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Laguerre-Freud Equations Associated with the H_q-Semiclassical Forms of Class One

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Abstract. We give the system of Laguerre-Freud equations for the recurrence coefficients β_n, $\gamma_{n+1}, n \geq 0$ of orthogonal polynomials with respect to a H_q-semiclassical form (linear functional) of class one. The system is solved in the case when $\beta_n = t_{n-1} - t_n$ and $\gamma_{n+1} = -t_n^2$ with $t_n \neq 0, n \geq 0$ and $t_{-1} = 0$. There are essentially three canonical cases.

1. Introduction and preliminary results

Let L to be a lowering operator, that is, a linear operator that decreases in one unit the degree of a polynomial and such that $L(1) = 0$. Among such lowering operators, we mention the derivative operator D, the difference operator D_w and the Hahn operator H_q. The concept of L-semiclassical orthogonal polynomials $\{S_n\}_{n\geq 0}$ of class $s \geq 0$ were extensively studied by Maroni and coworkers for $L \in \{D, D_w, H_q\}$ through the following distributional equation satisfied by the regular form v (linear functional) associated with a such sequence:

$$L(\Phi v) + \Psi v = 0 \,,$$

where Φ is a monic polynomial and Ψ a polynomial with $\deg \Psi \geq 1$. For $L \in \{D, D_w, H_q\}$, L-semiclassical of class zero are usually called L-classical and are completely described in [1, 8, 13]. For $L \in \{D, D_w\}$, the system satisfied by the coefficients of the recurrence relation of L-semiclassical orthogonal sequences of class one are established in [2, 10]. So, the aim of this paper is twofold. First, to establish the Laguerre-Freud equations corresponding to L-semiclassical orthogonal sequences of class one in the general case when $L = H_q$. Secondly, to solve the system in a special nonsymmetrical case (since the symmetric case is treated in [5]). Indeed, we exhaustively describe the family of H_q-semiclassical sequences $\{S_n\}_{n\geq 0}$ of class $s = 1$, verifying the following three-term recurrence relation:

$$\begin{aligned} &S_{n+2}(x) = \Big(x - (t_n - t_{n+1})\Big) S_{n+1}(x) + t_n^2 S_n(x) \,, \quad n \geq 0 \,, \\ &S_1(x) = x + t_0 \,, \qquad S_0(x) = 1 \,, \end{aligned}$$

with $t_n \neq 0$, $n \geq 0$. This family have been the subject of some works: for instance, Maroni [14, 15] characterized such sequences by a particular quadratic decomposition and by a perturbation of a symmetric

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form. see also [7, 11].

The structure of the manuscript is as follows. The first section is devoted to the preliminary results and notations used in the sequel. In the second section, the system of Laguerre-Freud equations is built. In the third section, first we give some properties of the sequence $\{S_n\}_{n\geq 0}$. Specially, we focus our attention on the case when it is H_q-semiclassical of class one. Second, using these proprieties and the system, we obtain all the sequences which we look for. Finally, we show that there are essentially three canonical cases.

Let $\mathcal{P}$ be the vector space of polynomials with complex coefficients and let $\mathcal{P}'$ be its dual. The elements of $\mathcal{P}'$ will be called either form or linear functional. We denote by $\langle v, f\rangle$ the action of $v \in \mathcal{P}'$ on $f \in \mathcal{P}$. For $n \geq 0$, $(v)_n = \langle v, x^n\rangle$ are the moments of v. In particular a form v is called symmetric if all of its moments of odd order are zero [3].

We define in the space $\mathcal{P}'$ the derivative v' of the form v by $\langle v', f\rangle := -\langle v, f'\rangle$, the left multiplication by a polynomial hv by $\langle hv, f\rangle := \langle v, hf\rangle$, the shifted form $h_a v$, by $\langle h_a v, f\rangle := \langle v, h_a f\rangle = \langle v, f(ax)\rangle$, the Dirac form at origin δ_0 by $\langle \delta_0, f\rangle := f(0)$ and the inverse multiplication by a polynomial of degree one $(x-c)^{-1}$, through

$$\Big\langle (x-c)^{-1}v, f\Big\rangle := \langle v, \theta_c f\rangle \text{ with } \Big(\theta_c f\Big)(x) := \frac{f(x)-f(c)}{x-c},\ f \in \mathcal{P},\ c \in \mathbb{C}.$$

.

Let us recall that a form v is said to be regular (quasi-definite) if there exists a sequence $\{S_n\}_{n\geq 0}$ of polynomials with $\deg S_n = n,\ n \geq 0$, such that

$$\langle v, S_n S_m\rangle = r_n \delta_{n,m} \quad , \quad r_n \neq 0, \quad n \geq 0\ . \tag{1}$$

We can always assume that each S_n is monic, i.e. $S_n(x) = x^n+$ lower degree terms. Then the sequence $\{S_n\}_{n\geq 0}$ is said to be orthogonal with respect to v (monic orthogonal polynomial sequence (MOPS) in short). It is a very well-known fact that the sequence $\{S_n\}_{n\geq 0}$ satisfies a three-term recurrence relation (see, for instance, the monograph by Chihara [3])

$$\begin{aligned} &S_{n+2}(x) = (x-\beta_{n+1})S_{n+1}(x) - \gamma_{n+1}S_n(x)\ , \quad n \geq 0\ ,\\ &S_1(x) = x-\beta_0\ , \qquad S_0(x) = 1\ , \end{aligned} \tag{2}$$

with $\Big(\beta_n, \gamma_{n+1}\Big) \in \mathbb{C} \times (\mathbb{C} - \{0\})\ , \quad n \geq 0$. By convention we set $\gamma_0 = (v)_0$.

The form v is said to be normalized if $(v)_0 = 1$. In this paper, we suppose that any form will be normalized.

Let us introduce the $q-$derivative operator [6]

$$(H_q f)(x) = \frac{f(qx)-f(x)}{(q-1)x}\ ,\ x \neq 0\ ,\ \Big(H_q f\Big)(0) = f'(0)\ ,\ f \in \mathcal{P}\ ,\ \ q \in \tilde{\mathbb{C}}\ ,$$

where $\tilde{\mathbb{C}} := \mathbb{C} - \Big(\{0\} \bigcup \Big(\bigcup_{n\geq 0}\{z \in \mathbb{C}\ ,\ \ z^n = 1\}\Big)\Big)$. When $q \longrightarrow 1$, we meet again the derivative D.

By duality, we can define H_q from $\mathcal{P}'$ to $\mathcal{P}'$ such that

$$\Big\langle H_q v, f\Big\rangle = -\Big\langle v, H_q f\Big\rangle\ , f \in \mathcal{P}\ , v \in \mathcal{P}'\ .$$

In particular, this yields $(H_q v)_n = -[n]_q (v)_{n-1}\ , n \geq 0$ with $(v)_{-1} = 0$ and $[n]_q := \frac{q^n-1}{q-1}\ ,\ \ n \geq 0$.

For $v \in \mathcal{P}'$ and $f,\ g \in \mathcal{P}$, we have the following results [5, 8, 9]

$$H_q(fv) = (h_{q^{-1}}f)H_q v + q^{-1}(H_{q^{-1}}f)v\ , \tag{3}$$

$$H_q(fg)(x) = (h_q f)(x)(H_q g)(x) + g(x)(H_q f)(x)\ , \tag{4}$$

$$H_q o h_{q^{-1}} = q^{-1}H_{q^{-1}}\ ,\ in\ \mathcal{P}\ . \tag{5}$$

Now, let us recall some features about the H_q-semiclassical character [5, 9].

Definition 1.1. *A form v is said to be H_q-semiclassical when it is regular and there exist two polynomials Φ (monic) and Ψ, $\deg(\Phi) = t \geq 0$, $\deg(\Psi) = p \geq 1$, such that*

$$H_q\Big(\Phi v\Big) + \Psi v = 0 \, . \tag{6}$$

The corresponding MOPS $\{S_n\}_{n\geq 0}$ is said to be H_q-semiclassical.

Proposition 1.2. *The H_q-semiclassical form v satisfying* (6) *is said to be of class $s = \max(t-2, p-1)$ if and only if the following condition is satisfied*

$$\prod_{c\in\mathcal{Z}_\Phi} \Big\{ \Big| qh_q\Psi(c) + (H_q\Phi)(c) \Big| + \Big| \langle v, (\theta_{cq} o \theta_c)\Phi + q\theta_{cq}\Psi \rangle \Big| \Big\} \neq 0 \, , \tag{7}$$

where $\mathcal{Z}_\Phi$ is the set of roots of Φ.

The H_q-semiclassical character of a form is kept by shifting. Indeed, the shifted form $\tilde{v} = h_{a^{-1}}v$, $a \in \mathbb{C} - \{0\}$ is also H_q-semiclassical having the same class as that v and fulfilling the equation

$$H_q\Big(\tilde{\Phi}\tilde{v}\Big) + \tilde{\Psi}\tilde{v} = 0 \, ,$$

where

$$\tilde{\Phi}(x) = a^{-t}\Phi(ax) \, , \; \tilde{\Psi}(x) = a^{1-t}\Psi(ax) \, .$$

The sequence $\{\tilde{S}_n\}_{n\geq 0}$, where $\tilde{S}_n(x) = a^{-n}S_n(ax)$, $n \geq 0$ is orthogonal with respect to $\tilde{v}$. The recurrence coefficients are given by

$$\tilde{\beta}_n = \frac{\beta_n}{a} \, , \quad \tilde{\gamma}_{n+1} = \frac{\gamma_{n+1}}{a^2} \, , \; n \geq 0 \, .$$

The next result [5] characterizes the elements of the functional equation satisfied by any symmetric H_q-semiclassical form.

Proposition 1.3. *Let v be a symmetric H_q-semiclassical form of class s satisfying* (6). *The following statements hold.*

(i) *When s is odd then Φ is odd and Ψ is even.*

(ii) *When s is even then Φ is even and Ψ is odd.*

2. The Laguerre-Freud equations

In this section we will establish the non-linear system satisfied by β_n and γ_n, simply by using the functional equation.

