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A Sharper Form of Half-Discrete Hilbert Inequality Related to Hardy Inequality

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Abstract. We introduce some new sharper forms of the half-discrete Hilbert inequality which are connected to the Hardy inequality.

1. Introduction

Suppose that $p > 1, \frac{1}{p} + \frac{1}{q} = 1$, f and g are positive functions such that $f \in L_p(\mathbb{R}_+)$ and $g \in L_q(\mathbb{R}_+)$,then the famous Hilbert inequality is given as

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \left\|f\right\|_p \left\|g\right\|_q . \tag{1.1}$$

The constant $\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}$ is the best possible [6]. Inequality (1.1) was extended in different ways.

Refinements of some Hilbert-type inequalities by virtue of various methods are obtained in [7] ,[8] and [9]. A survey of some recent results concerning Hilbert and Hilbert- type inequalities can be found in [17] and [18].

The corresponding discrete form of inequality (1.1) is given for two nonnegative sequences of real numbers $a = \{a_n\}$ and $b = \{b_n\}$ as

$$\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{a_m b_n}{m+n} < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \|a\|_{\ell_p} \|b\|_{\ell_q} , \tag{1.2}$$

where $a \in \ell_p$ and $b \in \ell_q$. Here the constant $\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}$ is also the best possible [6]. For extensions and generalizations of inequality (1.2) see for example [19] and [22].

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In [15] Yang introduced the following half-discrete Hilbert's inequality

$$\int_0^\infty f(x)\sum_{n=1}^\infty \frac{a_n}{(x+n)^\lambda}dx < B(\lambda_1,\lambda_2)\left(\int_0^\infty x^{p(1-\lambda_1)-1}f^p(x)dx\right)^{\frac{1}{p}}\left(\sum_{n=1}^\infty n^{q(1-\lambda_2)-1}a_n^q\right)^{\frac{1}{q}}, \tag{1.3}$$

here, $\lambda_1,\lambda_2 > 0, \lambda_1+\lambda_2=\lambda, 0<\lambda_1<1$, and the constant $B(\lambda_1,\lambda_2)$ is the best possible. In particular, for $\lambda = 1$ and $\lambda_1 = \frac{1}{q}, \lambda_2 = \frac{1}{p}$ we find

$$\int_0^\infty f(x)\sum_{n=1}^\infty \frac{a_n}{x+n}dx < \frac{\pi}{\sin\frac{\pi}{q}}\left(\int_0^\infty f^p(x)dx\right)^{\frac{1}{p}}\left(\sum_{n=1}^\infty a_n^q\right)^{\frac{1}{q}}. \tag{1.4}$$

For extensions and other half-discrete Hilbert's inequalities see for example [13], [14], [16], [20], [21] and [23].

If $p>1, f(x)>0$, and $F(x)=\int_0^x f(t)dt$, then the well-known Hardy inequality [6] is given as

$$\int_0^\infty \left(\frac{F(x)}{x}\right)^p dx < \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)dx, \tag{1.5}$$

where the constant $\left(\frac{p}{p-1}\right)^p$ is the best possible. A weighted form of (1.5) is given also by Hardy [6] as

$$\int_0^\infty x^a \left(\frac{F(x)}{x}\right)^p dx < \left(\frac{p}{p-1-a}\right)^p \int_0^\infty x^a f^p(x)dx, \tag{1.6}$$

where $p>1, a<p-1$ or $p<0, a>p-1$ and the constant $\left(\frac{p}{p-1-a}\right)^p$ is the best possible. Inequality (1.5) was discovered by Hardy while he was trying to introduce a simple proof of the Hilbert inequality. In the book [12] the following Hardy-type inequality is given

$$\int_{-\infty}^\infty e^{kpx}\left(\int_{-\infty}^x f(t)dt\right)^p dx \le (-k)^{-p}\int_{-\infty}^\infty e^{kpx}f^p(x)dx, \tag{1.7}$$

where $k<0$ and $p>1$ or $p<0$. If $0<p<1$, then the reverse form of (1.5) holds. The constant $(-k)^{-p}$ is the best possible.

