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Best Proximity Coincidence Point Theorems for Generalized Non-linear Contraction Mappings

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Abstract. In this paper, we obtained best proximity coincidence point theorems for α-Geraghty contractions in the setting of complete metric spaces by using weak P-property. Also we presented some examples to prove the validity of our results. Our results extended and unify many existing results in the literature. Moreover, in the last section as applications of our main results, we can apply some coincidence best proximity point and coupled coincidence proximity point on metric spaces endowed with an arbitrary binary relation.

1. Introduction

Fixed point theory is a branch of non-linear analysis which has attracted much attention in recent times due to its possible applications. The Banach contraction principle [1], which is a useful tool in the study of many branches of mathematics and mathematical sciences, is one of the earlier and fundamental result in fixed point theory. Because of its importance in non-linear analysis, a number of mathematicians have intensively investigated sufficient conditions to ensure that certain contractive mappings have a fixed point. They improved, generalized and extended this basic result either by defining a new contractive mappings in the context of a complete metric space or by investigating the existing contractive mappings in various abstract spaces; see, e.g., [2–11] and references therein. When a mapping from a metric space into itself has no fixed points, it could be interesting to study the existence and uniqueness of some points that minimize the distance between the origin and its corresponding image. These points are known as best proximity

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points and were introduced by [12] and modified by Sadiq Basha in [13]. Best proximity point theorems for several types of non-self mappings have been derived in [13–21]. Recently, Geraghty [6] obtained a generalization of the Banach contraction principle in the setting of complete metric spaces by considering an auxiliary function. Later, Amini-Harandi and Emami [2] characterized the result of Geraghty in the context of a partially ordered complete metric space. This result is of particular interest since many real world problems can be identified in a partially ordered complete metric space. Many mathematicians discussed the existence of a best proximity point of Geraghty contraction [22]. In this paper, we obtained best proximity coincidence point and fixed point theorems for α-Geraghty contractions in the setting of complete metric spaces. We obtain some examples to prove the validity of our results. Our results extend and unify many existing results in the literature.

2. Preliminaries

Definition 2.1. *[17] Let X be a metric space, A and B two nonempty subsets of X. Define*

$$\begin{aligned} d(A,B) &= \inf\{d(a,b) : a \in A, b \in B\}, \\ A_0 &= \{a \in A : \text{there exists some } b \in B \text{ such that } d(a,b) = d(A,B)\}, \\ B_0 &= \{b \in B : \text{there exists some } a \in A \text{ such that } d(a,b) = d(A,B)\}. \end{aligned}$$

In [18], the authors present sufficient conditions which determine when the sets A_0 and B_0 are nonempty.

Definition 2.2. *[23] Let $f : X \to X$ be a map and $\alpha : X \times X \to \mathbb{R}$ be a function. Then f is said to be α-admissible if $\alpha(x, y) \geq 1$ implies $\alpha(fx, fy) \geq 1$.*

Definition 2.3. *An $\alpha-$admissible map f is said to be triangular $\alpha-$admissible if $\alpha(x,z) \geq 1$ and $\alpha(z, y) \geq 1$ implies $\alpha(x, y) \geq 1$.*

Definition 2.4. *[24] A mapping $T : A \to B$ is said to be $\alpha-$proximal admissible, where $\alpha : A \times A \to [0,\infty)$ be a function, if*

$$\left.\begin{array}{l} \alpha(x_1, x_2) \geq 1 \\ d(u_1, Tx_1) = d(A,B) \\ d(u_2, Tx_2) = d(A,B) \end{array}\right\} \Longrightarrow \alpha(u_1, u_2) \geq 1$$

for all $x_1, x_2, u_1, u_2 \in A$.

We denote by $\mathcal{F}$ the class of all functions $\beta : [0,\infty) \to [0, 1)$ satisfying $\beta(t_n) \to 1$, implies $t_n \to 0$ as $n \to \infty$.

Definition 2.5. *[6] Let (X, d) be a metric space. A map $f : X \to X$ is called Geraghty contraction if there exists $\beta \in \mathcal{F}$ such that for all $x, y \in X$,*

$$d(fx, fy) \leq \beta(d(x, y))d(x, y).$$

By using such maps Geraghty et al. [6] proved the following fixed point result:

Theorem 2.6. *Let (X, d) be a complete metric space. A Mapping $f : X \to X$ is Geraghty contraction. Then f has a fixed point $x \in X$, and $\{f^n x\}$ converges to x.*

Cho et al. [25] generalized the concept of Geraghty contraction to α-Geraghty contraction and prove the fixed point theorem for such contraction.

Definition 2.7. *[25] Let (X, d) be a metric space, and let $\alpha : X \times X \to \mathbb{R}$ be a function. A map $f : X \to X$ is called α-Geraghty contraction if there exists $\beta \in \mathcal{F}$ such that for all $x, y \in X$,*

$$\alpha(x, y)d(fx, fy) \leq \beta(d(x, y))d(x, y).$$

Theorem 2.8. *[25] Let (X, d) be a complete metric space, $\alpha : X \times X \to \mathbb{R}$ be a function. Define a map $f : X \to X$ satisfying the following conditions:*

1. *f is continuous α-Geraghty contraction;*
2. *f be a triangular α-admissible;*
3. *there exists $x_1 \in X$ such that $\alpha(x_1, fx_1) \geq 1$.*

Then f has a fixed point $x \in X$, and $\{f^n x\}$ converges to x.

Definition 2.9. *[26] Let (A, B) be a pair of non-empty subsets of a metric space (X,d) with $A_0 \neq \emptyset$.Then the pair (A, B) is said to have the P-property if and only if for any $x_1, x_2, x_3, x_4 \in A_0$,*

$$\left.\begin{array}{l} d(x_1, fx_3) = d(A, B) \\ d(x_2, fx_4) = d(A, B) \end{array}\right\} \Rightarrow d(x_1, x_2) = d(fx_3, fx_4).$$

Definition 2.10. *[27] Let (A, B) be a part of nonempty subsets of a metric space (X, d) with $A_0 \neq \emptyset$. Then the pair (A, B) is said to have weak P-property if and only if for any $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$*

$$\left.\begin{array}{l} d(x_1, y_1) = d(A, B) \\ d(x_2, y_2) = d(A, B) \end{array}\right\} \Rightarrow d(x_1, x_2) \leq d(y_1, y_2).$$

Definition 2.11. *[20] Given a non-self mapping $f : A \to B$, then an element x^* is called best proximity point of the mappings if this condition satisfied:*

$$d(x^*, fx^*) = d(A, B).$$

We denote by $BPP(f)$, the set of best proximity points of f.

