Filomat 32:19 (2018), 6809–6818
https://doi.org/10.2298/FIL1819809L

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

A Note on the Radius of Robust Feasibility for Uncertain Convex Programs

Xiao-Bing Li[a], Qi-Lin Wang[a]

[a]*College of Mathematics and Statistics, Chongqing Jiaotong University, Chongqing, 400074, China.*

Abstract. In this paper, the notion of the radius of robust feasibility is considered for a convex program with general convex and compact uncertainty set. An exact calculating formula for the radius of robust feasibility is given for this uncertain convex program. Moreover, we give a necessary and sufficient condition for robust feasibility for uncertain convex programs to be positive. We also give some examples to illustrate our results.

1. Introduction

Robust optimization was first introduced by Soyster [1] to find the robust solution of uncertain linear program problem. But it is not until the last 20 years that it gathered the attention of the scientific community. In the late 1990s, Ben-Tal and Nemirovski [2, 3] and El-Ghaoui et al. [4, 5] independently have made an important step forward to developing a theory for robust optimization. They have given some elegant and significant results, such as the tractability and probability bounds, on robust optimization for linear as well as convex programs (see the monograph [6]). Recently, many authors have studied the optimality and duality in robust optimization for various mathematical programs, see [7–12] and other references therein.

In the framework of robust optimization, the key issue is to deal with the solutions of robust counterpart of uncertain programs. But, to get its robust counterpart, we must enforce the constraints for all possible uncertainties with the specified uncertainty sets. This may result that the robust counterpart is not feasible. To guarantee the feasibility, Goberna et al. [10] first introduced the notion of the radius of robust feasibility for robust semi-infinite linear programs. The main approach of [10] was inspired by the elegant work on the notion of consistency radius for linear semi-infinite programming in order to guarantee the feasibility of the nominal problem under perturbations preserving the number of constraints [13–15]. In [16], Goberna et al. employed a new proof way, which is different from the one of [10], to obtain the exact formula for the radius of robust multi-objective programs. Goberna et al. [17] extended the main results of [16] to robust convex programs. In 2017, Chuong and Jeyakumar [18] observed that the formula are valid only for programs with a ball uncertainty set, and then study the radius of robust feasibility for uncertain linear programs under a general convex and compact uncertainty.

2010 *Mathematics Subject Classification*. 49K99, 65K10,90C31
Keywords. Uncertain convex program; Robust optimization; Uncertainty set; Radius of robust feasibility
Received: 04 April 2017; Accepted: 10 October 2017
Communicated by Naseer Shahzad
Research supported by the Basic and Advanced Research Project of Chongqing (cstc2015jcyjBX0131 and cstc2015jcyjA30009), the Preparatory Project of Chongqing Jiaotong University (2018PY22) and the Program of Chongqing Innovation Team Project in University (CXTDX201601022)
Email addresses: cqjtulixb@yeah.net (Xiao-Bing Li), wangql97@126.com (Qi-Lin Wang)

Motivated by the work reported in [10, 16–18], this paper aims to establish some results of the radius of robust feasibility for uncertain convex programs under a general convex and compact uncertainty. The distinguishing feature of our work lies in the use of convex and compact uncertainty set which includes the ellipsoidal, ball, polytope and box uncertainty sets and so on. Under this uncertainty set, we first give an exact formula for radius of robust feasibility for uncertain convex programs. Then, by using the exact calculating formula, we obtain a necessary and sufficient condition for robust feasibility for uncertain convex programs to be positive. Moreover, it is shown that our results can be refined in the special cases of [10, 16–18].

The paper is organized as follows. In Section 2, we present some concepts and auxiliary results. In Section 3, we exhibit an exact formula for the radius of robust feasibility for uncertain convex programs, and a necessary and sufficient condition for robust feasibility for uncertain convex programs to be positive. Some examples are also given to illustrate the main results.

2. Preliminaries

In this section, we give some definitions and some auxiliary results which will be used in the sequel. Let 0_n and $\|\cdot\|$ be the vector of zeros and the Euclidean norm in $\mathbb{R}^n$, respectively. The inner product between $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$, is defined by $\langle x, y\rangle = x^\top y$. The closed unit ball and the distance associated the above norm are denoted by $\mathbb{B}_n$ and d, respectively. Given a nonempty set D of $\mathbb{R}^n$, the notations $\operatorname{int} D$, $\operatorname{cl} D$ and $\operatorname{conv} D$ will stand for the interior, the boundary and the convex hull of D, respectively. Let $\operatorname{cone} Z := \mathbb{R}_+ \operatorname{conv} D$ denote the convex conical hull of $D \cup \{0_n\}$. An extended real-valued function h on $\mathbb{R}^n$ is said to be proper if $h(x) > -\infty$ for all $x \in \mathbb{R}^n$ and there exists $x_0 \in \mathbb{R}^n$ such that $h(x_0) < +\infty$. We denote by $\Gamma(\mathbb{R}^n)$ the class of proper convex lower semicontinuous extended real-valued function. Let $h \in \Gamma(\mathbb{R}^n)$. The conjugate function $h^* : \mathbb{R}^n \to \mathbb{R}$ is defined as $h^*(y) := \sup\{y^\top x - h(x) : x \in \operatorname{dom} h\}$, where the effective domain of h, $\operatorname{dom} h$ is given by

$$\operatorname{dom} h := \{x \in \mathbb{R}^n : h(x) < +\infty\}.$$

The epigraph of h, $\operatorname{epi} h$ is defined as

$$\operatorname{epi} h := \{(x, r) \in \mathbb{R}^n \times \mathbb{R} : x \in \operatorname{dom} h, h(x) \le r\}.$$

If $\tilde{h}(x) := h(x) - \alpha$ for any $x \in \mathbb{R}^n, \alpha \in \mathbb{R}$, then $\operatorname{epi} \tilde{h}^* = \operatorname{epi} h^* + (0_n, \alpha)$. Moreover, for any $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$, let $p(x) := h(x) + a^\top x - b$ for all $x \in \mathbb{R}^n$, and hence $\operatorname{epi} p^* = \operatorname{epi} h^* + (a, b)$.