In [5](see also [12]) the following Hardy-type inequality is obtained for $p>1$

$$\int_1^\infty \frac{1}{x\,[\ln x]^p}\left(\int_1^x f(t)dt\right)^p dx \le \left(\frac{p}{p-1}\right)^p \int_1^\infty x^{p-1}f^p(x)dx. \tag{1.8}$$

About the Hardy inequality (1.5), its history and development, we recommend interested readers to see the papers [10] and [11].

In a recent paper [3] the following sharper form of the half-discrete Hilbert's inequality (1.3) is obtained

$$\int_0^\infty f(x)\sum_{n=1}^\infty \frac{a_n}{(x+n)^\lambda}dx < \frac{\lambda}{p}B\left(\frac{\lambda}{q},\frac{\lambda}{p}\right)\left(\int_0^\infty x^{-\lambda-1}F^p(x)dx\right)^{\frac{1}{p}}\left(\sum_{n=1}^\infty n^{q-\lambda-1}a_n^q\right)^{\frac{1}{q}}, \tag{1.9}$$

where the constant $\frac{\lambda}{p}B\left(\frac{\lambda}{q},\frac{\lambda}{p}\right)$ is the best possible. Note that, if we apply the weighted Hardy inequality (1.6) to (1.9) we get inequality (1.4) in the case of $\lambda_1 = \frac{\lambda}{p}, \lambda_2 = \frac{\lambda}{q}$. Another sharper form of inequality (1.10) with $\lambda = 1$ is given in the paper [1]. Some sharper forms of inequality (1.1) which are related to Hardy inequality are introduced in the papers [2] and [3].

Our goal in this paper is to give an extension of inequality (1.9) by introducing a new parameter and two strictly increasing functions with some restrictions.

The paper is divided into four sections. In section two, we give the main notations and lemmas that will be used in the paper. In section three, we introduce the main result of this paper which is in Theorem 3.1. In the last section, we give some examples and we introduce the connection between the obtained half-discrete inequalities and Hardy inequalities .

2. Preliminaries and Lemmas

Recall that the Gamma function $\Gamma(\theta)$ and the Beta function $B(\mu,\nu)$ are defined respectively by

$$\Gamma(\theta) = \int_0^\infty t^{\theta-1}e^{-t}dt, \quad \theta > 0,$$

$$B(\mu,\nu) = \int_0^\infty \frac{t^{\mu-1}}{(t+1)^{\mu+\nu}}dt, \quad \mu,\nu > 0.$$

By the definition of the gamma function, we may write

$$\frac{1}{(x+y)^\lambda} = \frac{1}{\Gamma(\lambda)}\int_0^\infty t^{\lambda-1}e^{-(x+y)t}dt. \tag{2.1}$$

In this paper, we will always assume that $u(x)(x \in (a,b), -\infty \le a < b \le \infty)$ *and* $v(n)(n \in [n_0-1,\infty), n_0 \in \mathbb{N})$ *are strictly increasing differentiable functions with* $u(a^+) = 0$, $v(n_0-1) = 0$, $u(b^-) = v(\infty) = \infty$.

We will need the following two Lemmas (the first one is given in the paper [2])

Lemma.2.1[see **[2]**, Lemma 2.1]. *Let* $r > 1$, $\frac{1}{r}+\frac{1}{s} = 1$, $f > 0$, $f \in L(a,b)$, $F(x) = \int_a^x f(w)dw$. *Then for* $t > 0$ *and* $\alpha > \frac{-1}{s}$ *we have*

$$\int_a^b e^{-tu(x)}f(x)dx \le t^{\frac{1}{r}-\alpha}\Gamma(\alpha s+1)^{\frac{1}{s}}\left\{\int_a^b [u(x)]^{-\alpha r}u'(x)e^{-tu(x)}F^r(x)dx\right\}^{\frac{1}{r}}.$$

Lemma 2.2. *Let $r > 1$, $\frac{1}{r} + \frac{1}{s} = 1$, $a_n > 0$, and suppose that the function $v(y)^{\beta r+1}$ is concave on its domain. Then for $t > 0$ and $\frac{-1}{r} < \beta \le 0$ we have*

$$\sum_{n=n_0}^{\infty} e^{-v(n)t} a_n < t^{-\beta-\frac{1}{r}} \Gamma(1+\beta r)^{\frac{1}{r}} \left\{ \sum_{n=n_0}^{\infty} \frac{v(n)^{-\beta s} e^{-v(n)t}}{v'(n)^{s-1}} a_n^s \right\}^{\frac{1}{s}}.$$

Proof.