Definition 2.12. *[28] Let (X, d) be a metric space and $A, B \subseteq X$, let $g : A \to A$ and $f : A \to B$ be mappings then a point $x \in A$ is a best proximity coincidence point of the pair (g, f) if $d(gx, fx) = d(A, B)$.*

3. Main Result

In this section, we prove the existence and uniqueness of best proximity coincidence point for α-Geraghty contraction in the field of complete metric space.

Our first result is the the existence of best proximity coincidence point for α-Geraghty contraction mappings.

Definition 3.1. *Let (X, d) be a metric space, and let $\alpha : X \times X \to \mathbb{R}$ be a function. A map $S : A \to B$ is called α-Geraghty contraction if there exists $\beta \in \mathcal{F}$ such that for all $x, y \in A$,*

$$\alpha(x, y)d(Sx, Sy) \leq \beta(d(x, y))d(x, y),$$

where $A, B \subseteq X$.

Now, we are in a position to prove our main result.

Theorem 3.2. *Let A, B be two nonempty closed subsets of a complete metric space (X, d) such that A_0 is nonempty and $g : A \to A$ is an isometry such that $A_0 \subseteq g(A_0)$, let $\alpha : A \times A \to \mathbb{R}$ be a function. Define a map $f : A \to B$ satisfying the following conditions:*

1. *f is continuous α-Geraghty contraction;*
2. *f be an α-proximal admissible;*
3. *for each $x, y \in A_0$ satisfying $d(x, f(y)) = d(A, B)$ and $\alpha(y, x) \geq 1$;*
4. *$f(A_0) \subseteq B_0$ and the pair (A, B) has the P-property.*

Then there exists x^* in A such that $d(gx^*, fx^*) = d(A, B)$.

Proof. Since A_0 is nonempty, we take $x_0 \in A_0$, since $f(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$, there exists $x_1 \in A_0$ such that

$$d(gx_1, fx_0) = d(A, B) \text{ and } \alpha(x_0, x_1) \geq 1. \tag{1}$$

Again, since $f(A_0) \subseteq B_0$, there exists $x_2 \in A_0$ such that

$$d(gx_2, fx_1) = d(A, B). \tag{2}$$

From (1), (2) and f be an α-proximal admissible, we obtain $\alpha(gx_1, gx_2) \geq 1$.
Since g is an isometry so

$$\alpha(x_1, x_2) \geq 1.$$

Thus we have

$$d(gx_2, fx_1) = d(A, B) \text{ and } \alpha(x_1, x_2) \geq 1. \tag{3}$$

Again, since $f(A_0) \subseteq B_0$, there exists $x_3 \in A_0$ such that

$$d(gx_3, fx_2) = d(A, B). \tag{4}$$

From (3), (4) and f be an α-proximal admissible, we obtain $\alpha(gx_2, gx_3) \geq 1$.
Since g is an isometry so

$$\alpha(x_2, x_3) \geq 1.$$

Repeating this process, we get a sequence $\{gx_n\}$ in A_0 satisfying

$$d(gx_{n+1}, fx_n) = d(A, B) \text{ with } \alpha(x_n, x_{n+1}) \geq 1,$$

for any $n \in \mathbb{N}$.
Since (A, B) has the P-property, we have that

$$d(gx_n, x_{n+1}) = d(fx_{n-1}, fx_n)$$

for any $n \in \mathbb{N}$.
Taking into account that f is α-Geraghty contraction and (A, B) has the P-property, so for any $n \in \mathbb{N}$, we have that

$$\begin{aligned} d(gx_n, gx_{n+1}) &= d(x_n, x_{n+1}) \\ &= d(fx_{n-1}, fx_n) \\ &\leq \alpha(x_{n-1}, x_n) d(fx_{n-1}, fx_n) \\ &\leq \beta(d(x_{n-1}, x_n)) d(x_{n-1}, x_n) \\ &< d(x_{n-1}, x_n) \\ &= d(gx_{n-1}, gx_n) \end{aligned}$$

where $\beta(d(x_{n-1}, x_n)) < 1$, g is an isometry and $\alpha(x_{n-1}, x_n) \geq 1$.

$$\Rightarrow d(gx_n, gx_{n+1}) < d(gx_{n-1}, gx_n),$$

so $\{d(gx_n, gx_{n+1})\}$ is strictly decreasing sequence of non-negative real numbers.
Suppose that there exists $n_0 \in \mathbb{N}$ such that $d(gx_{n_0}, gx_{n_0+1}) = 0$. In this case,

$$d(gx_{n_0}, gx_{n_0+1}) = 0 = d(x_{n_0}, x_{n_0+1}) = d(fx_{n_0-1}, fx_{n_0}),$$

and consequently

$$fx_{n_0-1} = fx_{n_0}.$$

Therefore,

$$d(A,B) = d(gx_{n_0}, fx_{n_0-1}) = d(gx_{n_0}, fx_{n_0}).$$

Note that $x_0 \in A_0,\ x_1 \in B_0$, and $x_0 = x_1$ so $A \cap B$ is non-empty, then $d(A,B) = 0$. Thus in this case, there exists unique best proximity coincidence point, i.e. there exists unique x^* in A for the pair (g, f) such that $d(gx^*, fx^*) = d(A,B)$.
In the contrary case, suppose that $d(fx_{n_0}, fx_{n_0-1}) > 0$ this implies that $d(gx_n, gx_{n+1}) > 0$ for any $n \in N$. Since $\{d(gx_n, gx_{n+1})\}$ is strictly decreasing sequence of nonnegative real numbers and hence there exists $r \geq 0$ such that

$$\lim_{n\to\infty} d(gx_n, gx_{n+1}) = \lim_{n\to\infty} d(x_n, x_{n+1}) = r.$$

We have to show that $r = 0$. Let $r \neq 0$ and $r > 0$, then from above inequality and since g is an isometry, we have

$$0 < \frac{d(gx_n, gx_{n+1})}{d(x_{n-1}, x_n)} \leq \beta(d(x_{n-1}, x_n)) < 1,$$