In the sequel we assume that $\{S_n\}_{n\geq 0}$ is a H_q-semiclassical sequence of class one verifying (2) and its corresponding form v satisfying (6) with

$$\Phi(x) = c_3x^3 + c_2x^2 + c_1x + c_0 \, , \quad \Psi(x) = a_2x^2 + a_1x + a_0 \, , \quad |c_3| + |a_2| \neq 0 \, . \tag{8}$$

Let us define for $n \geq 0$

$$\begin{aligned} &I_{n,k}(q) = \langle v, x^k S_n(x)S_n(q^{-1}x)\rangle \, , \; 0 \leq k \leq 2 \, , \\ &J_{n,k}(q) = \langle v, x^k S_n(x)S_{n+1}(q^{-1}x)\rangle \, , \; 0 \leq k \leq 2 \, , \\ &K_{n,k}(q) = \langle v, x^k H_q\Big(S_n(\xi)S_n(q^{-1}\xi)\Big)(x)\rangle \, , \; 0 \leq k \leq 3 \, , \\ &L_{n,k}(q) = \langle v, x^k H_q\Big(S_{n+1}(q^{-1}\xi)S_n(\xi)\Big)(x)\rangle \, , \; 0 \leq k \leq 3 \, . \end{aligned} \tag{9}$$

Lemma 2.1. *For $n \geq 0$, we have the following results:*

$$a_2I_{n,2}(q) + a_1I_{n,1}(q) + a_0I_{n,0}(q) - c_3K_{n,3}(q) - c_2K_{n,2}(q) - c_1K_{n,1}(q) - c_0K_{n,0}(q) = 0 \, , \tag{10}$$

$$a_2J_{n,2}(q) + a_1J_{n,1}(q) + a_0J_{n,0}(q) - c_3L_{n,3}(q) - c_2L_{n,2}(q) - c_1L_{n,1}(q) - c_0L_{n,0}(q) = 0 \, . \tag{11}$$

Proof. By (6), we get $\langle H_q(\Phi v) + \Psi v, S_n(x)S_n(q^{-1}x)\rangle = 0\ ,\ n \geq 0$ it is equivalent to

$$\langle \Psi v, S_n(x)S_n(q^{-1}x)\rangle - \langle \Phi v, H_q\Big(S_n(\xi)S_n(q^{-1}\xi)\Big)(x)\rangle = 0\ ,\ n \geq 0\ ,$$

then from (8) and (9), we can deduce (10).

We have $\langle H_q(\Phi v) + \Psi v, S_n(x)S_{n+1}(q^{-1}x)\rangle = 0\ ,\ n \geq 0$, then

$$\langle \Psi v, S_n(x)S_{n+1}(q^{-1}x)\rangle - \langle \Phi v, H_q\Big(S_n(\xi)S_{n+1}(q^{-1}\xi)\Big)(x)\rangle = 0\ ,\ n \geq 0\ ,$$

it follows (11). □

In order to determine $\{I_{n,k}(q)\}_{n\geq 0}$, $\{J_{n,k}(q)\}_{n\geq 0}$, $\{K_{n,k}(q)\}_{n\geq 0}$ and $\{L_{n,k}(q)\}_{n\geq 0}$, we need the following results:

Lemma 2.2. *We have the following formulas:*

$$\begin{aligned}
(v)_1 &= \beta_0\ ,\\
(v)_2 &= \gamma_1 + \beta_0^2\ ,\\
(v)_3 &= \beta_0^3 + (2\beta_0 + \beta_1)\gamma_1\ ,\\
(v)_4 &= \{\gamma_1 + \gamma_2 + 2\beta_0^2 + (\beta_0 + \beta_1)^2\}\gamma_1 + \beta_0^4\ .
\end{aligned}$$

Lemma 2.3. [10] *Let $\{x_n\}_{n\geq 0}$ with $x_n \neq 0\ ,\ n \geq 0$, $\{y_n\}_{n\geq 0}$ two sequences and $\{z_n\}_{n\geq 0}$ the sequence satisfying the recurrence relation:*

$$z_{n+1} = x_n z_n + y_n\ ,\ n \geq 0\ ,\ z_0 = a \in \mathbb{C} - \{0\}.$$

We have

$$z_{n+1} = \prod_{k=0}^{n} x_k\Big(a + \sum_{k=0}^{n}\Big(\prod_{\nu=0}^{k} x_\nu\Big)^{-1} y_k\Big),\ n \geq 0.$$

Lemma 2.4. *For $n \geq 0$ we have*

$$\langle v, x^{n+1}S_n(x)\rangle = \Big(\sum_{\nu=0}^{n}\beta_\nu\Big)\langle v, S_n^2\rangle\ , \tag{12}$$

$$\langle v, x^{n+2}S_n(x)\rangle = \Big(\sum_{\nu=0}^{n}\gamma_{\nu+1} + \sum_{\nu=0}^{n}\beta_\nu^2 + \sum_{\nu=0}^{n-1}\beta_\nu\sum_{k=\nu+1}^{n}\beta_k\Big)\langle v, S_n^2\rangle\ ,\ \sum_{0}^{-1} = 0\ , \tag{13}$$

$$\langle v, x^{n+3}S_n(x)\rangle = \Big(\sum_{\nu=0}^{n}\gamma_{\nu+1}\sum_{k=0}^{\nu+1}\beta_k + \sum_{\nu=0}^{n}\beta_\nu\sum_{k=0}^{\nu}\gamma_{k+1} + \sum_{\nu=0}^{n}\beta_\nu^3 + \sum_{\nu=0}^{n-1}\beta_\nu\sum_{k=\nu+1}^{n}\beta_k\sum_{i=\nu}^{k}\beta_i\Big)\langle v, S_n^2\rangle\ . \tag{14}$$

Proof. From the orthogonality of $\{S_n\}_{n\geq 0}$ and (2), we get respectively for $n \geq 0$
$\langle v, x^{n+2}S_{n+1}(x)\rangle = \gamma_{n+1}\langle v, x^{n+1}S_n(x)\rangle + \beta_{n+1}\langle v, S_{n+1}^2\rangle$,
$\langle v, x^{n+3}S_{n+1}(x)\rangle = \gamma_{n+1}\langle v, x^{n+2}S_n(x)\rangle + \beta_{n+1}\langle v, x^{n+2}S_{n+1}(x)\rangle + \langle v, S_{n+2}^2\rangle$,
and $\langle v, x^{n+4}S_{n+1}(x)\rangle = \gamma_{n+1}\langle v, x^{n+3}S_n(x)\rangle + \beta_{n+1}\langle v, x^{n+3}S_{n+1}(x)\rangle + \langle v, x^{n+3}S_{n+2}(x)\rangle$.
Thus, from the Lemma 2.3, we can deduce respectively (12), (13) and (14). □

Lemma 2.5. *We have*

$$\begin{aligned}
&S_{n+3}(x) = x^{n+3} + d_{n+2}x^{n+2} + e_{n+1}x^{n+1} + f_n x^n + \ldots\ ,\quad n \geq 0\ ,\\
&S_2(x) = x^2 + d_1 x + e_0\ ,\qquad S_1(x) = x + d_0\ ,
\end{aligned} \tag{15}$$

with for $n \geq 0$

$$d_n = -\sum_{\nu=0}^{n}\beta_\nu\ , \tag{16}$$

$$e_n = -\sum_{\nu=0}^{n} \gamma_{\nu+1} + \sum_{\nu=0}^{n} \beta_\nu \sum_{k=\nu+1}^{n+1} \beta_k \,, \tag{17}$$

$$f_n = \sum_{\nu=0}^{n} \gamma_{\nu+1} \sum_{k=\nu+2}^{n+2} \beta_k + \sum_{\nu=0}^{n} \beta_\nu \sum_{k=\nu+1}^{n+1} \gamma_{k+1} - \sum_{\nu=0}^{n} \beta_\nu \sum_{k=\nu+1}^{n+1} \beta_k \sum_{i=k+1}^{n+2} \beta_i \,. \tag{18}$$

Proof. The relations (16) and (17) are well known (see [3]).
According to the orthogonality of $\{S_n\}_{n\geq 0}$ with respect to v, by the relation (15) it follows that

$$f_n\langle v, x^n S_n(x)\rangle = -\langle v, x^{n+3} S_n(x)\rangle - d_{n+2}\langle v, x^{n+2} S_n(x)\rangle - e_{n+1}\langle v, x^{n+1} S_n(x)\rangle \,,\ n \geq 0 \,,$$

then using (12) − (14) and (16) − (17), we obtain (18). □

Lemma 2.6. *We have*

$$I_{n,0}(q) = q^{-n}\langle v, S_n^2\rangle \,,\ n \geq 0 \,, \tag{19}$$

$$I_{n,1}(q) = q^{-n}\Big\{\beta_n + (1-q)\sum_{\nu=0}^{n-1} \beta_\nu\Big\}\langle v, S_n^2\rangle \,,\ n \geq 0 \,, \tag{20}$$

$$I_{0,2}(q) = \gamma_1 + \beta_0^2 \,, \tag{21}$$

$$\begin{aligned} I_{n,2}(q) = q^{-n}\Big\{&\beta_n^2 + \gamma_n + \gamma_{n+1} + (1-q)\Big[(1+q)\sum_{\nu=0}^{n-2} \gamma_{\nu+1} + \beta_n \sum_{\nu=0}^{n-1} \beta_\nu \\ &+(1-q)\sum_{\nu=0}^{n-2} \beta_{\nu+1} \sum_{k=0}^{\nu} \beta_k + \sum_{\nu=0}^{n-1} \beta_\nu^2\Big]\Big\}\langle v, S_n^2\rangle \,,\ n \geq 1 \,, \end{aligned} \tag{22}$$

$$J_{n,0}(q) = q^{-n-1}(1-q)\Big(\sum_{\nu=0}^{n} \beta_\nu\Big)\langle v, S_n^2\rangle \,,\ n \geq 0 \,, \tag{23}$$

$$\begin{aligned} J_{n,1}(q) = q^{-n-1}\Big\{&\gamma_{n+1} + (1-q)\Big[(1+q)\sum_{\nu=0}^{n-1} \gamma_{\nu+1} + \sum_{\nu=0}^{n} \beta_\nu^2 \\ &+(1-q)\sum_{\nu=0}^{n-1} \beta_{\nu+1} \sum_{k=0}^{\nu} \beta_k\Big]\Big\}\langle v, S_n^2\rangle \,,\ n \geq 0 \,, \end{aligned} \tag{24}$$