Using the Hölder inequality, we get

$$\begin{aligned}
\sum_{n=n_0}^{\infty} e^{-v(n)t} a_n &= \sum_{n=n_0}^{\infty} \left\{ v(n)^{\beta} v'(n)^{\frac{1}{r}} e^{-\frac{v(n)t}{r}} \right\} \left\{ v(n)^{-\beta} v'(n)^{\frac{-1}{r}} e^{-\frac{v(n)t}{s}} a_n \right\} \\
&\le \left(\sum_{n=n_0}^{\infty} v(n)^{\beta r} v'(n) e^{-v(n)t} \right)^{\frac{1}{r}} \left(\sum_{n=n_0}^{\infty} v(n)^{-\beta s} v'(n)^{\frac{-s}{r}} e^{-v(n)t} a_n^s \right)^{\frac{1}{s}}
\end{aligned}$$

Since the function $v(y)^{\beta r+1}$ is concave, then its derivative $(\beta r + 1)\, v(n)^{\beta} v'(n)$ is decreasing $(\beta r + 1 > 0)$. Hence,

$$\begin{aligned}
\sum_{n=n_0}^{\infty} e^{-v(n)t} a_n &< \left(\int_{n_0-1}^{\infty} v(x)^{\beta r} v'(x) e^{-v(x)t} dx \right)^{\frac{1}{r}} \left(\sum_{n=n_0}^{\infty} \frac{v(n)^{-\beta s} e^{-v(n)t}}{v'(n)^{s-1}} a_n^s \right)^{\frac{1}{s}} \\
&= \left(\int_{0}^{\infty} \theta^{\beta r} e^{-\theta t} d\theta \right)^{\frac{1}{r}} \left(\sum_{n=n_0}^{\infty} \frac{v(n)^{-\beta s} e^{-v(n)t}}{v'(n)^{s-1}} a_n^s \right)^{\frac{1}{s}} \\
&= t^{-\beta-\frac{1}{r}} \Gamma(1+\beta r)^{\frac{1}{r}} \left\{ \sum_{n=n_0}^{\infty} \frac{v(n)^{-\beta s} e^{-v(n)t}}{v'(n)^{s-1}} a_n^s \right\}^{\frac{1}{s}}.
\end{aligned}$$

3. Main Results

Theorem 3.1. *Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, suppose that $0 < \lambda < q$, $\max(\frac{-\lambda}{p}, \frac{\lambda}{q} - 1) < \gamma < \frac{\lambda}{q}$, $f, a_n > 0$, u and v are functions as defined above and the function v satisfies the condition of Lemma 2.2. For $f \in L(a, b)$, define $F(x) = \int_a^x f(w) dw$. If $\int_a^b u(x)^{-p\gamma-\lambda-1} u'(x) F^p(x) dx < \infty$ and $\sum_{n=n_0}^{\infty} \frac{v(n)^{q\gamma+q-\lambda-1}}{v'(n)^{q-1}} a_n^q < \infty$, then*

$$\begin{aligned}
I := \int_a^b f(x) \sum_{n=n_0}^{\infty} \frac{a_n}{(u(x)+v(n))^{\lambda}} dx &= \sum_{n=n_0}^{\infty} a_n \int_a^b \frac{f(x)}{(u(x)+v(n))^{\lambda}} dx \\
&< C \left(\int_a^b u(x)^{-p\gamma-\lambda-1} u'(x) F^p(x) dx \right)^{\frac{1}{p}} \left(\sum_{n=n_0}^{\infty} \frac{v(n)^{q\gamma+q-\lambda-1}}{v'(n)^{q-1}} a_n^q \right)^{\frac{1}{q}},
\end{aligned} \tag{3.1}$$

where the constant $C = \left(\frac{\lambda}{p} + \gamma\right) B\left(\frac{\lambda}{q} - \gamma, \frac{\lambda}{p} + \gamma\right)$ is the best possible.