for any $n \in \mathbb{N}$. Which yields that

$$\lim_{n\to\infty} \beta(d(x_{n-1}, x_n)) = 1,$$

since $\beta \in \mathcal{F}$, above equation implies that

$$\lim_{n\to\infty} d(x_{n-1}, x_n) = 0,$$

where g is an isometry, so

$$\lim_{n\to\infty} d(gx_{n-1}, gx_n) = 0,$$

Hence $r = 0$ and this contradicts our assumption that $r > 0$. Therefore,

$$\lim_{n\to\infty} d(gx_n, gx_{n+1}) = 0.$$

Since $d(x_{n+1}, fx_n) = d(A,B)$ for any $n \in \mathbb{N}$, for fixed $p, q \in \mathbb{N}$, we have

$$d(gx_p, fx_{p-1}) = d(gx_q, fx_{q-1}) = d(A,B)$$

and since (A,B) satisfies P-property, so

$$d(gx_p, gx_q) = d(fx_{p-1}, fx_{q-1}).$$

Now we have to show that $\{gx_n\}$ is a Cauchy sequence.
On contrary, suppose that $\{gx_n\}$ is not a Cauchy sequence. Then there exists $\epsilon > 0$ such that for all $k > 0$, there exists $m(k) > n(k) > k$ with (the smallest number satisfying the condition below)

$$d(gx_{m(k)}, gx_{n(k)}) \geq \epsilon \ \text{ and } \ d(gx_{m(k)-1}, gx_{n(k)}) < \epsilon.$$

Then, we have

$$\begin{aligned} \epsilon &\leq d(gx_{m(k)}, gx_{n(k)}) \\ &\leq d(gx_{m(k)}, gx_{m(k)-1}) + d(gx_{m(k)-1}, gx_{n(k)}) \\ &< d(gx_{m(k}, gx_{m(k)-1}) + \epsilon. \end{aligned}$$

This implies that

$$\epsilon < d(gx_{m(k)}, gx_{m(k)-1}) + \epsilon. \tag{5}$$

Let $k \to \infty$ in the above inequality, we have

$$\lim_{k\to\infty} d(gx_{m(k)}, gx_{n(k)}) = \epsilon. \tag{6}$$

Now by using Triangular inequality, we have

$$d(gx_{m(k)}, gx_{n(k)}) \leq d(gx_{m(k)}, gx_{m(k)-1}) + d(gx_{m(k)-1}, gx_{n(k)-1}) + d(gx_{n(k)-1}, gx_{n(k)}).$$

Take limit on both sides, we get

$$\begin{aligned}\lim_{k\to\infty} d(gx_{m(k)-1}, gx_{n(k)-1}) \quad \geq \quad & \lim_{k\to\infty} d(gx_{m(k)}, gx_{n(k)}) - \lim_{k\to\infty} d(gx_{m(k)}, gx_{m(k)-1}) \\ & - \lim_{k\to\infty} d(gx_{n(k)-1}, gx_{n(k)}).\end{aligned}$$

By using (5) and (6), we obtain

$$\lim_{k\to\infty} d(gx_{m(k)-1}, gx_{n(k)-1}) = \epsilon.$$

Since $\alpha(x_{n(k)-1}, x_{m(k)-1}) \geq 1$, we have

$$\begin{aligned} d(gx_{m(k)}, gx_{n(k)}) &= d(x_{m(k)}, x_{n(k)}) \\ &= d(fx_{m(k)-1}, fx_{n(k)-1}) \\ &\leq \alpha(x_{n(k)-1}, x_{m(k)-1}) d(fx_{n(k)-1}, fx_{m(k)-1}) \\ &\leq \beta(d(x_{n(k)-1}, x_{m(k)-1})) d(x_{n(k)-1}, x_{m(k)-1}). \end{aligned}$$

Since g is an isometry. It follows that

$$\frac{d(gx_{m(k)}, gx_{n(k)})}{d(gx_{n(k)-1}, gx_{m(k)-1})} \leq \beta(d(x_{n(k)-1}), x_{m(k)-1}).$$

Letting $m, n \to \infty$ in the above inequality, we get

$$\lim_{k\to\infty} \beta(d(x_{n(k)-1}, x_{m(k)-1})) = 1,$$

and so

$$\lim_{n\to\infty} d(x_{n(k)-1}, x_{m(k)-1}) = 0.$$

Then one can write as $\lim_{n\to\infty} d(gx_{n(k)-1}, gx_{m(k)-1}) = 0$ Hence $\epsilon = 0$, which contradicts our supposition that $\epsilon > 0$. So we conclude that $\{gx_n\}$ is a Cauchy sequence in A. Since $\{gx_n\} \subseteq A$ and A is closed subset of a complete metric space (X, d). There is $x^* \in A$ such that $gx_n \to gx^*$ as $n \to \infty$. Since f is continuous, so we have

$$\begin{aligned} & fx_n \to fx^*. \\ & \Rightarrow d(gx_{n+1}, fx_n) \to d(gx^*, fx^*). \end{aligned}$$

Taking into account that $\{d(gx_{n+1}, fx_n)\}$ is a constant sequence with a value $d(A, B)$, we deduce

$$d(gx^*, fx^*) = d(A, B),$$

i.e., x^* is best proximity coincidence point of the pair (g, f). □

Remark 3.3. *The condition A and B are nonempty closed subsets of the metric space (X, d) is not a necessary condition for the existence of the unique best proximity coincidence point for α-Geraghty contraction $f : A \to B$ and $g : A \to A$. Since for any nonempty subset A of X, the pair (A, B) satisfied the (weak P-property) P-property, we have the following corollary.*

Corollary 3.4. *Let A be a nonempty closed subsets of a complete metric space (X, d) such that A_0 is nonempty, $\alpha : X \times X \to \mathbb{R}$ be a function. Define a map $f : A \to A$ and $g : A \to A$ such that $g(A) \subseteq A$ satisfying the following conditions:*

1. *f is continuous α-Geraghty contraction;*
2. *f be an α-proximal admissible;*
3. *g is an isometry;*
4. *for each $x, y \in A_0$ satisfying $d(x, f(y)) = d(A, B)$ and $\alpha(y, x) \geq 1$.*

Then (g, f) has a fixed coincidence point x^* in A and f is a Picard operator, that is, $f^n(x)$ converges to x^*.