$$\begin{aligned} J_{n,2}(q) = q^{-n-1}\Big\{&(\beta_n + \beta_{n+1})\gamma_{n+1} + (1-q)\Big[\sum_{k=0}^{n} \beta_k^3 + (1-q)\sum_{k=0}^{n-1} \beta_{k+1} \sum_{\nu=0}^{k} \beta_\nu^2 \\ &+(1-q)\sum_{k=0}^{n-1} \beta_{k+1}^2 \sum_{\nu=0}^{k} \beta_\nu + \sum_{k=0}^{n} \gamma_{k+1} \sum_{\nu=0}^{k} \beta_\nu + \sum_{k=0}^{n-1} \gamma_{k+1}\Big((2+q)\beta_{k+1} \\ &+(1+q)\beta_k\Big) + (1-q)^2 \sum_{k=0}^{n-1} \beta_{k+1} \sum_{\nu=0}^{k-1} \beta_{\nu+1} \sum_{i=0}^{\nu} \beta_i + (1-q^2)\sum_{k=0}^{n-1} \beta_{k+1} \sum_{\nu=0}^{k-1} \gamma_{\nu+1} \\ &-q^2 \sum_{k=0}^{n-1} \gamma_{k+1} \sum_{\nu=0}^{k-1} \beta_\nu\Big]\Big\}\langle v, S_n^2\rangle \,,\ n \geq 0 \,, \end{aligned} \tag{25}$$

$$K_{n,0}(q) = 0 \,,\ n \geq 0 \,, \tag{26}$$

$$K_{n,1}(q) = (q^{-n} + 1)[n]_q\langle v, S_n^2\rangle \,,\ n \geq 0 \,, \tag{27}$$

$$K_{n,2}(q) = \Big\{(1 + q^{-n})[n]_q\beta_n + (q^{n-1} + q^{-n})\sum_{k=0}^{n-1} \beta_k\Big\}\langle v, S_n^2\rangle \,,\ n \geq 0 \,, \tag{28}$$

$$K_{0,3}(q) = 0 \ , \tag{29}$$

$$\begin{aligned} K_{n,3}(q) = \Big\{&(q^{-n}+1)[n]_q(\gamma_n+\gamma_{n+1}) + (1+q)(q^{-n}+q^{n-2})\sum_{\nu=0}^{n-2}\gamma_{\nu+1} \\ &+q^{-n}[2n]_q\beta_n^2 + (q^{-n}+q^{n-1})\Big(\beta_n\sum_{\nu=0}^{n-1}\beta_\nu + \sum_{\nu=0}^{n-1}\beta_\nu^2\Big) \\ &+(q-1)(q^{n-2}-q^{-n})\sum_{\nu=0}^{n-2}\beta_\nu\sum_{k=\nu+1}^{n-1}\beta_k\Big\}\langle v, S_n^2\rangle \ , \ n \geq 1 \ , \end{aligned} \tag{30}$$

$$L_{n,0}(q) = q^{-n-1}[n+1]_q\langle v, S_n^2\rangle \ , \ n \geq 0 \ , \tag{31}$$

$$L_{n,1}(q) = q^{-n-1}\Big(\sum_{\nu=0}^{n}\beta_\nu\Big)\langle v, S_n^2\rangle \ , \ n \geq 0 \ , \tag{32}$$

$$L_{0,2}(q) = q^{-1}(\beta_0^2+\gamma_1) \ , \tag{33}$$

$$\begin{aligned} L_{n,2}(q) = q^{-n-1}\Big\{&[2n+1]_q\gamma_{n+1} + (1+q)\sum_{\nu=0}^{n-1}\gamma_{\nu+1} + \sum_{\nu=0}^{n}\beta_\nu^2 \\ &+(1-q)\sum_{\nu=0}^{n-1}\beta_{\nu+1}\sum_{k=0}^{\nu}\beta_k\Big\}\langle v, S_n^2\rangle \ , \ n \geq 1 \ , \end{aligned} \tag{34}$$

$$L_{0,3}(q) = q^{-1}\Big(\gamma_1(\beta_1+2\beta_0)+\beta_0^3\Big) \ , \tag{35}$$

$$\begin{aligned} L_{n,3}(q) = q^{-n-1}\Big\{&\gamma_{n+1}\Big([2n+1]_q\beta_{n+1} + [2n]_q\beta_n + (1+q^{2n})\sum_{\nu=0}^{n}\beta_\nu\Big) \\ &+(q+2)\sum_{\nu=0}^{n-1}\gamma_{\nu+1}(\beta_{\nu+1}+\beta_\nu) + \sum_{\nu=0}^{n}\beta_\nu^3 + (1-q^2)\Big(\sum_{\nu=0}^{n-2}\beta_\nu\sum_{k=\nu+1}^{n-1}\gamma_{k+1} \\ &+\sum_{\nu=0}^{n-2}\gamma_{\nu+1}\sum_{k=\nu+2}^{n}\beta_k\Big) + (1-q)^2\sum_{\nu=0}^{n-2}\beta_\nu\sum_{k=\nu+1}^{n-1}\beta_k\sum_{i=k+1}^{n}\beta_i \\ &+(1-q)\Big(\sum_{\nu=0}^{n-1}\beta_{\nu+1}^2\sum_{k=0}^{\nu}\beta_k + \sum_{\nu=0}^{n-1}\beta_{\nu+1}\sum_{k=0}^{\nu}\beta_k^2\Big)\Big\}\langle v, S_n^2\rangle \ , \ n \geq 1 \ . \end{aligned} \tag{36}$$

Proof. From the orthogonality of $\{S_n\}_{n\geq 0}$, we can deduce (19). We have $I_{0,1}(q) = (v)_1$. Thus, from the Lemma 2.2, we get

$$I_{0,1}(q) = \beta_0 \ . \tag{37}$$

For $n \geq 0$, by (2) we have

$$I_{n+1,1}(q) = \langle v, \{S_{n+2}(x) + \beta_{n+1}S_{n+1}(x) + \gamma_{n+1}S_n(x)\}S_{n+1}(q^{-1}x)\rangle \ .$$

By the orthogonality of $\{S_n\}_{n\geq 0}$, we can deduce that

$$I_{n+1,1}(q) = q^{-n-1}\beta_{n+1}\langle v, S_{n+1}^2\rangle + \gamma_{n+1}J_{n,0}(q) \ , \ n \geq 0 \ . \tag{38}$$

On other hand from (2) and the orthogonality of $\{S_n\}_{n\geq 0}$, we have

$$J_{n,0}(q) = q^{-1}I_{n,1}(q) - q^{-n}\beta_n\langle v, S_n^2\rangle \ , \ n \geq 0 \ .$$

By virtue of the last equation, (38) can be written

$$I_{n+1,1}(q) = q^{-1}\gamma_{n+1}I_{n,1}(q) + q^{-n}(q^{-1}\beta_{n+1}-\beta_n)\langle v, S_{n+1}^2\rangle \ , \ n \geq 0 \ .$$

Consequently, from the Lemma 2.3 and (37), we can deduce (20).
We remark that by (20) and (38), we obtain (23).
We have $I_{0,2}(q) = (v)_2$, from Lemma 2.2, we obtain (21).
For $n \geq 0$, by (2), we can write

$$I_{n+1,2}(q) = \langle v, x\{S_{n+2}(x) + \beta_{n+1}S_{n+1}(x) + \gamma_{n+1}S_n(x)\}S_{n+1}(q^{-1}x)\rangle \ , \ n \geq 0 \ .$$

Taking the orthogonality of $\{S_n\}_{n\geq 0}$ into account, we obtain

$$I_{n+1,2}(q) = q^{-n-1}\langle v, S_{n+2}^2\rangle + \beta_{n+1}I_{n+1,1}(q) + \gamma_{n+1}J_{n,1}(q) \ , \ n \geq 0 \ . \tag{39}$$

We can write $J_{0,1}(q) = q^{-1}(v)_2 - \beta_0(v)_1$, by the Lemma 2.2, we obtain

$$J_{0,1}(q) = q^{-1}\{\gamma_1 + (1-q)\beta_0^2\} \ . \tag{40}$$

Making $n = 0$ in (39), by (20) and (40), it follows that

$$I_{1,2}(q) = q^{-1}\gamma_1\{\gamma_1 + \gamma_2 + \beta_1^2 + (1-q)\beta_0(\beta_0 + \beta_1)\} \ . \tag{41}$$

When $n \geq 1$, by (2) and the orthogonality of $\{S_n\}_{n\geq 0}$, we get

$$J_{n,1}(q) = q^{-1}I_{n,2}(q) - \beta_n I_{n,1}(q) - q^{1-n}\gamma_n\langle v, S_n^2\rangle \ .$$

From (20), we can deduce that

$$J_{n,1}(q) = q^{-1}I_{n,2}(q) - q^{-n}\{q\gamma_n + \beta_n^2 + (1-q)\beta_n\sum_{\nu=0}^{n-1}\beta_\nu\}\langle v, S_n^2\rangle \ , \ n \geq 1 \ . \tag{42}$$

By virtue of (42), (39) becomes

$$\begin{aligned} I_{n+1,2}(q) = {} & q^{-1}\gamma_{n+1}I_{n,1}(q) + q^{-n-1}\{\gamma_{n+2} + \beta_{n+1}^2 - q^2\gamma_n - q\beta_n^2 \\ & +(1-q)\beta_{n+1}\sum_{\nu=0}^{n}\beta_\nu - q(1-q)\beta_n\sum_{\nu=0}^{n-1}\beta_\nu\}\langle v, S_{n+1}^2\rangle \ , \ n \geq 1 \ . \end{aligned}$$

Using (41) and the Lemma 2.3, we can deduce (22).
We can remark that by the relation (22), (40) and (42), we have (24).
We have $J_{0,2}(q) = q^{-1}(v)_3 - \beta_0(v)_2$, from Lemma 2.2, we get

$$J_{0,2}(q) = q^{-1}\{(\beta_0 + \beta_1)\gamma_1 + (1-q)\beta_0(\gamma_1 + \beta_0^2)\} \ . \tag{43}$$