Proof. By using (2.1) and applying the Hölder inequality, we obtain

$$\begin{aligned}
I &= \frac{1}{\Gamma(\lambda)}\int_a^b f(x)\sum_{n=n_0}^{\infty} a_n\left(\int_0^{\infty} t^{\lambda-1}e^{-(u(x)+v(n))t}dt\right)dx \\
&= \frac{1}{\Gamma(\lambda)}\int_0^{\infty}\left(t^{\frac{\lambda-1}{p}+\gamma}\int_a^b e^{-u(x)t}f(x)dx\right)\left(t^{\frac{\lambda-1}{q}-\gamma}\sum_{n=n_0}^{\infty}e^{-v(n)t}a_n\right)dt \\
&\leq \frac{1}{\Gamma(\lambda)}\left(\int_0^{\infty} t^{\lambda-1+p\gamma}\left(\int_a^b e^{-u(x)t}f(x)dx\right)^p dt\right)^{\frac{1}{p}}\left(\int_0^{\infty} t^{\lambda-1-q\gamma}\left(\sum_{n=n_0}^{\infty}e^{-v(n)t}a_n\right)^q dt\right)^{\frac{1}{q}}. \qquad (3.2)
\end{aligned}$$

By Lemma 2.1 for $r = p, s = q$, and by Lemma 2.2 for $r = p, s = q$, we obtain respectively,

$$\begin{aligned}
\left(\int_a^b e^{-u(x)t}f(x)dx\right)^p &\leq t^{1-\alpha p}\Gamma(\alpha q+1)^{\frac{p}{q}}\int_a^b u(x)^{-\alpha p}u'(x)e^{-u(x)t}F^p(x)dx, \\
\left(\sum_{n=n_0}^{\infty}e^{-v(n)t}a_n\right)^q &< t^{-\beta q-q+1}\Gamma(1+\beta p)^{\frac{q}{p}}\sum_{n=n_0}^{\infty}\frac{v(n)^{-\beta q}e^{-v(n)t}}{v'(n)^{q-1}}a_n^q.
\end{aligned}$$

Substituting these two inequalities in (3.2) we have

$$I < \frac{\Gamma(\alpha q+1)^{\frac{1}{q}}\Gamma(\beta p+1)^{\frac{1}{p}}}{\Gamma(\lambda)}\left(\int_a^b u(x)^{-\alpha p}u'(x)F^p(x)\left(\int_0^{\infty} t^{\lambda+p\gamma-\alpha p}e^{-u(x)t}dt\right)dx\right)^{\frac{1}{p}}\left(\sum_{n=n_0}^{\infty}\frac{v(n)^{-\beta q}}{v'(n)^{q-1}}a_n^q\left(\int_0^{\infty} t^{\lambda-q\gamma-\beta q-q+1}e^{-v(n)t}dt\right)\right)^{\frac{1}{q}}.$$

Since $\int_0^{\infty} t^{\lambda+p\gamma-\alpha p}e^{-u(x)t}dt = u(x)^{\alpha p-\gamma p-\lambda-1}\Gamma(\lambda+p\gamma-\alpha p+1)$ and
$\int_0^{\infty} t^{\lambda-q\gamma-q\beta-q+1}e^{-v(n)t}dt = v(n)^{q\gamma+q\beta+q-\lambda-1}\Gamma(\lambda-q\gamma-q\beta-q+1)$, we find

$$I < D\left(\int_a^b u(x)^{-p\gamma-\lambda-1}u'(x)F^p(x)dx\right)^{\frac{1}{p}}\left(\sum_{n=n_0}^{\infty}\frac{v(n)^{q\gamma+q-\lambda-1}}{v'(n)^{q-1}}a_n^q\right)^{\frac{1}{q}},$$