Proof. Following Theorem 3.2 by taking $A = B$, we obtained the desired result. □

Corollary 3.5. *Let A, B be two nonempty closed subsets of a complete metric space (X, d) such that A_0 is nonempty. Define a map $f : A \to B$ and an isometry $g : A \to A$ such that $g(A) \subseteq A$ satisfying the following conditions:*

1. *f is continuous;*
1. *f is Geraghty contraction;*
2. *$f(A_0) \subseteq B_0$ and the pair (A, B) has the P-property.*

Then there exists a unique x^* in A such that $d(gx^*, fx^*) = d(A, B)$.

Proof. From Theorem 3.2, we put $\alpha(x, y) = 1$, then we get desired result. □

Corollary 3.6. *Let A be a nonempty closed subsets of a complete metric space (X, d) such that A_0 is nonempty. Define a map $f : A \to A$ such that f is continuous Geraghty contraction and an isometry $g : A \to A$ such that $g(A) \subseteq A$. Then (g, f) has a fixed coincidence point x^* in A and f is a Picard operator, that is, $f^n(x)$ converges to x^*.*

Proof. In Corollary 3.4, taking $\alpha(x, y) = 1$ we have the desire result. □

Continuity of the mapping f can be omitted Theorem 3.2. We replace continuity of f with a suitable condition as follows:

(H) if $\{x_n\}$ is a sequence in A such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$,then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geq 1$ for all k.

Theorem 3.7. *Let A, B be two nonempty closed subsets of a complete metric space (X, d) such that A_0 is nonempty, $\alpha : X \times X \to \mathbb{R}$ be a function. Define a map $f : A \to B$ and $g : A \to A$ is an isometry, satisfying the following conditions:*

1. *f is α-Geraghty contraction;*
2. *f be an α-proximal admissible;*
3. *for each $x, y \in A_0$ satisfying $d(x, f(y)) = d(A, B)$ and $\alpha(y, x) \geq 1$;*
3. *$f(A_0) \subseteq B_0$ and the pair (A, B) has the P-property;*
4. (H) *holds.*

Then there exists x^* in A such that $d(gx^*, fx^*) = d(A, B)$.

Proof. Following Theorem 3.2, we have $\{gx_n\}$ is a Cauchy sequence such that $gx_n \to gx$ as $n \to \infty$.
Let $x_{m+1}, x_{n+1} \in A_0$ and $fx_m, fx_n \in B_0$, such that

$$d(gx_{m+1}, fx_m) = d(A, B)$$

and

$$d(gx_{n+1}, fx_n) = d(A, B),$$

where g is an isometry. Then by P-property, we obtain

$$d(gx_{m+1}, gx_{n+1}) = d(fx_m, fx_n).$$

So for $x_n, x_{n+1} \in A_0$,we have

$$\begin{aligned} d(gx_{n+1}, gx_n) &= d(x_{n+1}, x_n) \\ &= d(fx_n, fx_{n-1}) \\ &\leq \alpha(x_n, x_{n-1})d(fx_n, fx_{n-1}) \\ &\leq \beta(d(x_n, x_{n-1}))d(x_n, x_{n-1}) \\ &< d(x_n, x_{n-1}) = d(fx_{n-1}, fx_{n-2}), \end{aligned}$$

this implies $\{fx_n\}$ is a Cauchy sequence and $\{fx_n\} \to z$.
Thus $\{d(gx_n, gx)\} \to 0$ as $n \to \infty$, $\{d(fx_n, fx_{n-1})\} \to 0$ as $n \to \infty$ and

$$d(gx_{n+1}, fx_n) = d(A, B).$$

Taking limit as $n \to \infty$, we get

$$d(gx, z) = d(A, B).$$

Take a subsequence $\{gx_{n(k)}\}$ of $\{gx_n\}$, and $\alpha(x_{n(k)}, x) \geq 1$. It follows that

$$d(fx_{n(k)}, fx) \leq \alpha(x_{n(k)}, x)d(fx_{n(k)}, fx) \leq \beta(x_{n(k)}, x)d(gx_{n(k)}, gx).$$

By applying the limit $k \to \infty$

$$d(z, fx) = 0.$$

Thus $d(gx, fx) = d(A, B)$. □

Next result, we will give sufficient conditions in order to prove the uniqueness of best proximity coincidence point.

Definition 3.8. *Let $f : A \to B$, $\alpha : X \times X \to [0, \infty)$ be two mappings. A mapping f is called α-regular if for all $x, y \in A_0$ such that $\alpha(x, y) < 1$, there exists $z \in A_0$ such that $\alpha(x, z) \geq 1$ and $\alpha(y, z) \geq 1$.*

Theorem 3.9. *Under the hypothesis of Theorem 3.2 (resp. Theorem 3.7), assume that f is α-regular. Then for all best proximity coincidence points x^* and y^* of f in A_0 we have that $x^* = y^*$: In particular, f has a unique best proximity coincidence point.*

Proof. Let $x^*, y^* \in A_0$ be two best proximity coincidence points of f in A_0. Then $d(gx^*, fx^*) = d(gy^*, fy^*) = d(A, B)$ and f has P−property, we deduce that

$$d(gx^*, gy^*) = d(fx^*, fy^*).$$

We consider two cases:
Case-I: If $\alpha(x^*, y^*) \geq 1$. Using the fact that f is α-Geraghty contraction, we have

$$\begin{aligned} d(gx^*, gy^*) = d(x^*, y^*) = d(fx^*, fy^*) &\leq \alpha(x^*, y^*)d(fx^*, fy^*) \\ &\leq \beta(d(x^*, y^*))d(x^*, y^*) \\ &< d(gx^*, gy^*) \end{aligned}$$

$$\Rightarrow d(gx^*, gy^*) < d(gx^*, gy^*),$$

which is contradiction. So

$$gx^* = gy^*.$$

By g is isometry, so $x^* = y^*$.