Let Z be a closed and convex set in $\mathbb{R}^n$ with $0_n \in \operatorname{int} Z$. We define a function $\phi_Z : \mathbb{R}^n \to \mathbb{R}_+$ as

$$\phi_Z(x) := \inf\{t > 0 : x \in tZ\}.$$

This is the Minkowski functional in the convex analysis. It is well known that the Minkowski functional has many elegant properties. Now we recall the following properties of the Minkowski functional.

Lemma 2.1. [19, Lemma 1.3.13] Let Z be a closed and convex set in $\mathbb{R}^n$ with $0_n \in \operatorname{int} Z$. Then, ϕ_Z is sublinear (i.e., convex and positivity homogenous of degree one) and continuous, and $cl\, Z = \{x \in \mathbb{R}^n : \phi_Z(x) \le 1\}$. If in addition Z is bounded and symmetric (i.e., if it contains x it also contains $-x$), then $\phi_Z = \|\cdot\|$ is a normal on $\mathbb{R}^n$ generated by Z.

Consider the convex program in face of data uncertainty in the constraints

$$\text{(UP)} \qquad \min_{x\in\mathbb{R}^n} \ \bar{f}(x) \quad \text{s.t. } \bar{g}_j(x) + a_j^\top x - b_j \le 0,\ j \in J$$

where $\bar{f} : \mathbb{R}^n \to \mathbb{R}$ and $\bar{g}_j : \mathbb{R}^n \to \mathbb{R}, j \in J := \{1, \cdots, q\}$ are convex functions, $(a_j, b_j) \in \mathbb{R}^n \times \mathbb{R}, j \in J$ are uncertain vectors which belong to the uncertainty set U_j^α.

Following the robust optimization methodology in [6], the robust counterpart of the original problem (UP) can be given by

$$
\begin{aligned}
(\mathrm{RP}_\alpha) \qquad & \min_{x\in\mathbb{R}^n} \ \bar{f}(x) \\
& \text{s.t. } \bar{g}_j(x) + a_j^\top x - b_j \le 0, \forall (a_j, b_j) \in U_j^\alpha, j \in J,
\end{aligned}
$$

where the uncertain constraints are enforced for every possible value of the parameters (a_j, b_j), $j \in J$ within the uncertainty set U_j^α. Let $F^\alpha := \{x \in \mathbb{R}^n : \bar{g}_j(x) + a_j^\top x - b_j \le 0, \forall (a_j, b_j) \in U_j^\alpha, j \in J\}$ denote the feasible set of (RP_α). Then, the feasible solution set F is said to be the robust feasible set of (UP).

Throughout this paper, unless otherwise specified, assume that the uncertainty data $(a_j, b_j) \in U_j^\alpha$ with

$$U_j^\alpha := (\bar{a}_j, \bar{b}_j) + \alpha Z, j \in J,$$

where $(\bar{a}_j, \bar{b}_j) \in \mathbb{R}^{n+1}$, $j \in J$ are fixed and Z is a convex and compact set with $0_{n+1} \in \operatorname{int} Z$ and $\alpha \ge 0$. We also suppose that the nominal problem (RP_0) is feasible, i.e., $\{x \in \mathbb{R}^n : \bar{g}_j(x) + \bar{a}_j^\top x - \bar{b}_j \le 0, j \in J\}$ is nonempty.

We recall the following notion of the epigraphical set of constraint system in (RP_0).

Definition 2.2. The epigraphical set of the constraint system of the nominal problem (RP_0), $E(\bar{g}, \bar{a}, \bar{b})$, associated to $\bar{g} := (\bar{g}_1, \cdots, \bar{g}_q)$, $\bar{a} := (\bar{a}_1, \cdots, \bar{a}_q)$ and $\bar{b} := (\bar{b}_1, \cdots, \bar{b}_q)$, is defined to be

$$E(\bar{g}, \bar{a}, \bar{b}) := \operatorname{conv}\left(\bigcup_{j\in J}\{\operatorname{epi} \bar{g}_j^* + (\bar{a}_j, \bar{b}_j)\}\right).$$

As the constraint system, F^0, of the nominal problem (RP_0) is a convex system, it can be equivalently rewritten as an infinite linear system. Therefore, the notion of the epigraphical set was inspired by the concept of hypographical set for linear semi-infinite program of [14].

Following [10, 16–18], we define the concept of radius of the robust feasibility for problem (RP_α) as follows.

Definition 2.3. The radius of robust feasibility of problem (RP_α) is defined by

$$\rho = \sup\{\alpha \in \mathbb{R}_+ : (\mathrm{RP}_\alpha) \ \text{ is feasible}\}.$$

To discuss the radius of robust feasibility, we first recall the dual characterization of the solutions of a semi-infinite convex system.

Lemma 2.4. [20, Theorem 3.1] *Let $h_t \in \Gamma(\mathbb{R}^n)$ for all $t \in T$ (an arbitrary index set). Then, $\{x \in \mathbb{R}^n : h_t(x) \le 0, t \in T\} \ne \emptyset$ if and only if $(0_n, -1) \notin \operatorname{cl}\operatorname{cone}(\bigcup_{t\in T} \operatorname{epi} h_t^*)$.*

Goberna et al. [17] first established the next lemma to present the bounds for the radius of robust feasibility.