When $n \geq 0$, from (2) we have

$$\begin{aligned} J_{n+1,2}(q) = {} & \Big\langle v, x\{S_{n+2}(x) + \beta_{n+1}S_{n+1}(x) + \gamma_{n+1}S_n(x)\}\{(q^{-1}x - \beta_{n+1})S_{n+1}(q^{-1}x) \\ & -\gamma_{n+1}S_n(q^{-1}x)\}\Big\rangle \ . \end{aligned}$$

By the orthogonality of $\{S_n\}_{n\geq 0}$, we can deduce that for $n \geq 0$

$$\begin{aligned} J_{n+1,2}(q) = {} & q^{-1}\gamma_{n+1}J_{n,2}(q) + q^{-1}\beta_{n+1}I_{n+1,2}(q) - \beta_{n+1}^2 I_{n+1,1}(q) - \gamma_{n+1}^2 I_{n,1}(q) \\ & -\beta_{n+1}\gamma_{n+1}J_{n,1}(q) - q^{-n-1}\beta_{n+1}(q\gamma_{n+1} + \gamma_{n+2})\langle v, S_{n+2}^2\rangle \\ & +q^{-1}\langle v, x^2S_{n+2}(x)S_{n+1}(q^{-1}x)\rangle \ . \end{aligned} \tag{44}$$

Taking into account (2), we obtain

$$q^{-1}xS_{n+1}(q^{-1}x) = S_{n+2}(q^{-1}x) + \beta_{n+1}S_{n+1}(q^{-1}x) + \gamma_{n+1}S_n(q^{-1}x) \ , \ n \geq 0 \ .$$

Then, from the last equation and (44), we get

$$J_{n+1,2}(q) = q^{-1}\gamma_{n+1}J_{n,2}(q) + q^{-1}\beta_{n+1}I_{n+1,2}(q) + I_{n+2,1}(q) - \beta_{n+1}^2 I_{n+1,1}(q)$$
$$-\gamma_{n+1}^2 I_{n,1}(q) - \beta_{n+1}\gamma_{n+1}J_{n,1}(q) - q^{-n}\beta_{n+1}\gamma_{n+1}\langle v, S_{n+1}^2\rangle \ , \ n \geq 0 \ .$$

Consequently, from the Lemma 2.3, (20), (22), (24) and (43), we can deduce (25).
The relation (26) can be obtained directly from the definition of $K_{n,0}(q)$.
When $k \geq 1$, $n \geq 0$, we have

$$\begin{aligned} K_{n,k}(q) &= \langle v, x^k \frac{S_n(x)S_n(qx) - S_n(x)S_n(q^{-1}x)}{(q-1)x}\rangle \\ &= \frac{1}{q-1}\langle v, x^{k-1}S_n(x)S_n(qx)\rangle - \frac{1}{q-1}\langle v, x^{k-1}S_n(x)S_n(q^{-1}x)\rangle \\ &= \frac{1}{q-1}I_{n,k-1}(q) - \frac{1}{q-1}I_{n,k-1}(q^{-1}) \ . \end{aligned}$$

Which give $K_{n,k}(q)$, $1 \leq k \leq 3$ from (19) – (22).
On the other hand, we have

$$\begin{aligned} L_{n,0}(q) &= \langle v, \frac{S_{n+1}(x)S_n(qx) - S_n(x)S_{n+1}(q^{-1}x)}{(q-1)x}\rangle \\ &= \langle v, S_{n+1}(x)\frac{S_n(qx) - S_n(x)}{(q-1)x}\rangle + \langle v, S_n(x)\frac{S_{n+1}(x) - S_{n+1}(x)}{(q-1)x}\rangle \\ &= q^{n+1}[n+1]_q\langle v, S_n^2\rangle \ , \end{aligned}$$

$$\begin{aligned} L_{n,1}(q) &= \frac{1}{q-1}\langle v, S_{n+1}(x)S_n(qx)\rangle - \frac{1}{q-1}\langle v, S_n(x)S_{n+1}(q^{-1}x)\rangle \\ &= -\frac{1}{q-1}J_{n,0}(q) \ , \end{aligned}$$

$$\begin{aligned} L_{n,2}(q) &= \frac{1}{q-1}\langle v, xS_{n+1}(x)S_n(qx)\rangle - \frac{1}{q-1}\langle v, xS_n(x)S_{n+1}(q^{-1}x)\rangle \\ &= \frac{q^n}{q-1}\langle v, S_n^2\rangle - \frac{1}{q-1}J_{n,1}(q) \ , \end{aligned}$$

$$\begin{aligned} L_{n,3}(q) &= \frac{1}{q-1}\langle v, x^2S_{n+1}(x)S_n(qx)\rangle - \frac{1}{q-1}\langle v, x^2S_n(x)S_{n+1}(q^{-1}x)\rangle \\ &= \frac{1}{q-1}\langle v, x^2S_{n+1}(x)S_n(qx)\rangle - \frac{1}{q-1}J_{n,2}(q) \ . \end{aligned}$$

But, from (2) we have

$$\begin{aligned} \langle v, x^2S_{n+1}(x)S_n(qx)\rangle &= \langle v, x\{S_{n+2}(x) + \beta_{n+1}S_{n+1}(x) + \gamma_{n+1}S_n(x)\}S_n(qx)\rangle \\ &= \langle v, xS_{n+2}(x)S_n(qx)\rangle + \beta_{n+1}\langle v, xS_{n+1}(x)S_n(qx)\rangle + \gamma_{n+1}\langle v, xS_n(x)S_n(qx)\rangle \\ &= q^n\beta_{n+1}\gamma_{n+1}\langle v, S_n^2\rangle + \gamma_{n+1}I_{n,1}(q) \ , \end{aligned}$$

which implies

$$L_{n,3}(q)=\frac{q^n}{q-1}\beta_{n+1}\gamma_{n+1}\langle v,S_n^2\rangle+\frac{1}{q-1}\gamma_{n+1}I_{n,1}(q)-\frac{1}{q-1}J_{n,2}(q)\,.$$

Hence, we can deduce the desired results (31) − (36) from (20) and (23) − (25). □

The following is the main result of this section.

Proposition 2.7. *We have the following system*

$$a_2\gamma_1=-\Psi(\beta_0)\,, \tag{45}$$

$$\begin{aligned}
&\Big(a_2-[2n]_qc_3\Big)(\gamma_n+\gamma_{n+1})-(1+q)(1+q^{2n-2})c_3\sum_{\nu=0}^{n-2}\gamma_{\nu+1}=-\Psi(\beta_n)\\
&\quad+(1+q^{2n-1})\sum_{\nu=0}^{n-1}\Big(\theta_{\beta_n}\Phi\Big)(\beta_\nu)+(1-q)\Big\{(1-q^{2n-2})c_3\sum_{\nu=0}^{n-2}\beta_\nu\sum_{k=\nu+1}^{n-1}\beta_k\\
&\quad-(1+q)a_2\sum_{\nu=0}^{n-2}\gamma_{\nu+1}-(1-q)a_2\sum_{\nu=0}^{n-2}\beta_{\nu+1}\sum_{k=0}^{\nu}\beta_k-\sum_{\nu=0}^{n-1}\beta_\nu\Big(\theta_{\beta_n}\Psi\Big)(\beta_\nu)\Big\}\\
&\quad+\Big([2n]_q-(1+q^{2n-1})n\Big)c_3\beta_n^2+\Big([2n]_q-(1+q^{2n-1})n\Big)(c_2\beta_n+c_1)\,,\ n\geq 1\,,
\end{aligned} \tag{46}$$

$$\begin{aligned}
&\Xi_n\gamma_{n+1}-(1+q)c_2\sum_{\nu=0}^{n-1}\gamma_{\nu+1}-(q+2)c_3\sum_{\nu=0}^{n-1}\gamma_{\nu+1}(\beta_\nu+\beta_{\nu+1})=\sum_{\nu=0}^{n}\Phi(\beta_\nu)\\
&\quad+(1-q)\Big\{c_2\sum_{\nu=0}^{n-1}\beta_{\nu+1}\sum_{k=0}^{\nu}\beta_k+(1+q)c_3\sum_{\nu=0}^{n-2}\beta_\nu\sum_{k=\nu+1}^{n-1}\gamma_{k+1}\\
&\quad+(1+q)c_3\sum_{\nu=0}^{n-2}\gamma_{\nu+1}\sum_{k=\nu+2}^{n}\beta_k+(1-q)c_3\sum_{\nu=0}^{n-2}\beta_\nu\sum_{k=\nu+1}^{n-1}\beta_k\sum_{i=k+1}^{n}\beta_i+c_3\sum_{\nu=0}^{n-1}\beta_{\nu+1}^2\sum_{k=0}^{\nu}\beta_k\\
&\quad+c_3\sum_{\nu=0}^{n-1}\beta_{\nu+1}\sum_{k=0}^{\nu}\beta_k^2-\sum_{\nu=0}^{n}\beta_\nu\Psi(\beta_\nu)-(1+q)a_1\sum_{\nu=0}^{n-1}\gamma_{\nu+1}-(1-q)a_1\sum_{\nu=0}^{n-1}\beta_\nu\sum_{k=0}^{\nu}\beta_k\\
&\quad-a_2\sum_{\nu=0}^{n-1}\beta_{\nu+1}\gamma_{\nu+1}-(1+q)a_2\sum_{\nu=0}^{n-1}\gamma_{\nu+1}(\beta_\nu+\beta_{\nu+1})-a_2\sum_{\nu=0}^{n}\gamma_{\nu+1}\sum_{k=0}^{\nu}\beta_k\\
&\quad-(1-q)a_2\sum_{\nu=0}^{n-1}\beta_{\nu+1}\sum_{k=0}^{\nu}\beta_k^2-(1-q)a_2\sum_{\nu=0}^{n-1}\beta_{\nu+1}^2\sum_{k=0}^{\nu}\beta_k-(1-q^2)a_2\sum_{\nu=0}^{n-1}\beta_{\nu+1}\sum_{k=0}^{\nu-1}\gamma_{k+1}\\
&\quad+q^2a_2\sum_{\nu=0}^{n-1}\gamma_{\nu+1}\sum_{k=0}^{\nu-1}\beta_k-(1-q)^2a_2\sum_{\nu=0}^{n-1}\beta_{\nu+1}\sum_{k=0}^{\nu-1}\beta_{k+1}\sum_{i=0}^{k}\beta_i\Big\}\\
&\quad+\Big([n+1]_q-n-1\Big)c_0\,,\ n\geq 0\,,\ \sum_{0}^{-2}=0\,,
\end{aligned} \tag{47}$$

with for $n\geq 0$

$$\Xi_n=(a_2-[2n]_qc_3)(\beta_n+\beta_{n+1})+a_1-q^{2n}\beta_{n+1}c_3-[2n+1]_qc_2-(1+q^{2n})c_3\sum_{\nu=0}^{n}\beta_\nu\,.$$

Proof. Making $n=0$ in (10) and taking the relations (19) − (21) and (26) − (29) into account, we can deduce (45).