Case-II: If $\alpha(x^*, y^*) < 1$, then by the α-regularity of f, there exists $z_0 \in A_0$ such that $\alpha(x^*, z_0) \geq 1$ and $\alpha(y^*, z_0) \geq 1$. Based on z_0, we define a sequence $\{z_n\}$ and suppose that z_n converges to x and y, which proves the uniqueness. First, we shall prove that $\{gz_n\}$ converges to gx. Indeed, $fz_0 \in fA_0 \subseteq B_0$ implies that $z_1 \in A_0$ such that $d(gz_1, fz_0) = d(A, B)$. Now, we have

$$\alpha(x^*, z_0) \geq 1, \; d(gx^*, fx^*) = d(gz_1, fz_0) = d(A, B).$$

Since T is $\alpha-$proximal admissible, we get that $\alpha(gx^*, gz_1) \geq 1$.
Since g is an isometry, we get that $\alpha(x^*, z_1) \geq 1$.
Continuing this process, by induction, we can construct a sequence $\{gz_n\} \subseteq A_0$ such that

$$d(gz_{n+1}, fz_n) = d(A, B) \text{ and } \alpha(x^*, z_n) \geq 1 \text{ for all } n \geq 0.$$

Since $d(gz_{n+1}, fz_n) = d(gx^*, fx^*) = d(A, B)$, using the P-property, we obtain that

$$d(gx^*, gz_{n+1}) = d(fx^*, fz_n)$$

for all $n \geq 0$. Since f is α-Geraghty contraction, we have

$$\begin{aligned} d(gx^*, gz_{n+1}) &= d(fx^*, fz_n) \\ &\leq \alpha(x^*, z_n) d(fx^*, fz_n) \\ &\leq \beta(d(x^*, z_n)) d(x^*, z_n) \\ &< d(x^*, z_n). \end{aligned}$$

Since g is an isometry, so $d(gx^*, gz_n) = d(x^*, z_n)$, which shows that $\{d(gx^*, gz_{n+1})\}$ is a decreasing sequence of nonnegative real numbers, and there exists $r \geq 0$ such that $\lim_{n\to\infty} d(gx^*, gz_{n+1}) = r$. Assume $r > 0$, then we have

$$0 < \frac{d(gx^*, gz_{n+1})}{d(gx^*, gz_n)} \leq \beta(d(x^*, z_n)) < 1,$$

for any $n \in \mathbb{N}$.
The last inequality implies that $\lim_{n\to\infty} \beta(d(gx^*, gz_n)) = 1$ and since $\beta \in \mathcal{F}$ so $r = 0$ and this contradicts our assumption. Therefore $\lim_{n\to\infty} d(gx^*, gz_{n+1}) = 0$, that is $gz_{n+1} \to gx^*$ as $n \to \infty$
Repeating this argument,we have that $gz_n \to gx^*$ as $n \to \infty$, which proves that $\{gz_n\}$ is a sequence converging to gx^*. Similarly gz_n converges to gy^*. By uniqueness of limit we have $gx^* = gy^*$. Therefore $x^* = y^*$. □

Example 3.10. *Consider $X = \mathbb{R}^2$, with the usual metric d. Let $A = \{0\} \times [1, \infty)$ and $B = \{1\} \times [0, \infty)$. Obviously, $d(A, B) = 1$ and A, B are nonempty subsets of X, take $A_0 = A$ and $B_0 = B$.*
We define $f : A \to B$ as:

$$f(0, x) = (1, \ln x),$$

where $(0, x) \in A$ and $\ln x \in [0, \infty)$ and $g : A \to A$ is defined as $gx = x$.
Let $\alpha : \mathbb{R}^2 \times \mathbb{R}^2 \to [0, \infty)$ defined as:

$$\alpha((x_1, y_1), (x_2, y_2)) = \begin{cases} 1 & \text{if } 0 = x_1 = x_2 \text{ and } \infty > y_1, y_2 \geq 0, \\ 0 & \text{elsewhere.} \end{cases}$$

Clearly, for $(0, x_1) \in A_0$ one has $d((0, x_1), f(0, x_1)) = d(A, B)$ which implies that $\alpha((0, x_1), f(0, x_1)) = 1$. Also f is *α-Geraghty contraction as: for* $(0,x),(0,y) \in A$ *with* $x \neq y$ *and* $x > y$*, we have*

$$\begin{aligned}
\alpha((0,x),(0,y))d(f(0,x),f(0,y)) &= 1 \cdot d(f(0,x),f(0,y)) \\
&= |\ln(x) - \ln(y)| \\
&= \left|\ln\left(\frac{x}{y}\right)\right| \\
&= \left|\ln\left(\frac{(y)+(x-y)}{y}\right)\right| \\
&= \left|\ln\left(1+\frac{x-y}{y}\right)\right| \\
&\leq \ln(1+|x-y|) \\
&= \frac{\ln(1+|x-y|)}{|x-y|} \cdot |x-y| \\
&= \frac{\ln(1+d((0,x),(0,y)))}{d((0,x),(0,y))} \cdot d((0,x),(0,y)).
\end{aligned}$$

Take $\phi(t) = \ln(t)$ *for* $t \geq 0$*, we have*

$$\alpha((0,x),(0,y))d(f(0,x),f(0,y)) \leq \frac{\phi(d(0,x),d(0,y))}{d((0,x),(0,y))} \cdot d((0,x),(0,y)).$$

Setting $\beta(t) = \frac{\phi(t)}{t}$ *for* $t > 0$*, and* $\beta(0) = 0$*, we have*

$$\alpha((0,x),(0,y))d(f(0,x),f(0,y)) \leq \beta(d((0,x),(0,y))) \cdot d((0,x),(0,y)).$$

Obviously, when $x = y$ *the inequality is satisfied. Also* $\beta(t) = \frac{\ln(t)}{t} \in \mathcal{F}$*, by elementary calculus.*
The pair (A,B) *satisfied P-property. Thus, by Theorem 3.2 and Theorem 3.9, we get* $(0,1)$ *is the unique best proximity coincidence point of* (g,f)*.*

4. Some Applications

As applications of our main results, we can have some coincidence best proximity point and coupled coincidence proximity point on metric spaces endowed with an arbitrary binary relation.