Lemma 2.5. [17, Lemma 2.2] *Let $h_j : \mathbb{R}^n \to \mathbb{R}$, $j \in J$ be convex functions. Let $\alpha \ge 0$ and let $Z \subset \mathbb{R}^{n+1}$ be a convex and compact set with $0_{n+1} \in \operatorname{int} Z$. Assume that $(0_n, -1) \in \operatorname{cl}\operatorname{cone}(\bigcup_{j\in J} \operatorname{epi} h_j^* + \alpha Z)$. Then, for any $\delta > 0$, we have*

$$(0_n, -1) \in \operatorname{cl}\operatorname{cone}\left(\bigcup_{j\in J} \operatorname{epi} h_j^* + (\alpha + \delta) Z\right).$$

3. Main Results

The aim of this section is to study the radius of robust feasibility for an uncertain convex program with a compact and convex uncertainty set. First, we first give the formula for calculating the radius of robust feasibility.

Theorem 3.1. *Let the nominal problem* (RP_0) *be feasible.Then*

$$\rho = \inf_{(y,s)\in E(\bar{g},\bar{a},\bar{b})} \Phi_Z(-y,-s). \tag{1}$$

Proof. First, we show that

$$\rho \leq \inf_{(y,s)\in E(\bar{g},\bar{a},\bar{b})} \phi_Z(-y,-s). \tag{2}$$

Take any $(y,s) \in E(\bar{g},\bar{a},\bar{b})$. Then, by the definition of $E(\bar{g},\bar{a},\bar{b})$ and the Carathéodory Theorem, there exist $\lambda_l \geq 0$ with $\sum\limits_{l=1}^{n+2} \lambda_l = 1$, $j_l \in J$, $u_{j_l} \in \operatorname{dom}(\bar{g}_{j_l})^*$ and $r_{j_l} \geq 0$, $l = 1, \cdots, n+2$, such that

$$(y,s) = \sum_{l=1}^{n+2} \lambda_l \left(u_{j_l} + \bar{a}_{j_l}, \bar{g}^*_{j_l}(u_{j_l}) + r_{j_l} + \bar{b}_{j_l}\right). \tag{3}$$

Let $\varepsilon > 0$. Then, (3) and $\sum\limits_{l=1}^{n+2} \lambda_l = 1$ together implies that

$$(0_n, -\varepsilon) = \sum_{l=1}^{n+2} \lambda_l \left((u_{j_l} + \bar{a}_{j_l}, \bar{g}^*_{j_l}(u_{j_l}) + r_{j_l} + \bar{b}_{j_l}) + (-y, -s-\varepsilon)\right).$$

Dividing by ε on both sides, we have

$$(0_n, -1) = \sum_{l=1}^{n+2} \frac{\lambda_l}{\varepsilon} \left((u_{j_l} + \bar{a}_{j_l}, \bar{g}^*_{j_l}(u_{j_l}) + r_{j_l} + \bar{b}_{j_l}) + (-y, -s-\varepsilon)\right). \tag{4}$$

By the definition of ϕ_Z, for the above $\varepsilon > 0$, there exists $t_\varepsilon > 0$ such that

$$(-y, -s-\varepsilon) \in t_\varepsilon Z \text{ and } t_\varepsilon < \phi_Z(-y, -s-\varepsilon) + \varepsilon. \tag{5}$$

Now, let $\alpha \geq 0$ such that (RP_α) is feasible. Then, we have

$$\{x \in \mathbb{R}^n : \bar{g}_j(x) + a_j^\top x - b_j \leq 0, \forall (a_j, b_j) \in (\bar{a}_j, \bar{b}_j) + \alpha Z, j \in J\} \neq \emptyset. \tag{6}$$

We claim that

$$\alpha \leq \phi_Z(-y, -s-\varepsilon) + \varepsilon. \tag{7}$$

Noting that ϕ_Z is a continuous function, we take $\varepsilon \to 0$ in (7), and then can obtain $\alpha \leq \phi_Z(-y,-s)$. By the definition of radius ρ, we have $\rho \leq \phi_Z(-y,-s)$, and hence (2) holds.

Now, we turn our attention to the claim. Indeed, we assume by contradiction that $\alpha > \phi_Z(-y, -s-\varepsilon)$, which together with (5) implies that $\alpha > t_\varepsilon$. Moreover, from the first inclusion of (5), there exists $z \in Z$ such that $(-y, -s-\varepsilon) = t_\varepsilon z$. Since Z is convex and $0_{n+1} \in Z$, we have

$$(-y, -s-\varepsilon) \quad = \quad \alpha\left(\frac{t_\varepsilon}{\alpha} z + (1 - \frac{t_\varepsilon}{\alpha})0_{n+1}\right) \in \alpha Z. \tag{8}$$

So, (4) and (8) together yields that

$$\begin{aligned}(0_n, -1) \quad &\in \quad \operatorname{cone}\left(\bigcup_{j\in J} \{\operatorname{epi} \bar{g}^*_j + (\bar{a}_j, \bar{b}_j) + \alpha Z\}\right) \\ &\subset \quad \operatorname{cl}\operatorname{cone}\left(\bigcup_{(a_j,b_j)\in(\bar{a}_j,\bar{b}_j)+\alpha Z, j\in J} \{\operatorname{epi} \bar{g}^*_j + (a_j, b_j)\}\right).\end{aligned} \tag{9}$$

Let $h_j(x) := \bar{g}_j(x) + a_j^\top x - b_j$ for any $x \in \mathbb{R}^n$ and $j \in J$. Then, by the definition of epigraph, we have $\operatorname{epi} h_j^* = \operatorname{epi} \bar{g}_j^* + (a_j, b_j)$. Thus, (9) implies that

$$(0_n, -1) \quad \in \quad \operatorname{cl}\operatorname{cone}\left(\bigcup_{(a_j,b_j)\in(\bar{a}_j,\bar{b}_j)+\alpha Z, j\in J} \operatorname{epi} \bar{h}_j^* \right). \tag{10}$$

According to Lemma 2.4, (10) is equivalent to the following condition

$$\{x \in \mathbb{R}^n : h_j(x) = \bar{g}_j(x) + a_j^\top x - b_j \leq 0, \forall (a_j, b_j) \in (\bar{a}_j, \bar{b}_j) + \alpha Z, j \in J\} = \emptyset,$$

which contradicts (6).