Let $n\geq 1$, by virtue of the relations (19) − (21), (26) − (28) and (30), the equation (10) becomes

$$\Big(a_2-[2n]_qc_3\Big)(\gamma_n+\gamma_{n+1})-(1+q)(1+q^{2n-2})c_3\sum_{\nu=0}^{n-2}\gamma_{\nu+1}=-a_2\beta_n^2-a_1\beta_n-a_0+[2n]_qc_3\beta_n^2$$

$$+ (1+q^{2n-1})c_3\sum_{\nu=0}^{n-1}\beta_\nu(\beta_n+\beta_\nu) + [2n]_q c_2\beta_n + (1+q^{2n-1})c_2\sum_{\nu=0}^{n-1}\beta_\nu$$
$$+ ([2n]_q c_1 + (1-q)\{(q^{2n-2}-1)c_3\sum_{\nu=0}^{n-2}\beta_\nu\sum_{k=\nu+1}^{n-1}\beta_k - (1+q)a_2\sum_{\nu=0}^{n-2}\gamma_{\nu+1}$$
$$- a_2\sum_{\nu=0}^{n-1}\beta_\nu(\beta_n+\beta_\nu) - (1-q)a_2\sum_{\nu=0}^{n-2}\beta_{\nu+1}\sum_{k=0}^{\nu}\beta_k - a_1\sum_{\nu=0}^{n-1}\beta_\nu\}.$$

But, $\Big(\theta_{\beta_n}\Phi\Big)(\beta_\nu) = c_3(\beta_n^2+\beta_n\beta_\nu+\beta_\nu^2) + c_2(\beta_n+\beta_\nu) + c_1$ and $\Big(\theta_{\beta_n}\Psi\Big)(\beta_\nu) = a_2(\beta_n+\beta_\nu) + a_1$. Then, we can deduce (46).

Let $n=0$ in (11), by virtue of (23) − (25), (31) − (33) and (35), we get (47) for $n=0$. Where $n\geq 1$, on account of (23) − (25), (31) − (32), (34) and (36), (11) becomes

$$\{(a_2-[2n]_q c_3)(\beta_n+\beta_{n+1}) + a_1 - q^{2n-1}\beta_{n+1}c_3 - [2n+1]_q c_2 - (1+q^{2n})c_3\sum_{\nu=0}^{n}\beta_\nu\}\gamma_{n+1}$$
$$- (1+q)c_2\sum_{\nu=0}^{n-1}\gamma_{\nu+1} - (q+2)c_3\sum_{\nu=0}^{n-1}\gamma_{\nu+1}(\beta_\nu+\beta_{\nu+1}) = c_3\sum_{\nu=0}^{n}\beta_\nu^3 + c_2\sum_{\nu=0}^{n}\beta_\nu^2 + c_1\sum_{\nu=0}^{n}\beta_\nu$$
$$+ q^n[n+1]_{q^{-1}}c_0 + (1-q)\{c_2\sum_{\nu=1}^{n}\beta_\nu\sum_{k=0}^{\nu-1}\beta_k + (1+q)c_3\sum_{\nu=0}^{n-2}\beta_\nu\sum_{k=\nu+1}^{n-1}\gamma_{k+1}$$
$$+ (1+q)c_3\sum_{\nu=0}^{n-2}\gamma_{\nu+1}\sum_{k=\nu+2}^{n}\beta_k + (1-q)c_3\sum_{\nu=0}^{n-2}\beta_\nu\sum_{k=\nu+1}^{n-1}\beta_k\sum_{i=k+1}^{n}\beta_i + c_3\sum_{\nu=0}^{n-1}\beta_{\nu+1}^2\sum_{k=0}^{\nu}\beta_k$$
$$+ c_3\sum_{\nu=0}^{n-1}\beta_{\nu+1}\sum_{k=0}^{\nu}\beta_k^2 - a_0\sum_{\nu=0}^{n}\beta_\nu - (1+q)a_1\sum_{\nu=0}^{n-1}\gamma_{\nu+1} - a_1\sum_{\nu=0}^{n}\beta_\nu^2 - (1-q)a_1\sum_{\nu=0}^{n-1}\beta_\nu\sum_{k=0}^{\nu}\beta_k$$
$$- a_2\sum_{\nu=0}^{n-1}\beta_{\nu+1}\gamma_{\nu+1} - (1+q)a_2\sum_{\nu=0}^{n-1}\gamma_{\nu+1}(\beta_\nu+\beta_{\nu+1}) - a_2\sum_{\nu=0}^{n}\beta_\nu^3 - a_2\sum_{\nu=0}^{n}\gamma_{\nu+1}\sum_{k=0}^{\nu}\beta_k$$
$$- (1-q)a_2\sum_{\nu=0}^{n-1}\beta_{\nu+1}\sum_{k=0}^{\nu}\beta_k^2 - (1-q)a_2\sum_{\nu=0}^{n-1}\beta_{\nu+1}^2\sum_{k=0}^{\nu}\beta_k - (1-q^2)a_2\sum_{\nu=0}^{n-1}\beta_{\nu+1}\sum_{k=0}^{\nu-1}\gamma_{k+1}$$
$$- q^2a_2\sum_{\nu=0}^{n-1}\gamma_{\nu+1}\sum_{k=0}^{\nu-1}\beta_k - (1-q)^2a_2\sum_{\nu=0}^{n-1}\beta_{\nu+1}\sum_{k=0}^{\nu-1}\beta_{k+1}\sum_{i=0}^{k}\beta_i\}.$$

Here, we can deduce (47) for $n\geq 1$. □

Remark 1. If $q\to 1$ in (45) − (47), we obtain the result given in [2].

3. H_q-semiclassical forms of class one: specially case

From now on, let v be a H_q-semiclassical form of class $s_v = 1$ satisfying (6) and its corresponding MOPS $\{S_n\}_{n\geq 0}$ fulfils

$$\begin{aligned} &S_{n+2}(x) = \Big(x-(t_n-t_{n+1})\Big)S_{n+1}(x) + t_n^2S_n(x)\ , \quad n\geq 0\ ,\\ &S_1(x) = x+t_0\ , \qquad S_0(x)=1\ , \end{aligned} \tag{48}$$

with $t_n\neq 0\,,\ n\geq 0$.

The next Lemma will play an important role in the sequel.

Lemma 3.1. [15] *The following statements are equivalents*:
(a) The MOPS $\{S_n\}_{n\geq 0}$ satisfying (48).
(b) $(v)_{2n+2} = 0\ ,\ n \geq 0$ *and the form* xv *is regular.*
(c) There exists a regular symmetric form u *such that* $v = (v)_1 x^{-1}u + \delta_0$.
Remark 2. The form v is quasi-antisymmetric (i.e $(v)_{2n+2} = 0, n \geq 0$). For more information about these forms see [12, 15].

3.1. Class and functional equation of the form $u = (v)_1^{-1}xv$

In the sequel our aim is to characterize the structure of the polynomial elements of the functional equation (6) satisfied by the form v which its corresponding MOPS $\{S_n\}_{n\geq 0}$ fulfils (48). This is possible through the study of the H_q-semiclassical character of the symmetric form u define by

$$\lambda u = xv\ ,\ \lambda = (v)_1 \neq 0\ .$$

Consequently, according to [4], the form u is regular if and only if

$$S_n(0) \neq 0\ ,\ n \geq 0\ .$$

Now, multiplying equation (6) by $q^{-2}x^2$ and on account of (3), we obtain

$$H_q(Eu) + Fu = 0\ , \tag{49}$$

with

$$E(x) = x\Phi(x)\ , \quad F(x) = q^{-2}x\Psi(x) - q^{-1}(1 + q^{-1})\Phi(x)\ . \tag{50}$$

Theorem 3.2. *The form* u *is* H_q*-semiclassical of class* s_u *satisfying*

$$H_q(\tilde{E}u) + \tilde{F}u = 0\ . \tag{51}$$

Moreover,
(a) *If* $\Phi(0) \neq 0$, *then*

$$\tilde{E}(x) = E(x)\ , \quad \tilde{F}(x) = F(x)\ ,$$

and $s_u = 2$.
(b) *If* $\Phi(0) = 0$ *and* $\Psi(0) \neq 0$, *then*

$$\tilde{E}(x) = \Phi(x)\ , \quad \tilde{F}(x) = q^{-1}\Big(\Psi - (\theta_0\Phi)\Big)(x)\ ,$$

and $s_u = 1$.
(c) *If* $\Phi(0) = 0$ *and* $\Psi(0) = 0$, *then*

$$\tilde{E}(x) = (\theta_0\Phi)(x)\ , \quad \tilde{F}(x) = (\theta_0\Psi)(x)\ ,$$

and $s_u = 0$.

For the proof, we need the following lemma.

Lemma 3.3. (i) *For all root* c *of* Φ, *we have*

$$\langle u, \theta_{cq}(qF + \theta_c E)\rangle = \frac{c^2}{\lambda}\langle v, \theta_{cq}(q\Psi + \theta_c\Phi)\rangle - \frac{cq^{-1}}{\lambda}\Big((H_q\Phi) + q(h_q\Psi)\Big)(c)\ , \tag{52}$$

$$\Big((H_qE) + q(h_qF)\Big)(c) = q^{-1}c\Big((H_q\Phi) + q(h_q\Psi)\Big)(c)\ . \tag{53}$$

(ii) *The class of the form* u *depends only the root* $x = 0$ *of* E.