4.1. Best proximity coincidence point on metric spaces endowed with an arbitrary binary relation.

Before presenting our results, we need a few preliminaries. Let (X,d) be a metric space and $\mathcal{R}$ be a binary relation over X. Denote

$$\mathcal{S} = \mathcal{R} \cup \mathcal{R}^{-1};$$

this is the symmetric relation attached to $\mathcal{R}$. Clearly,

$$x,y \in X,\ x\mathcal{R}y \Longleftrightarrow x\mathcal{R}y \text{ or } y\mathcal{R}x.$$

Definition 4.1. *[24] We say that* $f : A \to B$ *is a proximal comparative mapping if*

$$\left.\begin{array}{l} x_1\mathcal{S}x_2 \\ d(u_1,fx_1) = d(A,B) \\ d(u_2,fx_2) = d(A,B) \end{array}\right\} \Longrightarrow u_1\mathcal{S}u_2,$$

for all $x_1,x_2,u_1,u_2 \in A$*.*

We also need the following hypothesis (H):

(H_B) if $\{x_n\}$ is a sequence in X and the point $x \in X$ are such that $x_n \mathcal{S} x_{n+1}$ for all n and $\lim_{n\to\infty} d(x_n, x) = 0$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $x_{n(k)} \mathcal{S} x$ for all k.

We have the following from coincidence best proximity point results.

Theorem 4.2. *Let A, B be two nonempty closed subsets of a complete metric space (X, d) such that A_0 is nonempty. Let $\mathcal{R}$ be a binary relation over X and $g : A \to A$ is isometry such that $A_0 \subseteq g(A_0)$. Define a map $f : A \to B$ is a continuous non-self-mapping satisfying the following conditions:*

1. *There exists $\beta \in \mathcal{F}$ such that*

$$x, y \in A,\ x\mathcal{S}y \Longrightarrow d(fx, fy) \leq \mathcal{B}(d(x, y))d(x, y);$$

2. *f be a proximal comparative mapping;*
3. *for each $x, y \in A_0$ satisfying $d(x, f(y)) = d(A, B)$ and $y\mathcal{S}x$;*
4. *$f(A_0) \subseteq B_0$ and the pair (A, B) has the $P-$property.*

Then there exists x^* in A such that $d(gx^*, fx^*) = d(A, B)$.

Proof. Define the mapping $\alpha : X \times X \to [0, \infty)$ by:

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x\mathcal{S}y, \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

Suppose that

$$\begin{cases} \alpha(x_1, x_2) \geq 1, \\ d(u_1, fx_1) = d(A, B), \\ d(u_2, fx_2) = d(A, B), \end{cases}$$

for some $x_1, x_2, u_1, u_2 \in A$. By the definition of α, we get that

$$\begin{cases} x_1\mathcal{S}x_2, \\ d(u_1, fx_1) = d(A, B), \\ d(u_2, fx_2) = d(A, B). \end{cases}$$

Condition (2) implies that $u_1\mathcal{S}u_2$, which gives us from the definition of α that $\alpha(u_1, u_2) \geq 1$.
Thus we proved that f is $\alpha-$proxiaml admissible. Condition (3) implies that

$$d(x, f(y)) = d(A, B) \text{ and } \alpha(y, x) \geq 1.$$

Finally, condition (1) implies that

$$\alpha(x, y)d(fx, fy) \leq \beta(d(x, y))d(x, y), \quad \forall x, y \in A$$

that is, f is an $\alpha-$Geraghty contraction. Now all the hypotheses of Theorem 3.2 are satisfied, and the desired results follows immediately form this theorem. □

Theorem 4.3. *Let (X, d) be two nonempty closed subsets of a complete metric space (X, d) such that A_0 is nonempty. Let $\mathcal{R}$ be a binary relation over X and $g : A \to A$ is isometry such that $A_0 \subseteq g(A_0)$. Define a map $f : A \to B$ is a non-self-mapping satisfying the following conditions:*

1. *There exists $\beta \in \mathcal{F}$ such that*

$$x, y \in A,\ x\mathcal{S}y \Longrightarrow d(fx, fy) \leq \mathcal{B}(d(x, y))d(x, y);$$

2. *f be a proximal comparative mapping;*

3. *for each $x, y \in A_0$ satisfying $d(x, f(y)) = d(A, B)$ and $y\mathcal{S}x$;*
4. *$f(A_0) \subseteq B_0$ and the pair (A, B) has the $P-$property;*
5. *(H_B) holds.*

Then there exists x^* in A such that $d(gx^*, fx^*) = d(A, B)$.

Proof. The result follows form Theorem 3.7 by considering the mapping α give by (7) and by observing that condition (H) implies condition $(\mathrm{H_B})$. □

Theorem 4.4. *Under the hypothesis of Theorem 4.2 (resp. Theorem 4.3), assume that the following condition holds: for all $(x, y) \in A \times A$ with $(x, y) \notin \mathcal{S}$, there exists $z \in A_0$ such that $x\mathcal{S}z$ and $y\mathcal{S}z$. Then for all best proximity coincidence points x^* and y^* of f in A_0 we have that $x^* = y^*$: In particular, f has a unique best proximity coincidence point.*

Proof. The result follows form Theorem 3.9 by considering the mapping α give by (7) and by observing that condition (H) implies condition $(\mathrm{H_B})$. □

4.2. Best proximity coupled coincidence point on metric spaces endowed with an arbitrary binary relation

We continuous the use of the notations of the previous subsection. Let $F : A\times A \to B$ be a given mapping and $g : A \to A$.

Definition 4.5. *We say that $(x^*, y^*) \in A \times A$ is is coupled coincidence best proximity point of F ang g if*

$$d(gx^*, F(x^*, y^*)) = d(gy^*, F(x^*, y^*)) = d(A, B).$$

We will use the following notations:

$$\mathcal{X} := X \times X, \quad \mathcal{A} := A \times A, \quad \mathcal{B} := B \times B.$$

Define the non-self-mapping $f : \mathcal{A} \to \mathcal{B}$ by:

$$f(x, y) = (F(x, y), F(y, x)), \quad \forall (x, y) \in \mathcal{A}.$$

We endow the product set $\mathcal{X}$ with the metric d_2 given by:

$$d_2((x, y), (u, v)) = \frac{d(x, u) + d(y, v)}{2}, \quad \forall (x, y), (u, v) \in \mathcal{X}.$$

Clearly, if (X, d) is complete, then $(\mathcal{X}, d_2)$ is complete.

Definition 4.6. *[24] We say that $F : A \times A \to B$ is bi-proximal comparative mapping if*

$$\left.\begin{array}{l} x_1\mathcal{S}x_2,\ y_1\mathcal{S}y_2 \\ d(u_1, F(x_1, y_1) = d(A, B), \\ d(u_2, F(x_2, y_2) = d(A, B). \end{array}\right\} \Longrightarrow u_1\mathcal{S}u_2,$$

for all $x_1, x_2, y_1, y_2, u_1, u_2 \in A$.