Last, we verify that

$$\rho \geq \inf_{(y,s)\in E(\bar{g},\bar{a},\bar{b})} \phi_Z(-y, -s). \tag{11}$$

Assume to the contrary that

$$\rho < \inf_{(y,s)\in E(\bar{g},\bar{a},\bar{b})} \phi_Z(-y, -s). \tag{12}$$

For each $\varepsilon > 0$, let $\bar{\alpha} := \rho + \varepsilon$. By the definition of ρ, we have

$$\{x \in \mathbb{R}^n : \bar{g}_j(x) + a_j^\top x - b_j \leq 0, \forall (a_j, b_j) \in (\bar{a}_j, \bar{b}_j) + \bar{\alpha} Z, j \in J\} = \emptyset. \tag{13}$$

Thus, according to Lemma 2.4, (13) is equivalent to the following condition

$$\begin{aligned}(0_n, -1) \quad &\in \quad \operatorname{cl}\operatorname{cone}\left(\bigcup_{(a_j,b_j)\in(\bar{a}_j,\bar{b}_j)+\bar{\alpha} Z, j\in J} \{\operatorname{epi} \bar{g}_j^* + (a_j, b_j)\} \right) \\ &\subset \quad \operatorname{cl}\operatorname{cone}\left(\bigcup_{j\in J} \{\operatorname{epi} \bar{g}_j^* + (\bar{a}_j, \bar{b}_j) + \bar{\alpha} Z\} \right),\end{aligned}$$

which together with Lemma 2.5 yields that

$$(0_n, -1) \in \operatorname{cone}\left(\bigcup_{j\in J} \{\operatorname{epi} \bar{g}_j^* + (\bar{a}_j, \bar{b}_j) + (\bar{\alpha} + \varepsilon) Z\} \right). \tag{14}$$

Then, by the Carathédory Theorem, there exist $\lambda_k \geq 0$, $j_k \in J$, $u_{j_k} \in \operatorname{dom}(\bar{g}_{j_k})^*$, $r_{j_k} \geq 0$ and $z_{j_k} \in Z$, $k = 1, \cdots, n+2$, such that

$$(0_n, -1) = \sum_{k=1}^{n+2} \lambda_k \left((u_{j_k} + \bar{a}_{j_k}, \bar{g}_{j_k}^*(u_{j_k}) + r_{j_k} + \bar{b}_{j_k}) + (\bar{\alpha} + \varepsilon) z_{j_k} \right). \tag{15}$$

Clearly, (15) implies that $\sum_{k=1}^{n+2} \lambda_k \neq 0$. Letting $\bar{\lambda}_k := \frac{\lambda_k}{\sum_{k=1}^{n+2} \lambda_k}$, we have $\bar{\lambda}_k \geq 0$ with $\sum_{k=1}^{n+2} \bar{\lambda}_k = 1$. Then, dividing by $\sum_{k=1}^{n+2} \lambda_k$ on both sides of (15) and rearranging terms, we can get

$$-(\bar{\alpha} + \varepsilon) \sum_{k=1}^{n+2} \bar{\lambda}_k z_{j_k} = \sum_{k=1}^{n+2} \bar{\lambda}_k \left(u_{j_k} + \bar{a}_{j_k} + g_{j_k}^*(u_{j_k}) + r_{j_k} + \bar{b}_{j_k} \right) + \frac{1}{\sum_{k=1}^{n+2} \lambda_k} (0_n, 1). \tag{16}$$

Letting $\bar{z} := \sum_{k=1}^{n+2} \bar{\lambda}_k z_{j_k}$, we have $\bar{z} \in Z$ due to the convexity of Z. Moreover, since $(0_n, 1)$ is a recession of $E(\bar{g}, \bar{a}, \bar{b})$, it follows from (16) that $-(\bar{\alpha} + \varepsilon)\bar{z} \in E(\bar{g}, \bar{a}, \bar{b})$. Thus, by Lemma 2.1, we have

$$\inf_{(y,s)\in E(\bar{g},\bar{a},\bar{b})} \phi_Z(-y, -s) \le \phi_Z\left((\bar{\alpha} + \varepsilon)\bar{z}\right) \le \bar{\alpha} + \varepsilon = \rho + 2\varepsilon. \tag{17}$$

Letting $\varepsilon \to 0$ in (17), we can get $\inf_{(y,s)\in E(\bar{g},\bar{a},\bar{b})} \phi_Z(-y, -s) \le \rho$, which contradicts (12). So, (11) holds and hence, the conclusion (1) follows from (2) and(11). □

Whenever the assumption on the uncertainty set Z is strengthened to be symmetric and bounded, Theorem 3.1 allows us to state the following result.

Corollary 3.2. *Let the nominal problem (RP_0) be feasible, and let the uncertainty set Z be symmetric. Then*

$$\rho = \min_{(y,s)\in E(\bar{g},\bar{a},\bar{b})} \|(y, s)\|$$

Proof. Since the uncertain set Z is symmetric and bounded, Lemma 2.1 gives $\phi_Z(z) = \|z\|$ for all $z \in Z$. Then, (1) becomes

$$\rho = \inf_{(y,s)\in E(\bar{g},\bar{a},\bar{b})} \|(y, s)\|. \tag{18}$$

To verify (18) is attained, it suffices to show the epigraph set $E(\bar{g}, \bar{a}, \bar{b})$ is closed as the norm $\|\cdot\|$ is coercive on any closed set. Indeed, let $(y_k, s_k) \in E(\bar{g}, \bar{a}, \bar{b})$ with $(y_k, s_k) \to (y, s)$. We only need to show $(y, s) \in E(\bar{g}, \bar{a}, \bar{b})$. Indeed, by the Carathédory Theorem, there exists $j_l \in J, l = 1, \cdots, n+2, \lambda_k^{j_l} \ge 0$ with $\sum_{l=1}^{n+2} \lambda_k^{j_l} = 1, u_k^{j_l} \in \operatorname{dom}(\bar{g}^*_{j_l})$ and $\varepsilon_k^{j_l} \ge 0$, such that

$$(y_k, s_k) = \sum_{l=1}^{n+2} \lambda_k^{j_l} \left(u_k^{j_l} + \bar{a}^{j_l}, \bar{g}^*_{j_l}(u_k^{j_l}) + \varepsilon_k^{j_l} + \bar{b}^{j_l}\right).$$

