! y_1(l,2k;\lambda) S_2(n-l,j;\lambda^{-1}).$$

By using (15), we obtain

$$\sum_{n=0}^{\infty} y_2(n,k;\lambda)\frac{t^n}{n!} = \sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j} j! \sum_{n=0}^{\infty} y_1(n,2k;\lambda)\frac{t^n}{n!}\sum_{n=0}^{\infty} y_1(n,j;\lambda^{-1})\frac{(-t)^n}{n!}.$$

By using the Cauchy product in the above equation, we get

$$\sum_{n=0}^{\infty} y_2(n,k;\lambda)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\sum_{l=0}^{n}\sum_{j=0}^{k}(-1)^{n+k-j-l}\binom{k}{j}\binom{n}{l} j! y_1(l,2k;\lambda) y_1(n-l,j;\lambda^{-1})\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the following theorem:

Theorem 2.10.

$$y_2(n,k;\lambda)\frac{t^n}{n!} = \sum_{l=0}^{n}\sum_{j=0}^{k}(-1)^{n+k-j-l}\binom{k}{j}\binom{n}{l} j! y_1(l,2k;\lambda) y_1(n-l,j;\lambda^{-1})$$

3. Partial differential equations of generating functions for the numbers $y_1(n,k;\lambda)$ and $y_2(n,k;\lambda)$

In this section, we give higher-order partial differential equation of generating functions for the numbers $y_1(n,k;\lambda)$ and $y_2(n,k;\lambda)$. By using these equations, we derive some identities and relations of these numbers.

In order to give some combinatorial sums including partial derivative of generating functions for the numbers $y_1(n,k;\lambda)$ and $y_2(n,k;\lambda)$, we need the following partial derivative equations:

$$\frac{\partial}{\partial t}\left\{F_{y_1}(t,2k;\lambda)\right\} = \lambda e^t F_{y_1}(t,2k-1;\lambda),$$

$$\frac{\partial^2}{\partial t^2}\left\{F_{y_1}(t,2k;\lambda)\right\} = \lambda e^t F_{y_1}(t,2k-1;\lambda) + \lambda^2 e^{2t} F_{y_1}(t,2k-2;\lambda),$$

$$\frac{\partial^3}{\partial t^3}\left\{F_{y_1}(t,2k;\lambda)\right\} = \lambda e^t F_{y_1}(t,2k-1;\lambda) + 3\lambda^2 e^{2t} F_{y_1}(t,2k-2;\lambda) + \lambda^3 e^{3t} F_{y_1}(t,2k-3;\lambda)$$

and

$$\frac{\partial}{\partial t}\left\{F_{y_1}(-t,j;\lambda^{-1})\right\} = -\lambda^{-1} e^{-t} F_{y_1}(-t,j-1;\lambda^{-1}),$$

$$\frac{\partial^2}{\partial t^2}\left\{F_{y_1}(-t,j;\lambda^{-1})\right\} = \lambda^{-1} e^{-t} F_{y_1}(-t,j-1;\lambda^{-1}) + \lambda^{-2} e^{-2t} F_{y_1}(-t,j-2;\lambda^{-1}),$$

$$\begin{aligned}\frac{\partial^3}{\partial t^3}\left\{F_{y_1}(-t,j;\lambda^{-1})\right\} &= -\lambda^{-1} e^{-t} F_{y_1}(-t,j-1;\lambda^{-1}) - 3\lambda^{-2} e^{-2t} F_{y_1}(-t,j-2;\lambda^{-1}) \\ &\quad -\lambda^{-3} e^{-3t} F_{y_1}(-t,j-3;\lambda^{-1}).\end{aligned}$$

Consequently, by using induction method, we get the following lemma:

Lemma 3.1.

$$\frac{\partial^m}{\partial t^m}\left\{F_{y_1}(t,2k;\lambda)\right\}=\sum_{l=1}^{m}\lambda^l e^{lt}S_2(m,l)\,F_{y_1}(t,2k-l;\lambda) \tag{16}$$

and

$$\frac{\partial^{v-m}}{\partial t^{v-m}}\left\{F_{y_1}(-t,j;\lambda^{-1})\right\}=\sum_{l=1}^{v-m}(-1)^{v-m}\lambda^{-l}e^{-lt}S_2(v-m,l)\,F_{y_1}(-t,j-l;\lambda^{-1}). \tag{17}$$