We have the following coupled coincidence best proximity point result.

Theorem 4.7. *Let A, B be two nonempty closed subsets of a complete metric space (X, d) such that A_0 is nonempty. Let $\mathcal{R}$ be a binary relation over X and $g : A \to A$ is isometry such that $A_0 \subseteq g(A_0)$. Define a map $F : A \times A \to B$ is a continuous non-self-mapping satisfying the following conditions:*

1. *There exists $\beta \in \mathcal{F}$ such that $x, y, u, v \in A$,*
$$x\mathcal{S}u,\ y\mathcal{S}v \Longrightarrow d(F(x, y), F(u, v)) \le \mathcal{B}\left(\frac{d(x, u) + d(y, v)}{2}\right)\left(\frac{d(x, u) + d(y, v)}{2}\right);$$

2. *F be a bi-proximal comparative mapping;*
3. *for each $x, y, u, v \in A_0$ satisfying*
   $d(u, F(x, y)) = d(v, F(y, x)) = d(A, B)$ and $x\mathcal{S}u,\ y\mathcal{S}v$;
4. *$F(A_0 \times A_0) \subseteq B_0$ and the pair (A, B) has the $P-$property.*

Then there exists x^*, y^* in A such that $d(gx^*, F(x^*, y^*)) = d(gy^*, F(y^*, x^*)) = d(A, B)$.

Proof. Define the binary relation $\mathcal{R}_2$ over $\mathcal{X}$ by:

$$(x, y),\ (u, v) \in \mathcal{X},\ (x, y)\mathcal{R}_2(u, v) \Longleftrightarrow x\mathcal{S}u,\ y\mathcal{S}v.$$

If we denote by $\mathcal{S}_2$ the symmetric relation attached to $\mathcal{R}_2$, clearly, we have $\mathcal{S}_2 = \mathcal{R}_2$.

We claim that $f : \mathcal{A} \to \mathcal{B}$ and $g : \mathcal{A} \to \mathcal{A}$ is an isometry such that $\mathcal{A}_0 \subseteq g(\mathcal{A}_0)$ have a coincidence best proximity point $(x^*, y^*) \in A_0 \times A_0$, that is, there exists $(x^*, y^*) \in A_0 \times A_0$ such that

$$d_2(g(x^*, y^*), f(x^*, y^*)) = d(\mathcal{A}, \mathcal{B}). \tag{8}$$

Denote by:

$$\mathcal{A}_0 := \{(a_1, a_2) \in \mathcal{A} : d_2((a_1, a_2), (b_1, b_2)) = d_2(\mathcal{A}, \mathcal{B}) \text{ for some } (b_1, b_2) \in \mathcal{B}\};$$

$$\mathcal{B}_0 := \{(b_1, b_2) \in \mathcal{A} : d_2((a_1, a_2), (b_1, b_2)) = d_2(\mathcal{A}, \mathcal{B}) \text{ for some } (a_1, a_2) \in \mathcal{A}\}.$$

We can observe that

$$d_2(\mathcal{A}, \mathcal{B}) = d(A, B).$$

In fact, we have

$$\begin{aligned} d_2(\mathcal{A}, \mathcal{B}) &= \inf\{d_2((a_1, a_2), (b_1, b_2)) : (a_1, a_2) \in \mathcal{A},\ (b_1, b_2) \in \mathcal{B}\} \\ &= \frac{1}{2}\inf\{d(a_1, b_1) + d(a_2, b_2) : (a_1, b_1) \in A \times B,\ (a_2, b_2) \in A \times B\} \\ &= \frac{1}{2}(\inf\{d(a_1, b_1) : (a_1, b_1) \in A \times B\} + \inf\{d(a_2, b_2) : (a_2, b_2) \in A \times B\}) \\ &= \frac{1}{2}(d(A, B) + d(A, B)) \\ &= d(A, B). \end{aligned}$$

Now, let $(a_1, a_2) \in \mathcal{A}_0$. Then there exists $(b_1, b_2) \in \mathcal{B}$ such that

$$d_2((a_1, a_2), (b_1, b_2)) = d_2(\mathcal{A}, \mathcal{B}),$$

that is,

$$d(a_1, b_1) + d(a_2, b_2) = 2d(A, B).$$

Thus we have

$$\begin{cases} d(a_1, b_1) + d(a_2, b_2) = 2d(A, B), \\ d(a_1, b_1) \geq d(A, B), \\ d(a_2, b_2) \geq d(A, B), \end{cases}$$

which implies that

$$d(a_1, b_1) = d(a_2, b_2) = d(A, B).$$

This implies that

$$(a_1, a_2) \in A_0 \times A_0.$$

Similarly, if $(a_1, a_2) \in A_0 \times A_0$, we have $(a_1, a_2) \in \mathcal{A}_0$. Thus we proved that

$$\mathcal{A}_0 = A_0 \times A_0.$$

Similarly, we can show that

$$\mathcal{B} = B_0 \times B_0.$$

Since A_0 is nonempty, then $\mathcal{A}_0$ is nonempty. On the other hand, from (4), we have

$$f(\mathcal{A}_0) = \{(F(x,y), F(y,x)) : (x,y) \in A_0 \times A_0\} \subset F(A_0 \times A_0) \times F(A_0 \times A_0) \subseteq \mathcal{B}.$$

Suppose now that for some $(a_1, a_2), (x_1, x_2) \in \mathcal{A}$ and $(b_1, b_2), (y_1, y_2) \in \mathcal{B}$, we have

$$d_2((a_1, a_2), (b_1, b_2)) = d_2(\mathcal{A}, \mathcal{B}),$$

$$d_2((x_1, x_2), (y_1, y_2)) = d_2(\mathcal{A}, \mathcal{B}).$$

This implies that

$$d(a_1, b_1) = d(a_2, b_2) = d(A, B),$$

$$d(x_1, y_1) = d(x_2, y_2) = d(A, B).$$

Since (A, B) satisfies the P-property, we get that

$$d(a_1, x_1) = d(b_1, y_1) \quad \text{and} \quad d(a_2, x_2) = d(b_2, y_2),$$

which implies that

$$d_2((a_1, a_2), (x_1, x_2)) = d_2((b_1, b_2), (y_1, y_2)).$$

Thus we proved that the $\mathcal{A}, \mathcal{B}$ satisfies the P-property.