Then, we get that

$$\begin{aligned} s_k &= \sum_{l=1}^{n+2} \lambda_k^{j_l} \left(\bar{g}^*_{j_l}(u_k^{j_l}) + \varepsilon_k^{j_l} + \bar{b}^{j_l}\right) \\ &\ge \sum_{l=1}^{n+2} \lambda_k^{j_l} \left(\bar{g}^*_{j_l}(u_k^{j_l}) + \bar{b}^{j_l}\right) \\ &\ge \sum_{l=1}^{n+2} \lambda_k^{j_l} \left(\langle y_k^{j_l} - \bar{a}^{j_l}, x\rangle - g_{j_l}(x) + \bar{b}^{j_l}\right), \forall x \in \mathbb{R}^n, \end{aligned} \tag{19}$$

where the last inequality follows from the definition of the conjugate function. Noting that $\lambda_k^{j_l} \ge 0$ and $\sum_{l=1}^{n+2} \lambda_k^{j_l} = 1$, without loss of generality, we can assume that $\lambda_k^{j_l} \to \lambda^{j_l}$ as $k \to \infty$. Clearly, $\sum_{l=1}^{n+2} \lambda^{j_l} = 1$. Letting $k \to \infty$ in (19), we have

$$s - \sum_{l=1}^{n+2} \lambda^{j_l} \bar{b}^{j_l} \ge \langle y - \sum_{l=1}^{n+2} \lambda^{j_l} \bar{a}^{j_l}, x\rangle - \sum_{l=1}^{n+2} \lambda^{j_l} g_{j_l}(x), \forall x \in \mathbb{R}^n. \tag{20}$$

Hence, (20) implies that $(y - \sum_{l=1}^{n+2} \lambda^{j_l} \bar{a}^{j_l}, s - \sum_{l=1}^{n+2} \lambda^{j_l} \bar{b}^{j_l}) \in \operatorname{epi}(\sum_{l=1}^{n+2} \lambda^{j_l} g_{j_l})^*$. As each $\bar{g}^{j_l}$ is a real-valued convex function, we have $\operatorname{epi}(\sum_{l=1}^{n+2} \lambda^{j_l} g_{j_l})^* = \sum_{l=1}^{n+2} \lambda^{j_l} \operatorname{epi} g^*_{j_l}$. Thus, $(y, s) \in \mathrm{E}(\bar{g}, \bar{a}, \bar{b})$ and the proof is complete. □

Remark 3.3. **(i)** Whenever the uncertainty set Z becomes the Euclidean unit closed ball $\mathbb{B}_{n+1}$, i.e., $Z = \mathbb{B}_{n+1}$, Goberna et al. [17] first have given the formula for the radius of robust feasibility of an uncertain convex program. Since the uncertainty set Z considered in this paper is a convex compact set, the ellipsoid and ball uncertainty sets are the special cases of the set Z. So, Theorem 3.1 extend the corresponding ones of [17].

(ii) If for any $j \in J$, the constraint function $\bar{g}_j$ is identical to zero , i.e., $\bar{g}_j(x) \equiv 0$, then corresponding epigraph set $E(\bar{g}, \bar{a}, \bar{b}) = \text{conv}\{(\bar{a}_j, \bar{b}_j) : j \in J\} + \mathbb{R}_+\{(0, 1)\}$. In this case, Chuong and Jeyakumar [18] have established an exact formula for radius of robust feasibility of uncertain linear program as: $\rho = \inf_{(y,s)\in -E(\bar{g},\bar{a},\bar{b})} \phi_Z(y, s)$. Hence, Theorem 1 and Corollary 3.2 extend the main results of [18] from linear program to convex case.

(iii) Whenever $Z = \mathbb{B}_{n+1}$ and $\bar{g}_j(x) \equiv 0$ for all $j \in J$, Goberna et al. [10, 16] have given the radius of robust feasibility of uncertain linear (semi-infinite) program: $\rho = d\left(0_n, E(\bar{g}, \bar{a}, \bar{b})\right)$. Thus, Theorem 3.1 also extends the corresponding ones of [10, 16].

In order to make reader to understand Remark 3.3, we give the following example.

Example 3.4. Let the uncertainty set Z be defined by

$$Z = \{(a, b) \in \mathbb{R}^2 : \frac{a^2}{2} + \frac{b^2}{4} \leq 1\}.$$

Let $\bar{f} : \mathbb{R} \to \mathbb{R}$ be a given convex function, $\bar{g}_j : \mathbb{R} \to \mathbb{R}$, $j = 1, 2$ be given by $\bar{g}_1(x) = x^2 - 1$ and $\bar{g}_2(x) = 2x^4 - 1$, and $\bar{a}_j = \bar{b}_j \equiv 0$, $j = 1, 2$. We can easily see that: (i) Z is an ellipsoidal ball and $\phi_Z = \|\cdot\|$; (ii) $\bar{g}_1$ and $\bar{g}_2$ are convex functions. We now consider the following robust convex program with ellipsoidal uncertainty:

$$\begin{aligned}
\text{(UP)} \qquad & \min_{x\in\mathbb{R}} \bar{f}(x) \\
& \text{s.t. } (x^2 - 1) + a_1 x - b_1 \leq 0, \ \forall (a_1, b_1) \in \alpha Z, \\
& \qquad (2x^4 - 1) + a_2 x - b_2 \leq 0, \ \forall (a_2, b_2) \in \alpha Z.
\end{aligned}$$