Proof. (i) Let c be a root of Φ. Then we can write

$$E(x) = x(x-c)(\theta_c\Phi)(x)\ . \tag{54}$$

Using the definition of the operator θ_c, it is easy to prove that, for $f\ ,g \in \mathcal{P}$, we have

$$\Big(\theta_c(fg)\Big)(x) = \Big(\theta_c f\Big)(x)g(x) + f(c)\Big(\theta_c g\Big)(x)\ . \tag{55}$$

From (50) and (54), we have

$$\langle u, \theta_{cq}\theta_c E\rangle = \frac{1}{\lambda}\langle xv, \theta_{cq}\Big((\xi)\Phi_c\Big)(x)\rangle\ .$$

Taking $f(x) = x$ and $g(x) = \Phi_c(x)$ in (55), we obtain

$$\langle xv, \theta_{cq}\Big(\xi\Phi_c\Big)(x)\rangle = c^2q^2\langle v, \theta_{cq}\theta_c\Phi\rangle + c(1+q)\langle v, \Phi_c\rangle + \langle v, \Phi\rangle - cq(H_q\Phi)(c)\ ,$$

since $x = (x-c) + c = (x-cq) + cq$. Therefore,

$$\langle u, \theta_{cq}\theta_c E\rangle = \frac{c^2q^2}{\lambda}\langle v, \theta_{cq}\theta_c\Phi\rangle + \frac{(1+q)c}{\lambda}\langle v, \Phi_c\rangle + \frac{1}{\lambda}\langle v, \Phi\rangle - \frac{cq}{\lambda}(H_q\Phi)(c)\ . \tag{56}$$

Proceeding as in (56), we can easily prove that

$$\begin{aligned}\langle u, q\theta_{cq}F\rangle = {} & \tfrac{c^2q}{\lambda}\langle v, \theta_{cq}\Psi\rangle + \tfrac{1}{\lambda}\langle v, (q^{-1}x+c)\Psi\rangle - (1+q)\tfrac{c}{\lambda}\langle v, \theta_{cq}\Phi\rangle \\ & -\tfrac{1+q^{-1}}{\lambda}\langle v, \Phi\rangle - \tfrac{c}{\lambda}(h_q\Psi)(c) + \tfrac{1+q^{-1}}{\lambda}(h_q\Phi)(c)\ .\end{aligned} \tag{57}$$

Adding (56) and (57), we get

$$\begin{aligned}\langle u, \theta_{cq}(\theta_c E + qF)\rangle = {} & \tfrac{c^2}{\lambda}\langle v, q^2\theta_{cq}\theta_c\Phi + \theta_{cq}\Psi\rangle + \tfrac{(1+q)c}{\lambda}\langle v, (\theta_c - \theta_{cq})\Phi\rangle \\ & + \tfrac{1}{\lambda}\{-cq(H_q\Phi)(c) + (1+q^{-1})(h_q\Phi)(c)\} - \tfrac{c}{\lambda}(h_q\Psi)(c) \\ & + \tfrac{1}{\lambda}\langle v, -q^{-1}\Phi(x) + (q^{-1}x+c)\Psi(x)\rangle\ .\end{aligned}$$

This yields (52), since
$(h_q\Phi)(c) = (q-1)c(H_q\Phi)(c)$, $(\theta_c - \theta_{cq})\Phi = -(q-1)c\theta_{cq}\theta_c\Phi$, and
$\langle v, -q^{-1}\Phi(x) + (q^{-1}x+c)\Psi(x)\rangle = \langle H_q(\Phi v) + \Psi v, q^{-1}x + c\rangle = 0.$
Next, it is easy to find (53) from (50).

(ii) Let c be a root of E such that $c \neq 0$. According to (50) we get $\Phi(c) = 0$. We suppose $\Big(q(h_qF)+(H_qE)\Big)(c) = 0$. According to (53), we obtain

$$\Big((H_q\Phi) + q(h_q\Psi)\Big)(c) = 0\ ,$$

and

$$\langle u, \theta_{cq}(\theta_c E + qF)\rangle = \frac{c^2}{\lambda}\langle v, \theta_{cq}(\theta_c\Phi + q\Psi)\rangle \neq 0\ ,$$

since v is H_q-semi-classical and so satisfies (7). Therefore, equation (49) is not simplified by $x-c$ for $c \neq 0$. □

Proof. (of Theorem 3.2) We may write $\Big((H_qE) + q(h_qF)\Big)(0) = -q^{-1}\Phi(0)$.

(a) If $\Phi(0) \neq 0$, then $\Big((H_qE) + q(h_qF)\Big)(0) \neq 0$. Thus, equation (51) cannot be simplified and so the form u is of class

$$s_u = \max\Big(\deg(E) - 2, \deg(F) - 1\Big) = \max\Big(\deg(\Phi) - 1, \deg(\Psi)\Big)\ .$$

Hence, $s_u = 2$.

(b) If $\Phi(0) = 0$, then

$$\Big((H_qE)+q(h_qF)\Big)(0)=0 \text{ and } \langle u,\theta_0(\theta_0E+qF)\rangle=0,$$

according to (52) and (53). So, equation (51) can be simplified by the polynomial x and becomes

$$H_q(\tilde{E}u)+\tilde{F}u=0\,, \tag{58}$$

where

$$\tilde{E}(x)=\Phi(x)\,,\quad \tilde{F}(x)=q^{-1}\Big(\Psi-\theta_0\Phi\Big)(x)\,. \tag{59}$$

It is easy to see that (58) is not simplified, since from (59), we have $\Big((H_q\tilde{E})+q(h_q\tilde{F})\Big)(0)=\Psi(0)\neq 0$. Therefore $s_u=1$.

(c) If $\Phi(0)=0$ and $\Psi(0)=0$, then

$$\Big((H_q\tilde{E})+q(h_q\tilde{F})\Big)(0)=0\,.$$

A simple calculation gives $\langle u,\theta_0(\theta_0\tilde{E}+q\tilde{F})\rangle=\frac{1}{\lambda}\langle v,\Psi\rangle=0$. So, (58) is simplified by the polynomial x and it becomes

$$H_q(\check{E}u)+\check{F}u=0\,, \tag{60}$$

where

$$\check{E}(x)=(\theta_0\Phi)(x)\,,\quad \check{F}(x)=(\theta_0\Psi)(x)\,. \tag{61}$$

If 0 is a root of $\theta_0\Phi$, then $\Big((H_q\Phi)+q(h_q\Psi)\Big)(0)=0$. Assuming that $\Big((H_q\check{E})+q(h_q\check{F})\Big)(0)=0$.

A simple calculation gives from (61) $\langle u,\theta_0(\theta_0\check{E}+q\check{F})\rangle=\frac{1}{\lambda}\langle v,\theta_0(\theta_0\Phi+q\Psi)\rangle\neq 0$, since v is a H_q-semiclassical and it satisfies (7). Hence, equation (60) is not simplified and so, $s_u=0$. □

3.2. *Structure of the polynomials Φ and Ψ*

Let us spilt up each polynomial form Φ, Ψ, $\theta_0\Phi$ and $\theta_0\Psi$ according to its odd and even parts that is to say

$$\begin{aligned}&\Phi(x)=\Phi^e(x^2)+x\Phi^o(x^2)\,,\ \Psi(x)=\Psi^e(x^2)+x\Psi^o(x^2)\,,\\&(\theta_0\Phi)(x)=\Phi_1^e(x^2)+x\Phi_1^o(x^2)\,,\ (\theta_0\Psi)(x)=\Psi_1^e(x^2)+x\Psi_1^o(x^2)\,.\end{aligned} \tag{62}$$

Proposition 3.4. *If v be a H_q-semiclassical form of class one satisfying* (6) *and $\{S_n\}_{n\geq 0}$ be its corresponding MOPS fulfilling* (48), *then*

$$\Phi^e=0\,,\quad \Psi^o=0\,. \tag{63}$$

Proof. Writing

$$\tilde{E}(x)=\tilde{E}^e(x)+x\tilde{E}^o(x)\,,\ \tilde{F}(x)=\tilde{F}^e(x)+x\tilde{F}^o(x)\,.$$

We have to examine the following situations:

(a) $\Phi(0)\neq 0$. According to (62) and from the expression of polynomials $\tilde{E}$ and $\tilde{F}$ given in Theorem 3.2, we get

$$\begin{aligned}&\tilde{E}^e(x)=x\Phi^o(x)\,,\quad \tilde{E}^o(x)=\Phi^e(x)\,,\\&\tilde{F}^e(x)=q^{-2}x\Psi^o(x)-q^{-1}(1+q^{-1})\Phi^e(x)\,,\quad \tilde{F}^o(x)=q^{-2}\Psi^e(x)-q^{-1}(1+q^{-1})\Phi^o(x)\,.\end{aligned}$$

Then, $\tilde{E}^o=\tilde{F}^e=0$, from Proposition 1.3, since $s_u=2$. This gives (63).

In the other cases, we are going to proceed with the same techniques.

(b) $\Phi(0)=0$ and $\Psi(0)\neq 0$. Similar as above,

$$\begin{aligned}&\tilde{E}^e(x)=\Phi^e(x)\,,\quad \tilde{E}^o(x)=\Phi^o(x)\,,\\&\tilde{F}^e(x)=q^{-1}\Psi^e(x)-q^{-1}\Phi_1^e(x)\,,\quad \tilde{F}^o(x)=q^{-1}\Psi^o(x)-q^{-1}\Phi_1^o(x)\,.\end{aligned}$$

If $s_u = 1$, then $\tilde{E}^e = \tilde{F}^o = 0$. This leads to result (63), since $\Phi(x) = x(\theta_0\Phi)(x)$.
(c) $\Phi(0) = 0$ and $\Psi(0) = 0$. In this case, we obtain

$$\tilde{E}^e(x) = \Phi_1^e(x)\ , \quad \tilde{E}^o(x) = \Phi_1^o(x)\ ,$$
$$\tilde{F}^e(x) = \Psi_1^e(x)\ , \quad \tilde{F}^o(x) = \Psi_1^o(x)\ .$$

Since u is of even class, $\tilde{E}^o = \tilde{F}^e = 0$. This gives the desired result (63), since $\Psi(x) = x(\theta_0\Psi)(x)$. □

Theorem 3.5. *If v is a H_q-semiclassical form of class one satisfying* (6) *and $\{S_n\}_{n\geq 0}$ be its corresponding MOPS fulfilling* (48)*, then*

$$\Phi(x) = x\ , \quad \Psi(x) = a_2x^2\ ,\ a_2 \neq 0\ , \tag{64}$$

or

$$\Phi(x) = x^3 + c_1x\ , \quad \Psi(x) = a_2x^2\ ,\ a_2c_1 \neq 0\ . \tag{65}$$

Proof. We have to consider four cases:
A. $\deg(\Phi) = 0$. We have $\Phi^e(x) = 1$ and $\Phi^o(x) = 0$. From Proposition 3.4, we get $\Phi^e(x) = 0$ which yields a contradiction.
B. $\deg(\Phi) = 1$. In this case, we have $c_1 = 1$, $c_2 = c_3 = 0$ and $a_2 \neq 0$. Then, we obtain $\Phi^e(x) = c_0$, $\Phi^o(x) = 1$, $\Psi^e(x) = a_2x + a_0$ and $\Psi^o(x) = a_1$. Following Proposition 3.4, three situations to establish.