where the constant $D = \frac{\Gamma(\alpha q+1)^{\frac{1}{q}}\Gamma(\beta p+1)^{\frac{1}{p}}\Gamma(\lambda+p\gamma-\alpha p+1)^{\frac{1}{p}}\Gamma(\lambda-q\gamma-q\beta-q+1)^{\frac{1}{q}}}{\Gamma(\lambda)}$. Now, if we set $\alpha = \frac{\lambda+p\gamma}{pq}$ and $\beta = \frac{\lambda-q\gamma-q}{pq}$ and use the following formulas for the gamma function: $\Gamma(u+1) = u\Gamma(u)$ and $\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = B(x,y)$, we deduce that $D = C$. Inequality (3.1) is proved. To prove that the constant factor C in (3.1) is the best possible, we define $\widetilde{f}(x) = \frac{\lambda+p\gamma-\varepsilon}{p}u(x)^{\frac{\lambda+p\gamma-\varepsilon}{p}-1}u'(x)$ for $x \in [a_1, b)\,(0 < \varepsilon < \lambda+p\gamma)$, $\widetilde{f}(x) = 0$ for $x \in (a, a_1)$, where a_1 is such that $u(a_1) = 1$, and we let $\widetilde{a_n} = v(n)^{\frac{\lambda-q\gamma-q-\varepsilon}{q}}v'(n)\,(n \geq n_0)$. Then, we get $\widetilde{F}(x) = (u(x)^{\frac{\lambda+p\gamma-\varepsilon}{p}} - 1)$ for $x \in [a_1, b)$, $\widetilde{F}(x) = 0$ for $x \in (a, a_1)$. Suppose that the constant C is not the best possible, then we may find a constant K such that $0 < K < C$ and