Then, from direct calculation it follows that $\bar{g}_1^*(y) = \frac{y^2}{4} + 1$ and $\bar{g}_2^*(y) = \frac{3y^{\frac{4}{3}}}{8} + 1$. So, $\text{epi}\, \bar{g}_1^* = \text{epi}\, \bar{g}_2^* = \mathbb{R}\times[1, +\infty)$, and hence $E(\bar{g}, \bar{a}, \bar{b}) = \text{conv}\left\{\text{epi}\, \bar{g}_1^* \bigcup \text{epi}\, \bar{g}_2^*\right\} = \mathbb{R} \times [1, +\infty)$. Applying Corollary 3.2, we get that

$$\rho = \min_{(y,s)\in E(\bar{g},\bar{a},\bar{b})} \|(y, s)\| = \min_{(y,s)\in\mathbb{R}\times[1,\infty)} \sqrt{y^2 + s^2} = 1.$$

However, we can not employ the main results of [10, 16–18] to establish the estimates for the radius of robust feasibility of problem (UP). The main reasons are the uncertainty set Z is not ball and the constraint functions g_j for all $j \in J$ are not linear. Thus, Theorem 1 and Corollary 3.2 extend the corresponding ones of [10, 16–18].

Now, we establish the necessary and sufficient condition for the radius of robust feasibility for (UP) to be positive.

Theorem 3.5. *Let the nominal problem (RP_0) be feasible. Consider the radius of robust feasibility for (RP_α) defined as in Definition 2.3. Then, it holds that*

$$\sup_{(x,\delta)\in\mathbb{R}^n\times\mathbb{R}} \{\delta : \bar{g}_j(x) + \bar{a}_j^\top x - \bar{b}_j^\top + \delta \leq 0, \forall j \in J\} > 0 \Longleftrightarrow \rho > 0.$$

Proof. $[\Longrightarrow]$ Assume that $\sup_{(x,\delta)\in\mathbb{R}^n\times\mathbb{R}} \{\delta : \bar{g}_j(x) + \bar{a}_j^\top x - \bar{b}_j^\top + \delta \leq 0, \forall j \in J\} > 0$. Let $\bar{h}_j(x) := \bar{g}_j(x) + \bar{a}_j^\top x - \bar{b}_j^\top$ for any $x \in \mathbb{R}^n$ and $j \in J$. Then, there exist $x_0 \in \mathbb{R}^n$ and $\delta_0 > 0$ such that

$$\bar{h}_j(x_0) + \delta_0 \leq 0, \forall j \in J. \tag{21}$$

To show $\rho > 0$, we proceed by the method of contradiction and suppose that $\rho = 0$. Then, for any $k \in \mathbb{N}$, there exists $(y_k, s_k) \in E(\bar{g}, \bar{a}, \bar{b})$ such that

$$\phi_Z(-y_k, -s_k) < \frac{1}{k}.$$

By the definition of ϕ_Z, there exist $t_k > 0$ such that

$$(-y_k, -s_k) \in t_k Z \text{ and } t_k < \phi_Z(-y_k, -s_k) \leq \frac{1}{k}.$$

Hence, there exists $z_k \in Z$ such that $(-y_k, -s_k) = t_k z_k$. Then, we have

$$(-y_k, -s_k) = \frac{1}{k}(kt_k z_k + (1 - kt_k)0_{n+1}) \in \frac{1}{k}Z,$$

because Z is convex and $0_{n+1} \in Z$. Thus, there exists $\hat{z}_k \in Z$ such that

$$(-y_k, -s_k) = \frac{1}{k}\hat{z}_k. \tag{22}$$

Noting that Z is compact, without loss of generality, we can assume that $\hat{z}_k \to \hat{z}$. So, letting $k \to \infty$ in (22), we have $(-y_k, -s_k) \to 0_{n+1}$, and hence $(y_k, s_k) \to 0_{n+1}$.

From the proof of Theorem 3.1, we observe that $E(\bar{g}, \bar{a}, \bar{b})$ is closed. Granting this, $(y_k, s_k) \in E(\bar{g}, \bar{a}, \bar{b})$ and $(y_k, s_k) \to 0_{n+1}$ imply that $0_{n+1} \in E(\bar{g}, \bar{a}, \bar{b})$. Then, by the definition of $E(\bar{g}, \bar{a}, \bar{b})$ and the Carathédory Theorem, there exist $\lambda_l \geq 0$ with $\sum_{l=1}^{n+2} \lambda_l = 1$, $j_l \in J$, $u_{j_l} \in \operatorname{dom}(\bar{g}_{j_l})^*$ and $\varepsilon_{j_l} \geq 0, l = 1, \cdots, n+2$, such that

$$0_{n+1} = \sum_{l=1}^{n+2} \lambda_l \left(u_{j_l} + \bar{a}_{j_l}, \bar{g}^*_{j_l}(u_{j_l}) + \varepsilon_{j_l} + \bar{b}_{j_l}\right).$$

Noting that $\sum_{l=1}^{n+2} \lambda_l = 1$ and $\delta_0 > 0$, we get that

$$(0_n, -\delta_0) = \sum_{l=1}^{n+2} \lambda_l \left(u_{j_l} + \bar{a}_{j_l}, \bar{g}^*_{j_l}(u_{j_l}) + \varepsilon_{j_l} + \bar{b}_{j_l} - \delta_0\right).$$