B$_1$. $\Phi(0) \neq 0$, then $c_0 = 0$ and $a_1 = 0$ which yields a contradiction.

B$_2$. $\Phi(0) = 0$ and $\Psi(0) \neq 0$, here $c_0 = 0$ and $a_1 = 0$. Thus, $a_0 = 0$ since from (6), we have $\langle H_q(\Phi v)+\Psi v, 1\rangle = 0$, which yields a contradiction.

B$_3$. $\Phi(0) = 0$ and $\Psi(0) = 0$, then $\Phi_1^o(x) = 0$ and $a_1 = 0$. This gives (64).

C. $\deg(\Phi) = 2$. We have $c_2 = 1$, $c_3 = 0$ and $a_2 \neq 0$. Then, we obtain $\Phi^e(x) = x + c_0$ and $\Phi^o(x) = c_1$. By virtue of Proposition 3.4, such assumptions lead us to a contradiction.
D. $\deg(\Phi) = 3$. In this case, we get $\Phi^e(x) = c_2x + c_0$, $\Phi^o(x) = x + c_1$, $\Psi^e(x) = a_2x + a_0$ and $\Psi^o(x) = a_1$, because $1 \leq \deg(\Psi) \leq 2$. We have to examine three subcases:

D$_1$. $\Phi(0) \neq 0$, then $c_2 = c_0 = 0$ and $a_1 = 0$, which yields a contradiction.

D$_2$. $\Phi(0) = 0$ and $\Psi(0) \neq 0$, here $c_2 = c_0 = 0$ and $a_1 = 0$. Thus, $a_0 = 0$ since from (6), we have $\langle H_q(\Phi v) + \Psi v, 1\rangle = 0$,which yields a contradiction.

D$_3$. $\Phi(0) = 0$ and $\Psi(0) = 0$, thus $c_2 = 0$ and $a_1 = 0$.

Now, we assume that $c_1 = 0$. Then, from (6) we have $(a_2 - [n+1]_q)(v)_{2n+3} = 0$, $n \geq 0$. So, after a certain rang the Hankel determinants [12] associated with v are equal to zero, by following v is not regular. This leads to result (65). □

3.3. *Recurrence coefficients of $\{S_n\}_{n\geq 0}$*

First, let us recall the following standard material needed to the sequel [3]

$$(a;q)_0 = 1\ , \quad (a;q)_n = \prod_{\nu=1}^{n}(1 - aq^{\nu-1})\ ,\ n \geq 1\ .$$

Second, we assume that $\{S_n\}_{n\geq 0}$ be a H_q-semiclassical sequence of class $s_v = 1$ satisfying (48). By virtue of the Theorem 3.5, it follows that v satisfying (6) with

$$\Phi(x) = c_3x^3 + c_1x\ , \quad \Psi(x) = a_2x^2\ . \tag{66}$$

Proposition 3.6. *The sequence $\{t_n\}_{n\geq 0}$ is define by*

$$t_{2n} = t_0\Lambda_n(c_3, c_1)\ , \quad t_{2n+1} = t_0^{-1}\Omega_n(c_3, c_1)\ , \tag{67}$$

where

$$\Lambda_n(c_3,c_1)=\begin{cases} q^n\frac{(q^2;q^2)_n}{(q;q^2)_n}\ ,\ if\ (c_3,c_1)=(0,1)\ ,\\ q^n\frac{(q^2;q^2)_n(w^{-1};q^2)_n(q^{-1}w^{-1};q^4)_n}{(q;q^2)_n(q^{-1}w^{-1};q^2)_n(qw^{-1};q^4)_n}\ ,\ if\ c_3=1\ ,\ c_1w\neq 0\ ,\\ \frac{(q^2;q^2)_n}{q^{2n}(q;q^2)_n}\ ,\ if\ c_3=1\ ,\ w=0\ ,\ c_1\neq 0\ ,\end{cases}$$

$$\Omega_n(c_3,c_1)=\begin{cases} -q^n\frac{(q^3;q^2)_n}{a_2(q^2;q^2)_n}\ ,\ if\ (c_3,c_1)=(0,1)\ ,\\ -w^{-1}c_1q^n(q-1)\frac{(q^3;q^2)_n(qw^{-1};q^2)_n(qw^{-1};q^4)_n}{(1-w^{-1}q^{4n+1})(q^2;q^2)_n(w^{-1};q^2)_n(q^3w^{-1};q^4)_n}\ ,\ if\ c_3=1\ ,\ c_1w\neq 0\ ,\\ c_1(q-1)\frac{(q^3;q^2)_n}{q^{2n+1}(q^2;q^2)_n}\ ,\ if\ c_3=1\ ,\ w=0\ ,\ c_1\neq 0\ ,\end{cases}$$

with

$$w=(q-1)a_2+1\ .$$

Proof. Taking into account (48) and (66), the system (45) − (47) becomes for $n\geq 0$

$$(q+1)\{(1+q^{2n-2})c_3+(q-1)a_2\}\sum_{\nu=0}^{n-1}t_\nu t_{\nu+1}+q(q+1)\{[2n-3]_qc_3-a_2\}t_nt_{n-1}-[2n]_qc_1=0\ , \tag{68}$$

$$(q^2-1)\{c_3+(q-1)a_2\}\sum_{\nu=0}^{n-1}t_\nu t_{\nu+1}-\{[2n+1]_qc_3-a_2\}t_nt_{n+1}+q^3\{[2n-3]_qc_3-a_2\}t_nt_{n-1}+c_1=0\ . \tag{69}$$

Subtracting identities (68) and (69) after multiplying respectively by q^2 and $q+1$, we obtain

$$\{(1+q^{2n})c_3+(q-1)a_2\}\sum_{\nu=0}^{n}t_\nu t_{\nu+1}+q\{[2n-1]_qc_3-a_2\}t_nt_{n+1}-[n+1]_{q^2}c_1=0\ ,\ n\geq 0\ .$$

Equivalently,

$$\{[2n+1]_qc_3-a_2\}T_n-q\{[2n-1]_qc_3-a_2\}T_{n-1}=[n+1]_{q^2}c_1\ ,\ n\geq 0\ , \tag{70}$$

where

$$T_n=\sum_{\nu=0}^{n}t_\nu t_{\nu+1}\ ,\ n\geq 0\ . \tag{71}$$

So, we get from (70)

$$\{[2n+1]_qc_3-a_2\}T_n=c_1\sum_{k=0}^{n}q^{n-k}[k+1]_{q^2}\ ,\ n\geq 0\ .$$

Hence,

$$T_n=\frac{[n+1]_q[n+2]_qc_1}{(q+1)\{[2n+1]_qc_3-a_2\}}\ ,\ n\geq 0\ . \tag{72}$$

Now, from (71), we have

$$t_{2n}t_{2n+1}=T_{2n}-T_{2n-1}\ ,\quad t_{2n+1}t_{2n+2}=T_{2n+1}-T_{2n}\ ,\ n\geq 0\ ,\ T_{-1}=0\ .$$

Then, from (72) we obtain for $n\geq 0$

$$\begin{aligned} t_{2n}t_{2n+1}&=q^{2n}\frac{[2n+1]_q\{[2n-1]_qc_3-a_2\}c_1}{\{[4n-1]_qc_3-a_2\}\{[4n+1]_qc_3-a_2\}}\ ,\\ t_{2n+1}t_{2n+2}&=q^{2n+1}\frac{[2n+2]_q\{[2n]_qc_3-a_2\}c_1}{\{[4n+1]_qc_3-a_2\}\{[4n+3]_qc_3-a_2\}}\ .\end{aligned} \tag{73}$$

This leads to

$$\frac{t_{2n+2}}{t_{2n}} = q\frac{[2n+2]_q\{[2n]_q c_3 - a_2\}\{[4n-1]_q c_3 - a_2\}}{[2n+1]_q\{[4n+3]_q c_3 - a_2\}\{[2n-1]_q c_3 - a_2\}} \ .$$

Here,

$$t_{2n+2} = t_0\Lambda_{n+1}(c_3, c_1) \ , \ n \geq 0 \ .$$

On account of (73) and the above equation, we have

$$t_{2n+1} = q^n \frac{[2n+1]_q\{[2n-1]_q c_3 - a_2\}c_1}{t_0\{[4n-1]_q c_3 - a_2\}\{[4n+1]_q c_3 - a_2\}\Lambda_n(c_3, c_1)} \ , \ n \geq 0 \ .$$

Hence the desired result (67). □

3.4. *The canonical cases*

Before quoting the different canonical situations, let us proceed to the general transformation.

$$\{\tilde{S}_n(x) = a^{-n}S_n(ax)\}_{n\geq 0} \text{ MOPS satisfies (48) with } \tilde{t}_n = a^{-1}t_n \ , \ n \geq 0 \ .$$

Then, the form $\tilde{v} = h_{a^{-1}}v$ fulfils

$$H_q(a^{-t}\Phi(ax)\tilde{v}) + a^{1-t}\Psi(ax)\tilde{v} = 0 \ , \ t = \deg(\Phi) \ . \tag{74}$$

Any so-called canonical case will be denoted by $\tilde{t}_n$, $\tilde{v}$.