Substituting $t=0$ into (16) and (17), we get two combinatorial sums by the following theorem:

Theorem 3.2.

$$\frac{\partial^m}{\partial t^m}\left\{F_{y_1}(t,2k;\lambda)\right\}\bigg|_{t=0}=\sum_{i=1}^{m}\frac{\lambda^i(\lambda+1)^{2k-i}}{(2k-i)!}S_2(m,i) \tag{18}$$

and

$$\frac{\partial^{v-m}}{\partial t^{v-m}}\left\{F_{y_1}(-t,j;\lambda^{-1})\right\}\bigg|_{t=0}=\sum_{l=1}^{v-m}(-1)^{v-m}\frac{\lambda^{-l}\left(\lambda^{-1}+1\right)^{j-l}}{(j-l)!}S_2(v-m,l)\,. \tag{19}$$

Applying Leibnitz's formula for the vth derivative, with respect to t, to (15) and combining with (16) and (17), we obtain higher-order partial differential equation by the following theorem:

Theorem 3.3.

$$\frac{\partial^v}{\partial t^v}\left\{F_{y_2}(t,k;\lambda)\right\}=\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j!\sum_{m=0}^{v}\binom{v}{m}\frac{\partial^m}{\partial t^m}\left\{F_{y_1}(t,2k;\lambda)\right\}\frac{\partial^{v-m}}{\partial t^{v-m}}\left\{F_{y_1}\left(-t,j;\lambda^{-1}\right)\right\}. \tag{20}$$

By substituting $t=0$ into equation (20) and combining with (18) and (19), we get combinatorial sums for higher-order partial derivative of the number $y_2(n,k;\lambda)$ by the following theorem:

Theorem 3.4.

$$\begin{aligned}\frac{\partial^v}{\partial t^v}\left\{F_{y_2}(t,k;\lambda)\right\}\bigg|_{t=0} &= \sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j!\sum_{m=0}^{v}\binom{v}{m}\sum_{i=1}^{m}\frac{\lambda^i(\lambda+1)^{2k-i}}{(2k-i)!}S_2(m,i)\\ &\quad\times\sum_{l=1}^{v-m}(-1)^{v-m}\frac{\lambda^{-l}\left(\lambda^{-1}+1\right)^{j-l}}{(j-l)!}S_2(v-m,l)\,.\end{aligned} \tag{21}$$

4. Computation algorithm for the central factorial numbers $T(n,k)$

In this section, we firstly give a computation algorithm for the numbers $E_n^{*(-k)}$. By using (13), we construct a computation algorithm for the central factorial numbers $T(n,k)$ of the second kind with the help of explicit formula for the numbers $E_n^{*(-2j)}$.

Algorithm 1 This algorithm will return the values of the second kind Euler numbers of negative order, $E_n^{*(-k)}$ given by equation (3).

procedure SECOND_KIND_EULER_NUMBERS_ORDER_NEGATIVE(n: nonnegative integer, k: natural numbers)
Begin
Lobal variables:
$j \leftarrow 0, E \leftarrow 0$
for all j in $\{0, 1, 2, \ldots, k\}$ **do**
$E \leftarrow E + Binomial\_Coef(k, j) * Power(j - (k/2), n)$
end for
return $2^{n-k} * E$
end procedure

By using Algorithm 1, we give the following algorithm to compute the numbers $T(n, k)$:

Algorithm 2 This algorithm will return the values of the central factorial numbers, $T(n, k)$ of the second kind given by equation (13) with the help of the Algorithm 1.

procedure SECOND_KIND_CENTRAL_FACTORIAL_NUMBERS(n: nonnegative integer, k: natural numbers)
Begin
Lobal variables:
$j \leftarrow 0, T \leftarrow 0$
for all j in $\{0, 1, 2, \ldots, k\}$ **do**
$T \leftarrow T + Power(-1, k - j) * Binomial\_Coef(k, j) * Second\_Kind\_Euler\_Numbers\_Order\_Negative(n, 2j)$
end for
return $\left(2^{2k-n} / (2k)!\right) * T$
end procedure

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