Then, we have that

$$\begin{aligned}
(0_n, -1) &= \sum_{l=1}^{n+2} \frac{\lambda_l}{\delta_0} \left((u_{j_l}, \bar{g}^*_{j_l}(u_{j_l}) + \varepsilon_{j_l}) + (\bar{a}_{j_l}, \bar{b}_{j_l} - \delta_0)\right) \\
&\in \operatorname{cone}\left(\bigcup_{j \in J} \{\operatorname{epi} \bar{g}^*_j + (\bar{a}_j, \bar{b}_j) + (0_n, -\delta_0)\}\right) \\
&= \operatorname{cone}\left(\bigcup_{j \in J} \{\operatorname{epi} \bar{h}^*_j + (0_n, -\delta_0)\}\right) \\
&= \operatorname{cone}\left(\bigcup_{j \in J} \{\operatorname{epi} (\bar{h}_j + \delta_0)^*\}\right).
\end{aligned}$$

Thus, it together with Lemma 2.4 implies that

$$\{x \in \mathbb{R}^n : \bar{h}_j(x) + \delta_0 \leq 0, j \in J\} = \emptyset,$$

which contradicts (21).

[$\Longleftarrow$] Let $\rho > 0$ and set $X(\delta) := \{x \in \mathbb{R}^n : \bar{h}_j(x) + \delta \leq 0, \forall j \in J\}$, where $\delta > 0$ and $\bar{h}_j(x) = \bar{g}_j(x) + \bar{a}_j^\top x - \bar{b}_j$, $j \in J$. To finish the proof, it suffices to show that there exists $\hat{\delta} > 0$ such that $X(\hat{\delta}) \neq \emptyset$.

Arguing by contradiction, we suppose that $X(\delta) = \emptyset$ for all $\delta > 0$. Then, Lemma 2.4 implies that

$$\begin{aligned}
(0_n, -1) &\in \operatorname{cl}\operatorname{cone}\left(\bigcup_{i\in J} \operatorname{epi}(\bar{h}_j + \delta)^*\right) \\
&= \operatorname{cl}\operatorname{cone}\left(\bigcup_{j\in J} \{\operatorname{epi}\bar{g}_j^* + (\bar{a}_j, \bar{b}_j) + (0, -\delta)\}\right) \\
&\subset \operatorname{cl}\operatorname{cone}\left(\bigcup_{j\in J} \{\operatorname{epi}\bar{g}_j^* + (\bar{a}_j, \bar{b}_j) + \delta\mathbb{B}_{n+1}\}\right).
\end{aligned}$$

Applying Lemma 2.5 to $\bar{g}_j$, $j \in J$, for any $\varepsilon > 0$, we have

$$(0_n, -1) \in \operatorname{cone}\left(\bigcup_{j\in J} \{\operatorname{epi}\bar{g}_j^* + (\bar{a}_j, \bar{b}_j) + (\delta + \varepsilon)\mathbb{B}_{n+1}\}\right).$$

Then, by the Carathédory Theorem, there exists $\lambda_l \geq 0$, $j_l \in J$, $u_{j_l} \in \operatorname{dom}(\bar{g}_{j_l})^*$, $r_{j_l} \geq 0$, $(w_{j_l}, t_{j_l}) \in \mathbb{B}_{n+1}$, $l = 1, \cdots, n+2$ such that

$$(0_n, -1) = \sum_{l=1}^{n+2} \lambda_l \left((u_{j_l} + \bar{a}_{j_l}, \bar{g}_{j_l}^*(u_{j_l}) + r_{j_l} + \bar{b}_{j_l}) + (\delta + \varepsilon)(w_{j_l}, t_{j_l}) \right). \tag{23}$$

Clearly, $\sum_{l=1}^{n+2} \lambda_l \neq 0$. Letting $\tilde{\lambda}_l := \frac{\lambda_l}{\sum_{l=1}^{n+2} \lambda_l} \geq 0, l = 1, \cdots, n+2$, we get that $\sum_{l=1}^{n+2} \tilde{\lambda}_l = 1$. Dividing by $\sum_{l=1}^{n+2} \lambda_l \neq 0$ on both sides of (23) and rearranging terms, it follows that

$$(\delta + \varepsilon) \sum_{l=1}^{n+2} \tilde{\lambda}_l(-w_{j_l}, -t_{j_l}) = \sum_{l=1}^{n+2} \tilde{\lambda}_l(u_{j_l} + \bar{a}_{j_l}, \bar{g}_{j_l}^*(u_{j_l}) + r_{j_l} + \bar{b}_{j_l}) + \frac{1}{\sum_{l=1}^{n+2} \lambda_l}(0_n, 1).$$

Let $(w^*, t^*) := \sum_{l=1}^{n+2} \tilde{\lambda}_l(-w_{j_l}, -t_{j_l}) \in \mathbb{B}_{n+1}$. Then, it together with the fact: $(0_n, 1)$ is a recession direction of $E(\bar{g}, \bar{a}, \bar{b})$, implies that

$$(\delta + \varepsilon)(-w^*, -t^*) \in E(\bar{g}, \bar{a}, \bar{b}). \tag{24}$$

Keeping in mind the arbitrariness of δ and ε, we take $\delta \to 0$ and $\varepsilon \to 0$ in (24), and then get that $0_{n+1} \in E(\bar{g}, \bar{a}, \bar{b})$ as $E(\bar{g}, \bar{a}, \bar{b})$ is closed set. Therefore, we have that

$$\rho = \inf_{(y,s)\in E(\bar{g},\bar{a},\bar{b})} \phi_Z(y, s) \leq \phi_Z(0_{n+1}) = 0, \tag{25}$$

where the last equality holds as inasmuch $0_{n+1} \in Z$. Therefore, (25) contradicts our assumption $\rho > 0$ and hence, the proof is complete. □

Remark 3.6. In the special case of $\bar{g}_j(x) \equiv 0$ for all $j \in J$, the proof of Theorem 3.5 leads to Proposition 2.1 [18], which presents the necessary and sufficient condition for the radius of robust feasibility for uncertain linear program with a convex and compact uncertainty. So, Theorem 3.5 extends Proposition 2.1 [18] from the linear program to convex case.

