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Order-Lipschitz Mappings Restricted with Linear Bounded Mappings in Normed Vector Spaces without Normalities of Involving Cones via Methods of Upper and Lower Solutions

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Abstract. In this paper, without assuming the normalities of cones, we prove some new fixed point theorems of order-Lipschitz mappings restricted with linear bounded mappings in normed vector space in the framework of w-convergence via the method of upper and lower solutions. It is worth mentioning that the unique existence result of fixed points in this paper, presents a characterization of Picard-completeness of order-Lipschitz mappings.

1. Introduction

Let P be cone of a normed vector space $(E, \|\cdot\|)$, $D \subset E$ and $\preceq$ the partial order on E introduced by P. Recall that a mapping $T : D \to E$ is called an order-Lipschitz mapping restricted with linear bounded mappings if there exist linear bounded mappings $A, B : P \to P$ such that

$$-B(x-y) \preceq Tx - Ty \preceq A(x-y),\ \forall\, x, y \in D,\ y \preceq x. \tag{1}$$

The research on fixed points of order-Lipschitz mappings was initiated by Krasnoselskii and Zabreiko [1]. Assuming that P is a normal solid cone of a Banach space E, Krasnoselskii and Zabreiko [1] investigated the unique existence result of fixed points for order-Lipschitz mappings restricted with linear bounded mappings provided that $A = B$ and $\|A\| < 1$, which was then improved by Zhang and Sun [2] to the case that the spectral radius $r(A) < 1$. Without assuming the solidness of P, Sun [3] studied fixed points of order-Lipschitz mappings restricted with nonnegative real numbers via the method of upper and lower solutions.

Note that in [1-3], it is necessarily assumed that the cone is normal. Recently, without assuming the normality of the cone, Jiang and Li [4] proved the following fixed point result of order-Lipschitz mappings

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restricted with vectors in Banach algebras in the framework of w-convergence.

Theorem 1 ([4, Theorem 3]) *Let P be a solid cone of a Banach algebra $(E, \|\cdot\|)$, $u_0, v_0 \in E$ with $u_0 \preceq v_0$ and $T : D = [u_0, v_0] \to E$ an order-Lipschitz mapping restricted with vectors $A \in P$ and $B = \theta$. Assume that $u_0 \preceq Tu_0, Tv_0 \preceq v_0$ (i.e., u_0 and v_0 are a pair of lower and upper solutions of T), $r(A) < 1$ and T is Picard-complete at u_0 and v_0. Then T has a unique fixed point $u^* \in [u_0, v_0]$, and for each $x_0 \in [u_0, v_0]$, $x_n \xrightarrow{w} u^*$, where $\{x_n\} = O(T, x_0)$ and $O(T, x_0)$ denotes the Picard iteration sequence of T at x_0 (i.e., $x_n = T^n x_0$ for each n).*

In this paper, we shall consider fixed point theory of order-Lipschitz mappings restricted with linear bounded mappings in normed vector spaces (instead of, Banach spaces [1,2] and Banach algebras [3]) with non-normal cones. We first prove some fixed point theorems under the assumption that the order-Lipschitz mapping T has only a lower solution or an upper solution. Furthermore, we investigate the unique existence of fixed points in the case that T has a pair of lower and upper solutions. It is worth mentioning that the unique existence result of fixed points in this paper, presents a characterization of Picard-completeness of order-Lipschitz mappings (see Proposition 2). In addition, we present a suitable example which shows the usability of our theorems.