$$
\begin{aligned}
\int_a^b \widetilde{f}(x) \sum_{n=n_0}^{\infty} \frac{\widetilde{a_n}}{(u(x)+v(n))^{\lambda}} &< K\left(\int_{a_1}^{b} u(x)^{-p\gamma-\lambda-1} u'(x)\left[u(x)^{\frac{\lambda+p\gamma-\varepsilon}{p}}-1\right]^p dx\right)^{\frac{1}{p}} \left(\sum_{n=n_0}^{\infty} \frac{v(n)^{q\gamma+q-\lambda-1}}{v'(n)^{q-1}} v(n)^{\lambda-q\gamma-q-\varepsilon} v'(n)^q\right)^{\frac{1}{q}} \\
&< K\left(\int_{a_1}^{b} u(x)^{-p\gamma-\lambda-1} u'(x)\left[u(x)^{\frac{\lambda+p\gamma-\varepsilon}{p}}\right]^p dx\right)^{\frac{1}{p}} \left(\sum_{n=n_0}^{\infty} v(n)^{-1-\varepsilon} v'(n)\right)^{\frac{1}{q}} \\
&= K\left(\int_1^{\infty} \delta^{-1-\varepsilon} d\delta\right)^{\frac{1}{p}} \left(\frac{v'(n_0)}{v(n_0)^{\varepsilon+1}} + \sum_{n=n_0+1}^{\infty} v(n)^{-1-\varepsilon} v'(n)\right)^{\frac{1}{q}} \\
&< \frac{K}{\varepsilon^{\frac{1}{p}}}\left[\frac{v'(n_0)}{v(n_0)^{\varepsilon+1}} + \int_{n_0}^{\infty} \frac{v'(\tau)}{v(\tau)^{\varepsilon+1}} d\tau\right]^{\frac{1}{q}} \\
&= \frac{K}{\varepsilon^{\frac{1}{p}}}\left[\frac{v'(n_0)}{v(n_0)^{\varepsilon+1}} + \frac{1}{\varepsilon v(n_0)^{\varepsilon}}\right]^{\frac{1}{q}} = \frac{K}{\varepsilon}\left[\frac{\varepsilon v'(n_0)}{v(n_0)^{\varepsilon+1}} + \frac{1}{v(n_0)^{\varepsilon}}\right]^{\frac{1}{q}} \qquad (3.3)
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
\widetilde{I} &= \frac{\lambda+p\gamma-\varepsilon}{p} \int_{a_1}^{b} \sum_{n=n_0}^{\infty} \frac{u(x)^{\frac{\lambda+p\gamma-\varepsilon}{p}-1} u'(x) v(n)^{\frac{\lambda-q\gamma-q-\varepsilon}{q}} v'(n)}{(u(x)+v(n))^{\lambda}} dx \\
&= \frac{\lambda+p\gamma-\varepsilon}{p} \sum_{n=n_0}^{\infty} v'(n) v(n)^{-\varepsilon-1} \int_{\frac{1}{v(n)}}^{\infty} \frac{\delta^{\frac{\lambda+p\gamma-\varepsilon}{p}-1}}{(\delta+1)^{\lambda}} d\delta \\
&= \frac{\lambda+p\gamma-\varepsilon}{p} \sum_{n=n_0}^{\infty} v'(n) v(n)^{-\varepsilon-1} \left\{\int_0^{\infty} \frac{\delta^{\frac{\lambda+p\gamma-\varepsilon}{p}-1}}{(\delta+1)^{\lambda}} d\delta - \int_0^{\frac{1}{v(n)}} \frac{\delta^{\frac{\lambda+p\gamma-\varepsilon}{p}-1}}{(\delta+1)^{\lambda}} d\delta\right\} \\
&= \frac{\lambda+p\gamma-\varepsilon}{p} \sum_{n=n_0}^{\infty} v'(n) v(n)^{-\varepsilon-1} \left\{\left[B\left(\frac{\lambda-\varepsilon}{q}, \frac{\lambda-\varepsilon}{p}\right) + o(1)\right] - \int_0^{\frac{1}{v(n)}} \frac{\delta^{\frac{\lambda+p\gamma-\varepsilon}{p}-1}}{(\delta+1)^{\lambda}} d\delta\right\} \\
&> \frac{\lambda+p\gamma-\varepsilon}{p}\left[B\left(\frac{\lambda-\varepsilon}{p}+\gamma, \frac{\lambda+\varepsilon}{q}-\gamma\right)\right] \int_{n_0}^{\infty} v'(\theta) v(\theta)^{-\varepsilon-1} d\theta - \int_{n_0}^{\infty} v'(\theta) v(\theta)^{-\varepsilon-1} d\theta \int_0^{\frac{1}{v(n)}} \delta^{\frac{\lambda+p\gamma-\varepsilon}{p}-1} d\delta \\
&= \frac{\lambda+p\gamma-\varepsilon}{p}\left[B\left(\frac{\lambda-\varepsilon}{p}+\gamma, \frac{\lambda+\varepsilon}{q}-\gamma\right)\right] \int_{v(n_0)}^{\infty} r^{-\varepsilon-1} dr - O(1) \\
&= \frac{\lambda+p\gamma-\varepsilon}{\varepsilon v(n_0)^{\varepsilon} p} B\left(\frac{\lambda-\varepsilon}{p}+\gamma, \frac{\lambda+\varepsilon}{q}-\gamma\right) - O(1). \qquad (3.4)
\end{aligned}
$$

Hence, if we let $\varepsilon \to 0^+$from (3.3) and (3.4) we obtain a contradiction. Thus, the proof of the theorem is completed.

4. Some Examples

Here we give some examples of inequality (3.1) by considering some special functions u and v that satisfies the properties presented earlier. Precisely, u and v should be increasing functions on their domains and $v(n)^{\beta p+1} = v(n)^{\frac{\lambda}{q}-\gamma}$ must be concave ($\beta = \frac{\lambda-q\gamma-q}{pq}$).

1. Let $u(x) = x$, $x \in (0,\infty)$ and $v(n) = n$, $n \geq 1$, then we find by (3.1)

$$\int_0^\infty f(x)\sum_{n=1}^\infty \frac{a_n}{(x+n)^\lambda}dx < C\left(\int_0^\infty x^{-p\gamma-\lambda-1}F^p(x)dx\right)^{\frac{1}{p}}\left(\sum_{n=1}^\infty n^{q\gamma+q-\lambda-1}a_n^q\right)^{\frac{1}{q}}, \tag{4.1}$$

here $F(x) = \int_0^x f(w)dw$. If we put $\gamma = 0$ in (4.1) we get (1.9). If we let $\lambda = 1, \gamma = \frac{p-2}{p}$ we obtain the following form

$$\int_0^\infty f(x)\sum_{n=1}^\infty \frac{a_n}{x+n}dx < \frac{\pi}{q\sin\frac{\pi}{q}}\left(\int_0^\infty \left(\frac{F(x)}{x}\right)^p dx\right)^{\frac{1}{p}}\left(\sum_{n=1}^\infty a_n^q\right)^{\frac{1}{q}}. \tag{4.2}$$