Suppose now that for some $(a_1, a_2), (x_1, x_2), (u_1, u_2), (v_1, v_2) \in \mathcal{A}$, we have

$$\begin{aligned} &(a_1, a_2)\mathcal{S}_2(x_1, x_2), \\ &d_2((u_1, u_2), f(a_1, a_2)) = d_2(A, B), \\ &d_2((v_1, v_2), f(x_1, x_2)) = d_2(A, B). \end{aligned}$$

This implies that

$$\begin{aligned} &a_1\mathcal{S}x_1, \ a_2\mathcal{S}x_2, \\ &d(u_1, F(a_1, a_2)) = d(A, B), \\ &d(v_1, F(x_1, x_2)) = d(A, B), \end{aligned}$$

and

$$\begin{aligned} &a_2\mathcal{S}x_2, \ a_1\mathcal{S}x_1, \\ &d(u_2, F(a_2, a_1)) = d(A, B), \\ &d(v_2, F(x_2, x_1)) = d(A, B). \end{aligned}$$

Since F is a bi-proximal comparative mapping, we get that

$$u_1\mathcal{S}v_1 \quad \text{and} \quad u_2\mathcal{S}v_2,$$

that is, $(u_1, u_2)\mathcal{S}_2(v_1, v_2)$. Thus we proved that f is a proximal comparative mapping. Now, from condition (3), we have

$$d(x_1, F(x_0, y_0)) + d(y_1, F(y_0, x_0)) = 2d(A, B) \quad \text{and} \quad (x_0, y_0)\mathcal{S}_2(x_1, y_1),$$

which implies that

$$d_2((x_1, y_1), f(x_0, y_0)) = d_2(\mathcal{A}, \mathcal{B}) \quad \text{and} \quad (x_0, y_0)\mathcal{S}_2(x_1, y_1).$$

Moreover, if $(x, y), (u, v) \in \mathcal{A}$ are such that $(x, y)\mathcal{S}_2(u, v)$, that is $x\mathcal{S}u$ and $y\mathcal{S}v$, from condition (1), we get that

$$d(F(x, y), F(u, v)) \leq \beta\left(\frac{d(x, u) + d(y, v)}{2}\right)\left(\frac{d(x, u) + d(y, v)}{2}\right) \tag{9}$$

and

$$d(F(y,x),F(v,u)) \leq \beta\left(\frac{d(x,u)+d(y,v)}{2}\right)\left(\frac{d(x,u)+d(y,v)}{2}\right) \quad (10)$$

Adding (9) to (10) we obtain that

$$d_2((F(x,y),F(y,x)),(F(u,v),F(v,u))) \leq \beta(d_2((x,y),(u,v)))d_2((x,y),(u,v)),$$

that is,

$$d_2(f(x,y),f(y,x)) \leq \beta(d_2((x,y),(u,v)))d_2((x,y),(u,v)),$$

Now, all the conditions of Theorem 4.2 are satisfied, we deduce that f and g have coincidence best proximity point $(x^*,y^*) \in \mathcal{A}_0$, that is $(x^*,y^*) \in A_0 \times A_0$ and satisfies

$$d_2(g(x^*,y^*),f(x^*,y^*)) = d_2(\mathcal{A},\mathcal{B}).$$

Thus we proved our claim (8). Since $d_2(\mathcal{A},\mathcal{B}) = d(A,B)$ and g is isometry, the above equality implies immediately that $d(gx^*,F(x^*,y^*)) = d(gy^*,F(y^*,x^*)) = d(A,B)$. This completes the proof. □

Similarly, form Theorem 4.3, we get the following result.

Theorem 4.8. *Let A,B be two nonempty closed subsets of a complete metric space (X,d) such that A_0 is nonempty. Let $\mathcal{R}$ be a binary relation over X and $g : A \to A$ is isometry such that $A_0 \subseteq g(A_0)$. Define a map $F : A \times A \to B$ is a non-self-mapping satisfying the following conditions:*

1. *There exists $\beta \in \mathcal{F}$ such that $x,y,u,v \in A$,*
   $x\mathcal{S}u,\ y\mathcal{S}v \Longrightarrow d(F(x,y),F(u,v)) \leq \mathcal{B}\left(\frac{d(x,u)+d(y,v)}{2}\right)\left(\frac{d(x,u)+d(y,v)}{2}\right);$
2. *F be a bi-proximal comparative mapping;*
3. *for each $x,y,u,v \in A_0$ satisfying*
   $d(u,F(x,y)) = d(v,F(y,x)) = d(A,B)$ and $x\mathcal{S}u,\ y\mathcal{S}v$;
4. *$F(A_0 \times A_0) \subseteq B_0$ and the pair (A,B) has the $P-$property;*
5. (H_B) *holds.*

Then there exists x^*,y^* in A such that $d(gx^*,F(x^*,y^*)) = d(gy^*,F(y^*,x^*)) = d(A,B)$.

Next, we have the following uniqueness result of a coupled coincidence best proximity point.

Theorem 4.9. *Under the hypothesis of Theorem 4.7 (resp. Theorem 4.8), assume that the following condition holds: for all $(x,y) \in A \times A$, there exists $z \in A_0$ such that $x\mathcal{S}z$ and $y\mathcal{S}z$. Then f and g have a unique coincidence best proximity points $(x^*,y^*) \in A \times A$. Moreover, we have $x^* = y^*$.*

Proof. Let $(x,y),(u,v) \in \mathcal{A}$. By hypothesis, there exists $z_1 \in A_0$ such that $x\mathcal{S}z_1$ and $u\mathcal{S}z_1$. Similarly, there exists $z_2 \in A_0$ such that $y\mathcal{S}z_2$ and $v\mathcal{S}z_2$. Then, we have $(x,y)\mathcal{S}_2(z_1,z_2)$ and $(u,v)\mathcal{S}_2(z_1,z_2)$ where $(z_1,z_2) \in \mathcal{A}_0$. Now, applying Theorem 4.4, we obtain that f and g have a unique coincidence best proximity point that is unique coupled coincidence best proximity point of F and g. For the equality $x^* = y^*$ we have only to remark that if (x^*,y^*) is a coupled coincidence best proximity point, then (y^*,x^*) is also a coupled coincidence best proximity point. □

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