Theorem 3.7. *The following canonical cases arise:*

(a) *When* $\Phi(x) = x$, *we have*

$$\begin{cases} \tilde{t}_{2n} = -\lambda q^n \frac{(q^2;q^2)_n}{(q;q^2)_n} \ , \ n \geq 0 \ , \\ \tilde{t}_{2n+1} = q^n \frac{(q^3;q^2)_n}{2\lambda(q^2;q^2)_n} \ , \ n \geq 0 \ , \\ H_q(x\tilde{v}) + 2x^2\tilde{v} = 0 \ . \end{cases} \tag{75}$$

(b) *When* $\Phi(x) = x^3 + c_1x$, $c_1 \neq 0$, *we have the following canonical cases*

(i) $1 + (q-1)a_2 \neq 0$

$$\begin{cases} \tilde{t}_{2n} = -\lambda q^n \frac{(q^2;q^2)_n(b^2q^2;q^2)_n(b^2q;q^4)_n}{(q;q^2)_n(b^2q;q^2)_n(b^2q^3;q^4)_n} \ , \ n \geq 0 \ , \\ \tilde{t}_{2n+1} = -b^2q^{n+2}(q-1)\frac{(q^3;q^2)_n(b^2q^3;q^2)_n(b^2q^3;q^4)_n}{\lambda(1-b^2q^{4n+3})(q^2;q^2)_n(b^2q^2;q^2)_n(b^2q^5;q^4)_n} \ , \ n \geq 0 \ , \\ H_q\Big((x^3 - x)\tilde{v}\Big) - \frac{b^2q^2-1}{b^2q^2(q-1)}x^2\tilde{v} = 0 \ , \\ b^2 - 1 \neq 0 \ . \end{cases} \tag{76}$$

(ii) $a_2 = -(q-1)^{-1}$

$$\begin{cases} \tilde{t}_{2n} = -\lambda \frac{(q^2;q^2)_n}{q^{2n}(q;q^2)_n} \ , \ n \geq 0 \ , \\ \tilde{t}_{2n+1} = -(q-1)\frac{(q^3;q^2)_n}{\lambda q^{2n+1}(q^2;q^2)_n} \ , \ n \geq 0 \ , \\ H_q\Big((x^3 + x)\tilde{v}\Big) - (q-1)^{-1}x^2\tilde{v} = 0 \ . \end{cases} \tag{77}$$

Proof. (a) In this case (67) and (74) become

$$\begin{cases} t_{2n} = t_0 q^n \frac{(q^2;q^2)_n}{(q;q^2)_n} \ , \ n \geq 0 \ , \\ t_{2n+1} = -t_0^{-1} q^n \frac{(q^3;q^2)_n}{a_2(q^2;q^2)_n} \ , \ n \geq 0 \ , \\ H_q\Big(xv\Big) + a_2x^2v = 0 \ . \end{cases}$$

With the choice $a = \sqrt{2a_2^{-1}}$ and putting $\lambda = -t_0$, we get (75).

(b) (i) In this case (67) and (74) reduce to

$$\begin{cases} t_{2n} = t_0 q^n \frac{(q^2;q^2)_n (w;q^2)_n (q^{-1}w;q^4)_n}{(q;q^2)_n (q^{-1}w;q^2)_n (qw;q^4)_n} \ , \ n \geq 0 \ , \\ t_{2n+1} = t_0^{-1} - wc_1 q^n (q-1) \frac{(q^3;q^2)_n (qw;q^2)_n (qw;q^4)_n}{(1-wq^{4n+1})(q^2;q^2)_n (w;q^2)_n (q^3 w;q^4)_n} \ , \ n \geq 0 \ , \\ H_q\Big((x^3 + c_1 x)v\Big) + a_2 x^2 v = 0 \ . \end{cases}$$

The choice $a = \sqrt{-c_1}$ and putting $-\frac{b^2q^2-1}{b^2q^2(q-1)} = a_2$, $\lambda = -t_0$, we obtain (76).

(ii) In this case (67) and (74) become

$$\begin{cases} t_{2n} = t_0 \frac{(q^2;q^2)_n}{q^{2n}(q;q^2)_n} \ , \ n \geq 0 \ , \\ t_{2n+1} = t_0^{-1} c_1 (q-1) \frac{(q^3;q^2)_n}{q^{2n+1}(q^2;q^2)_n} \ , \ n \geq 0 \ , \\ H_q\Big((x^3 + c_1 x)v\Big) + a_2 x^2 v = 0 \ . \end{cases}$$

The choice $a = \sqrt{c_1}$ and putting $\lambda = -t_0$, we obtain (77). □

Remarks 3. (i) The form defined by (76) is regular if and only if $b^2 \neq q^{-(n+1)}$, $n \geq 0$.

(ii) The canonical cases (75) and (77) are studied in [9] and the corresponding linear forms have respectively the following integral representations:

$$\langle \tilde{v}, f \rangle = \begin{cases} f(0) + P \displaystyle\int_{-\infty}^{+\infty} \frac{f(x)}{x\Big(-2(q-1)x^2; q^{-2}\Big)_\infty} dx \ , \ q > 1 \ , \ f \in ¶ \ , \\ f(0) + P \displaystyle\int_{-\frac{1}{q\sqrt{2(1-q)}}}^{\frac{1}{q\sqrt{2(1-q)}}} \frac{\Big(2q^2(1-q)x^2; q^2\Big)_\infty}{x} f(x)\, dx \ , \ 0 < q < 1 \ , \ f \in ¶ \ , \end{cases}$$

$$\langle \tilde{v}, f \rangle = f(0) + P \int_{-\infty}^{+\infty} \frac{1}{x\Big(-x^2; q^2\Big)_\infty} f(x)\, dx \ , \ 0 < q < 1 \ , \ f \in ¶ \ ,$$

where

$$P \int_{-\infty}^{+\infty} \frac{V(x)}{x} dx = \lim_{\varepsilon \to 0^+} \Big(\int_{-\infty}^{-\varepsilon} \frac{V(x)}{x} dx + \int_{+\varepsilon}^{+\infty} \frac{V(x)}{x} dx \Big) \ , \tag{78}$$

with V is a locally integrable function with rapid decay and continuous at the point $x = 0$ and

$$(a;q)_\infty = \prod_{\nu=0}^{+\infty} (1 - aq^\nu) \ , \ |q| < 1 \ .$$

Proposition 3.8. *The form $\tilde{v}$ define by (76) has the following integral representation*

$$\langle \tilde{v}, f \rangle = \begin{cases} f(0) + P \displaystyle\int_{-q}^{q} \frac{(-q^{-1}x; q^{-1})_\infty (q^{-1}x; q^{-1})_\infty}{x(b^{-1}q^{-1}x; q^{-1})_\infty (-b^{-1}q^{-1}x; q^{-1})_\infty} f(x)\, dx \ , \ q > 1 \ , \ b > 1 \ , \\ f(0) + P \displaystyle\int_{-b}^{b} \frac{(b^{-1}x; q)_\infty (-b^{-1}x; q)_\infty}{x(-x; q)_\infty (x; q)_\infty} f(x)\, dx \ , \ 0 < q < 1 \ , \ 0 < b < 1 \ . \end{cases} \tag{79}$$

Proof. From (76), we have

$$\tilde{v} = \lambda^{-1} x u + \delta_0 \ , \tag{80}$$

where u is symmetric H_q-classical form satisfying the q-Pearson equation [8]

$$H_q\Big((x^2-1)u\Big) - \frac{b^2q^2-1}{b^2q^2(q-1)}xu = 0\ .$$

This form has the following integral representation [8]

$$\langle u, f\rangle = \begin{cases} K_1 \displaystyle\int_{-q}^{q} \frac{(-q^{-1}x;q^{-1})_\infty (q^{-1}x;q^{-1})_\infty}{(b^{-1}q^{-1}x;q^{-1})_\infty(-b^{-1}q^{-1}x;q^{-1})_\infty} f(x)\,dx\ ,\ q>1\ ,\ b>1\ , \\ K_2 \displaystyle\int_{-b}^{b} \frac{(b^{-1}x;q)_\infty(-b^{-1}x;q)_\infty}{(-x;q)_\infty(x;q)_\infty} f(x)\,dx\ ,\ 0<q<1\ ,\ 0<b<1\ , \end{cases} \tag{81}$$

with

$$K_1 = \Big(\int_{-q}^{q} \frac{(-q^{-1}x;q^{-1})_\infty (q^{-1}x;q^{-1})_\infty}{(b^{-1}q^{-1}x;q^{-1})_\infty(-b^{-1}q^{-1}x;q^{-1})_\infty} dx\Big)^{-1}\ ,$$

$$K_2 = \Big(\int_{-b}^{b} \frac{(b^{-1}x;q)_\infty(-b^{-1}x;q)_\infty}{(-x;q)_\infty(x;q)_\infty} dx\Big)^{-1}\ .$$

Thus, by (80) and (81), we have

$$\langle \tilde{v}, f\rangle = \begin{cases} f(0) + \lambda^{-1}K_1 P \displaystyle\int_{-q}^{q} \frac{(-q^{-1}x;q^{-1})_\infty (q^{-1}x;q^{-1})_\infty}{x(b^{-1}q^{-1}x;q^{-1})_\infty(-b^{-1}q^{-1}x;q^{-1})_\infty} (f(x)-f(0))\,dx\ ,\ q>1\ ,\ b>1\ , \\ f(0) + \lambda^{-1}K_2 P \displaystyle\int_{-b}^{b} \frac{(b^{-1}x;q)_\infty(-b^{-1}x;q)_\infty}{x(-x;q)_\infty(x;q)_\infty} (f(x)-f(0))\,dx\ ,\ 0<q<1\ ,\ 0<b<1\ . \end{cases}$$

But from (78) it easy to see that

$$P\int_{-q}^{q} \frac{(-q^{-1}x;q^{-1})_\infty (q^{-1}x;q^{-1})_\infty}{x(b^{-1}q^{-1}x;q^{-1})_\infty(-b^{-1}q^{-1}x;q^{-1})_\infty} dx = 0\ ,$$

and

$$P\int_{-b}^{b} \frac{(b^{-1}x;q)_\infty(-b^{-1}x;q)_\infty}{x(-x;q)_\infty(x;q)_\infty} dx = 0\ .$$

Therefore, with the choosing

$$\lambda = \begin{cases} K_1\ ,\ q>1\ ,\ b>1\ , \\ K_2\ ,\ 0<q<1\ ,\ 0<b<1\ , \end{cases}$$

we obtain the result (79). □

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