Applying Hardy inequality (1.5) to (4.2) we get (1.4). If we apply the weighted Hardy inequality (1.6) to (4.1) we get

$$\int_0^\infty f(x)\sum_{n=1}^\infty \frac{a_n}{(x+n)^\lambda}dx < C_1\left(\int_0^\infty x^{p-\lambda-1-p\gamma}f^p(x)dx\right)^{\frac{1}{p}}\left(\sum_{n=1}^\infty n^{q\gamma+q-\lambda-1}a_n^q\right)^{\frac{1}{q}}, \tag{4.3}$$

where $C_1 = B\left(\frac{\lambda}{p}+\gamma, \frac{\lambda}{q}-\gamma\right)$. Inequality (4.3) is equivalent to inequality (1.3) if we set $\gamma + \frac{\lambda}{p} = \lambda_1$ and $\frac{\lambda}{q} - \gamma = \lambda_2$.

2. If $u(x) = e^x, x \in (-\infty,\infty)$,and $v(n) = n, n \geq 1$, we obtain by (3.1)

$$\int_{-\infty}^\infty f(x)\sum_{n=1}^\infty \frac{a_n}{(e^x+n)^\lambda}dx < C\left(\int_{-\infty}^\infty e^{-(\lambda+p\gamma)x}F^p(x)dx\right)^{\frac{1}{p}}\left(\sum_{n=1}^\infty n^{q\gamma+q-\lambda-1}a_n^q\right)^{\frac{1}{q}}, \tag{4.4}$$

here $F(x) = \int_{-\infty}^x f(t)dt$. If we apply (1.9) to the integral on the right of (4.4), we have

$$\int_{-\infty}^\infty f(x)\sum_{n=1}^\infty \frac{a_n}{(e^x+n)^\lambda}dx < C_1\left(\int_{-\infty}^\infty e^{-(\lambda+p\gamma)x}f^p(x)dx\right)^{\frac{1}{p}}\left(\sum_{n=1}^\infty n^{q\gamma+q-\lambda-1}a_n^q\right)^{\frac{1}{q}}.$$

3. If $u(x) = \ln x, x \in (1,\infty)$, $v(n) = \ln n, n \geq 2$, then we have

$$\int_1^\infty f(x)\sum_{n=2}^\infty \frac{a_n}{(\ln x+\ln n)^\lambda}dx < C\left(\int_1^\infty \frac{F^p(x)}{x\,[\ln x]^{\lambda+p\gamma+1}}dx\right)^{\frac{1}{p}}\left(\sum_{n=2}^\infty \frac{a_n^q}{(\ln n)^{\lambda+1-q\gamma-q\lambda}\,n^{1-q}}\right)^{\frac{1}{q}},$$

here $F(x) = \int_1^x f(t)dt$. In particular for $\lambda = 1, \gamma = \frac{p-2}{p}$, we get

$$\int_1^\infty f(x) \sum_{n=2}^\infty \frac{a_n}{\ln x + \ln n} dx < \frac{\pi}{q \sin \frac{\pi}{p}} \left(\int_1^\infty \frac{F^p(x)}{x [\ln x]^p} dx \right)^{\frac{1}{p}} \left(\sum_{n=2}^\infty \frac{a_n^q}{n^{1-q}} \right)^{\frac{1}{q}}, \tag{4.5}$$

if we apply (1.10) to (4.5) we get the following half-discrete Hilbert-type inequality

$$\int_1^\infty f(x) \sum_{n=2}^\infty \frac{a_n}{\ln x + \ln n} dx < \frac{\pi}{\sin \frac{\pi}{p}} \left(\int_1^\infty x^{p-1} f^p(x) dx \right)^{\frac{1}{p}} \left(\sum_{n=2}^\infty n^{q-1} a_n^q \right)^{\frac{1}{q}}.$$

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