2. Preliminaries

Let P be a cone of a normed vector space $(E, \|\cdot\|)$. A cone P induces partial order $\preceq$ on E by $x \preceq y \Leftrightarrow y - x \in P$ for each $x, y \in X$. In this case, E is called an ordered normed vector space. For each $u_0, v_0 \in E$ with $u_0 \preceq v_0$, and set $[u_0, v_0] = \{u \in E : u_0 \preceq u \preceq v_0\}$, $[u_0, +\infty) = \{x \in E : u_0 \preceq x\}$ and $(-\infty, v_0] = \{x \in E : x \preceq v_0\}$. A cone P is solid [5] if $\mathrm{int}P \neq \emptyset$, where $\mathrm{int}P$ denotes the interior of P. For each $x, y \in E$ with $y - x \in \mathrm{int}P$, we write $x \ll y$. A cone P of a normed vector space E is normal [5] if there is a positive number N such that $x, y \in E$ and $\theta \preceq x \preceq y$ implies that $\|x\| \leq N\|y\|$, and the minimal N is called a normal constant of P. Note that a cone P of a normed vector space E is non-normal if and only if there exist $\{u_n\}, \{v_n\} \subset P$ such that $u_n + v_n \xrightarrow{\|\cdot\|} \theta \nRightarrow u_n \xrightarrow{\|\cdot\|} \theta$. Consequently, if P is non-normal then the sandwich theorem in the sense of norm-convergence does not hold. While, it has been shown in [6] that the sandwich theorem in the sense of w-convergence still holds even if P is non-normal.

Let P be a solid cone of a normed vector space E. A sequence $\{x_n\} \subset E$ is w-convergent [6] if for each $\epsilon \in \mathrm{int}P$, there exist a positive integer n_0 and $x \in E$ such that $x - \epsilon \ll x_n \ll x + \epsilon$ for each $n \geq n_0$ (denote $x_n \xrightarrow{w} x$ and x is called a w-limit of $\{x_n\}$). A sequence $\{x_n\} \subset E$ is w-Cauchy [4] if for each $\epsilon \in \mathrm{int}P$, there exists a positive integer n_0 such that $-\epsilon \ll x_n - x_m \ll \epsilon$ for each $m, n \geq n_0$, i.e., $x_n - x_m \xrightarrow{w} \theta (m, n \to \infty)$. A subset $D \subset E$ is w-closed [4] if for each $\{x_n\} \subset D$, $x_n \xrightarrow{w} x$ implies $x \in D$.

Lemma 1 ([4,6]) *Let P be a solid cone of a normed vector space $(E, \|\cdot\|)$ and $u_0, v_0 \in E$ with $u_0 \preceq v_0$. Then*

(i) *each sequence $\{x_n\} \subset E$ has a unique w-limit;*

(ii) *the partial order intervals $[u_0, v_0]$, $[u_0, +\infty)$ and $(-\infty, v_0]$ are w-closed;*

(iii) *for each $\{x_n\}, \{y_n\}, \{z_n\} \subset E$ with $x_n \preceq y_n \preceq z_n$ for each n, $x_n \xrightarrow{w} z$ and $z_n \xrightarrow{w} z$ imply $y_n \xrightarrow{w} z$, where $z \in E$.*

Lemma 2 ([6]) *Let P be a solid cone of a normed vector space $(E, \|\cdot\|)$ and $x_n \subset E$. Then $x_n \xrightarrow{\|\cdot\|} x$ implies $x_n \xrightarrow{w} x$. Moreover, if P is normal then $x_n \xrightarrow{w} x \Leftrightarrow x_n \xrightarrow{\|\cdot\|} x$.*

Let P be a solid cone of a normed vector space E, $D \subset E$, $x_0 \in D$ and $T : E \to E$. If the Picard iteration sequence $O(T, x_0)$ is w-convergent provided that it is w-Cauchy, then T is said to be Picard-complete at x_0. If T is Picard-complete at each $x \in D$ then it is said to be Picard-complete on D.

Remark 1 It is clear that if $O(T, x_0)$ is w-convergent then T is certainly Picard-complete at x_0. In particular when P is a normal cone of a Banach space E, each mapping $T : E \to E$ is Picard-complete on E by Lemma

2.