From the proof of Theorem 3.5, we can immediately obtain the following conclusion.

Corollary 3.7. *Let the nominal problem* (PU_0) *be feasible. Then, it holds that*

$$0_{n+1} \in E(\bar{g}) \Longleftrightarrow \rho = 0.$$

The following example is given to illustrate Corollary 3.7.

Example 3.8. Let Z and $\bar{f}$ be given as in Example 3.4. Let $\bar{g}_j : \mathbb{R} \to \mathbb{R}, j = 1,2$ be given by $\bar{g}_1(x) = x^2$ and $\bar{g}_2(x) = 2x^4$. Let $\bar{a}_j = \bar{b}_j \equiv 0, j = 1,2.$. Consider the following robust convex program with ellipsoidal uncertainty:

$$\begin{aligned}
(\text{UP}) \qquad & \min_{x\in\mathbb{R}} \bar{f}(x) \\
& \text{s.t. } x^2 + a_1x - b_1 \le 0, \ \forall (a_1, b_1) \in \alpha Z, \\
& \qquad 2x^4 + a_2x - b_2 \le 0, \ \forall (a_2, b_2) \in \alpha Z.
\end{aligned}$$

Then, from direct calculation it follows that $\bar{g}_1^*(y) = \frac{y^2}{4}$ and $\bar{g}_2^*(y) = \frac{3y^{\frac{4}{3}}}{8}$. So, $0_2 \in E(\bar{g}, \bar{a}, \bar{b}) = \text{conv}\left\{\text{epi}\,\bar{g}_1^* \bigcup \text{epi}\,\bar{g}_2^*\right\} = \mathbb{R} \times \mathbb{R}_+$ and $\rho = \min_{(y,s)\in\mathbb{R}\times[0,\infty)} \sqrt{y^2 + s^2} = 0$. Thus, Corollary 3.7 is applicable.

Acknowledgments

The authors express their deep gratitude to the anonymous referees and the associate editor for their valuable comments and suggestions that helped to improve this article. Xiao-Bing Li would like to thank Professor Jen-Chih Yao for his hospitality during his stay from July 2017 to August 2017 at the Center for Fundamental Science of Kaohsiung Medical University in Taiwan.

References

[1] A. L. Soyster, Convex Programming with Set-Inclusive Constraints and Applications to Inexact Linear Programming. Oper. Res., 21(1973):1154-1157.
[2] A. Ben-Tal and A. Nemirovski, Robust convex optimization. Math. Oper. Res., 23(1998):769-805.
[3] A. Ben-Tal and A. Nemirovski, Robust solutions to uncertain programs. Oper. Res. Lett., 25(1999):1-13.
[4] L. El-Ghaoui and H. Lebret, Robust solutions to least-square problems to uncertain data matrices. SIAM J. Matrix Anal. Appl., 18(1997):1035-1064.
[5] L. El-Ghaoui, F. Oustry and H. Lebret, Robust solutions to uncertain semidefinite programs. SIAM J. Optim., 9(1998):33-52.
[6] A. Ben-Tal, L. El-Ghaoui and A. Nemirovski, Robust Optimization, Princeton Series in Applied Mathemathics, 2009.
[7] A. Beck and A. Ben-Tal, Duality in robust optimization: primal worst equals dual best. Oper. Res. Lett. 37(2009):1-6.
[8] M. A. Goberna, V. Jeyakumar, G. Li and M. A. López, Robust linear semi-infinite programming duality under uncertainty. Mathematical programming, series B, 139(2013):185-203.
[9] X. K. Sun and Y. Cai, On robust duality for fractional programming with uncertainty data. Positivity, 18(2014):9-28.
[10] M. A. Goberna, V. Jeyakumar, G. Li and J. Vicente-Perez, Robust solutions of multi-objective linear semi-infinite programs under constraint data uncertainty. SIAM J. Optim. 24(2014):1402-1419.
[11] D. H. Fang, C. Li, J. C. Yao, Stable lagrage dualities for robust conical programming. J. Nonlinear Convex A., 16(2015):2141-2158.
[12] T. D. Chuong, Optimality and duality for robust multiobjecitve optimization problems. Nonlinear Anal. 134(2016):127-143,.
[13] M. J. Cánovas, M. A. López, J. Parra and M. I. Todorov, Stability and well-posedness in linear semiinfinite programming. SIAM J. Optim. 10(1999):82-98.
[14] M. J. Cánovas, M. A. López, J. Parra and F. J. Toledo, Distance to ill-posedness and the consistency value of linear semi-infinite inequality systems. Math. Program. 103A (2005):95-126.
[15] M. J. Cánovas, M. A. López, J. Parra and F. J. Toledo, Distance to ill-posedness for linear inequality systems under block perturbations: convex and infinite-dimensional cases. Optimization 60(2011):925-946.
[16] M. A. Goberna, V. Jeyakumar, G. Li and J. Vicente-Perez, Robust solutions to multiobjective linear programs with uncertain data. Eur. J. Oper. Res. 242(2015):730-743.
[17] M. A. Goberna, V. Jeyakumar, G. Li, and N. Linh, Radius of robust feasibility formulas for classes of convex programs with uncertain polynomial constraints. Oper. Res. Lett. 44(2016):67-73.
[18] T. D. Chuong and V.Jeyakumar, An exact fomula for radius of robust feasiblity of uncertain linear programs. J. Optim. Thoery Appl., 173(2017):203-226.
[19] W. Shirotzek, Nonsmooth Analysis, Universitext Pringer, Berlin, 2007.
[20] N. Dinh, M. A. Goberna and M. A. López, From linear to convex systems: Consistency, Farkas' Lemma and applications. J. Convex Anal. 13(2006): 279-290.