Let $(E, \|\cdot\|)$ be a normed vector space, $\{x_n\} \subset E$ and $D \subset E$. If $x_n \preceq x_m$ or $x_m \preceq x_n$ for each $m \neq n$ then $\{x_n\}$ is said to be comparable. If there exists some $c > 0$ such that $\|x_n\| \leq c$ for each n then $\{x_n\}$ is said to be bounded. For each mapping $T : D \to E$, set

$$T_{B-C-D} = \{x \in D : O(T, x) \text{ is bounded and comparable}\}.$$

Proposition 1 *For each $\alpha > 1$ and each $a, b \geq 0$ with $a \leq b$, we have*

$$b^\alpha - a^\alpha \leq 2^{k_0} b^\alpha (b^{\frac{\alpha}{2^{k_0}}} - a^{\frac{\alpha}{2^{k_0}}}), \tag{2}$$

where $k_0 = \min\{k \in \mathbb{N}_+ : \frac{\alpha}{2^k} \leq 1\}$ and $\mathbb{N}_+$ denotes the set of all positive integers.

Proof. Direct calculation that

$$\begin{aligned}
b^\alpha - a^\alpha =& (b^{\frac{\alpha}{2}} + a^{\frac{\alpha}{2}})(b^{\frac{\alpha}{2}} - a^{\frac{\alpha}{2}}) \\
=& (b^{\frac{\alpha}{2}} + a^{\frac{\alpha}{2}})(b^{\frac{\alpha}{4}} + a^{\frac{\alpha}{4}})(b^{\frac{\alpha}{4}} - a^{\frac{\alpha}{4}}) \\
=& (b^{\frac{\alpha}{2}} + a^{\frac{\alpha}{2}})(b^{\frac{\alpha}{4}} + a^{\frac{\alpha}{4}}) \cdots (b^{\frac{\alpha}{2^{k_0}}} + a^{\frac{\alpha}{2^{k_0}}})(b^{\frac{\alpha}{2^{k_0}}} - a^{\frac{\alpha}{2^{k_0}}}) \\
\leq& 2^{k_0} b^{\frac{\alpha}{2}} b^{\frac{\alpha}{4}} \cdots b^{\frac{\alpha}{2^{k_0}}} (b^{\frac{\alpha}{2^{k_0}}} - a^{\frac{\alpha}{2^{k_0}}}) = 2^{k_0} b^{\frac{\alpha}{2} + \frac{\alpha}{4} + \cdots + \frac{\alpha}{2^{k_0}}} (b^{\frac{\alpha}{2^{k_0}}} - a^{\frac{\alpha}{2^{k_0}}}) \\
\leq& 2^{k_0} b^\alpha (b^{\frac{\alpha}{2^{k_0}}} - a^{\frac{\alpha}{2^{k_0}}}).
\end{aligned}$$

□

The following example will show that there exists some mapping $T : D \to E$ such that it is Picard-complete on D.

Example 1 Let $E = C^1_{\mathbb{R}}[0, 1]$ be endowed with the norm $\|x\| = \|x\|_\infty + \|x'\|_\infty$ and $P = \{x \in E : x(t) \geq 0, \forall\, t \in [0, 1]\}$, where $\|x\|_\infty = \max\limits_{t \in [0,1]} x(t)$ for each $x \in C_{\mathbb{R}}[0, 1]$. Then $(E, \|\cdot\|)$ is a Banach space and P is a non-normal solid cone [5]. Let $(Tx)(t) = \int_0^t x^\alpha(s)ds$ for each $x \in E$ and each $t \in [0, 1]$, where $\alpha > 1$.

Let $x_0(t) \equiv 1$ and $\{x_n\} = O(T, x_0)$. Clearly, $T : P \to P$ is a nondecreasing mapping and $(Tx_0)(t) = \int_0^t x_0^\alpha(s)ds = t \leq 1 = x_0(t)$ for each $t \in [0, 1]$, and so $\{x_n\}$ is comparable. By induction, for each $t \in [0, 1]$ and $n \geq 2$ we have

$$\begin{aligned}
x_n(t) = (T^n x_0)(t) =& \int_0^t x_{n-1}^\alpha(s)ds \\
=& \frac{t^{1+\alpha+\alpha^2+\cdots+\alpha^{n-1}}}{(1+\alpha)^{\alpha^{n-2}}(1+\alpha+\alpha^2)^{\alpha^{n-3}} \cdots (1+\alpha+\alpha^2+\cdots+\alpha^{n-1})} \\
\leq& 1,
\end{aligned}$$

and so

$$\|x_n\| = \|x_n\|_\infty + \|x_n'\|_\infty = \|x_n\|_\infty + \|x_{n-1}^\alpha\|_\infty \leq 2,$$

which together with $\|Tx_0\| = 2$ implies that $\{x_n\}$ is bounded. This shows $T_{B-C-P} \neq \emptyset$.

For each $x_0 \in T_{B-C-P}$, set $\{x_n\} = O(T, x_0)$. Then $\{x_n\} \subset P$ since $T(P) \subset P$, and there exists $c > 0$ such that $\|x_n\| \leq c$ for each n. For each $\varepsilon > 0$, there exists $\epsilon \in \mathrm{int}P$ such that $\|\epsilon\| < \min\{\frac{\varepsilon}{2}, (\frac{\varepsilon}{2^{k_0+1}c^\alpha})^{\frac{2^{k_0}}{\alpha}}\}$, where k_0 is the one given in Proposition 1. Suppose that $\{x_n\}$ is w-Cauchy, then there exists a positive integer n_0 such that $-\epsilon \ll x_n - x_m \ll \epsilon$ for each $m > n \geq n_0$, *i.e.*, $-\epsilon(t) \leq x_n(t) - x_m(t) \leq \epsilon(t)$ for each $t \in [0, 1]$. Thus we have

$\|x_n - x_m\|_\infty = \max_{t\in[0,1]} |x_n(t) - x_m(t)| \le \max_{t\in[0,1]} |\epsilon(t)| = \|\epsilon\|_\infty \le \|\epsilon\| < \min\{\frac{\varepsilon}{2}, (\frac{\varepsilon}{2^{k_0+1}c^\alpha})^{\frac{2^{k_0}}{\alpha}}\}$ for each $m, n \ge n_0$. Since $\{x_n\}$ is comparable, we have from $\{x_n\} \subset P$ and $0 < \frac{\alpha}{2^{k_0}} \le 1$ that $0 \le |x_m^{\frac{\alpha}{2^{k_0}}}(t) - x_n^{\frac{\alpha}{2^{k_0}}}(t)| \le |x_m(t) - x_n(t)|^{\frac{\alpha}{2^{k_0}}}$ for each $m > n$ and each $t \in [0,1]$. Thus it follows from (2) and $(Tx_n)'(t) = x_{n-1}^\alpha(t)$ that, for each $m > n \ge n_0 + 1$,

$$
\begin{aligned}
\|x_n - x_m\| &= \|x_n - x_m\|_\infty + \|(x_n - x_m)'\|_\infty \\
&= \|x_n - x_m\|_\infty + \max_{t\in[0,1]} |(x_n(t) - x_m(t))'| \\
&= \|x_n - x_m\|_\infty + \max_{t\in[0,1]} |(Tx_{n-1}(t) - Tx_{m-1}(t))'| \\
&= \|x_n - x_m\|_\infty + \max_{t\in[0,1]} |x_{n-1}^\alpha(t) - x_{m-1}^\alpha(t)| \\
&\le \|x_n - x_m\|_\infty + 2^{k_0}c^\alpha \max_{t\in[0,1]} |x_{n-1}^{\frac{\alpha}{2^{k_0}}}(t) - x_{m-1}^{\frac{\alpha}{2^{k_0}}}(t)| \\
&\le \|x_n - x_m\|_\infty + 2^{k_0}c^\alpha \max_{t\in[0,1]} |x_{n-1}(t) - x_{m-1}(t)|^{\frac{\alpha}{2^{k_0}}} \\
&= \|x_n - x_m\|_\infty + 2^{k_0}c^\alpha \|x_{n-1} - x_{m-1}\|_\infty^{\frac{\alpha}{2^{k_0}}} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
\end{aligned}
$$

which implies that $x_n - x_m \xrightarrow{\|\cdot\|} \theta$. Therefore, there exists $x^* \in P$ such that $x_n \xrightarrow{\|\cdot\|} x^*$, and hence $x_n \xrightarrow{w} x^*$ by Lemma 2. This shows that T is Picard-complete on T_{B-C-P}.

Let P be a solid cone of a normed vector space $(E, \|\cdot\|)$ and $D \subset E$. A mapping $T : D \to E$ is w-continuous at $x_0 \in D$ if for each $\{x_n\} \subset E$, $x_n \xrightarrow{w} x_0$ implies $Tx_n \xrightarrow{w} Tx_0$. If T is w-continuous at each $x \in D$ then T is said to be w-continuous on D.

Lemma 4 *Let P be a solid cone of a normed vector space $(E, \|\cdot\|)$ and $A : E \to E$ a linear bounded mapping with $A(P) \subset P$. Then A is w-continuous on E.*

Proof. Let $x \in E$ and $\{x_n\}$ be a sequence in E such that $x_n \xrightarrow{w} x$. For each $\epsilon \in \text{int}P$, it is clear that $\frac{\epsilon}{m} \in \text{int}P$ for each m, and hence there exists n_m such that $-\frac{\epsilon}{m} \ll x_n - x \ll \frac{\epsilon}{m}$ for each $n \ge n_m$. Note that A is a linear mapping with $A(P) \subset P$, then $-\frac{A\epsilon}{m} \le Ax_n - Ax \le \frac{A\epsilon}{m}$ for each $n \ge n_m$. It is clear that $\frac{A\epsilon}{m} \xrightarrow{\|\cdot\|} \theta (m \to \infty)$ since A is a bounded mapping, and hence $\frac{A\epsilon}{m} \xrightarrow{w} \theta (m \to \infty)$ by Lemma 2. Moreover, by (iii) of Lemma 1 we obtain $Ax_n - Ax \xrightarrow{w} \theta$, *i.e.*, A is continuous at x. □

3. Main results

We first state and prove some existence results of fixed points of order-Lipschitz mappings in normed vector spaces without assumption of the normality of the cone.

Theorem 2 *Let P be a solid cone of a normed vector space $(E, \|\cdot\|)$, $u_0 \in E$ and $T : D = [u_0, +\infty) \to E$ an order-Lipschitz mapping restricted with linear bounded mappings $A : P \to P$ and $B = \hat{\theta}$, where $\hat{\theta}$ denotes the zero mapping. Assume that $u_0 \le Tu_0$, $r(A) < 1$ and T is Picard-complete at u_0. Then T has a fixed point $u^* \in [u_0, +\infty)$. Moreover, let $u \in [u_0, +\infty)$ be a fixed point of T such that it is comparable to u^* (i.e., $u \le u^*$ or $u^* \le u$), then $u = u^*$.*

Proof. Since $A : P \to P$ is a linear bounded mapping with $r(A) < 1$, the inverse of $I - A$ exists, denote it by $(I - A)^{-1}$. Moreover by Neumann's formula,

$$(I - A)^{-1} = \sum_{n=0}^{\infty} A^n = I + A + A^2 + \cdots + A^n + \cdots, \tag{3}$$

which implies that $(I-A)^{-1}: P \to P$ is a linear bounded mapping. It follows from $r(A) < 1$ and Gelfand's formula that there exist a positive integer n_0 and $\beta \in (r(A), 1)$ such that

$$\|A^n\| \le \beta^n, \ \forall\, n \ge n_0. \tag{4}$$

Set $\{u_n\} = O(T, u_0)$. Note that (1) holds for $D = [u_0, +\infty)$, then $B = \hat{\theta}$ implies that T is nondecreasing on $[u_0, +\infty)$. Thus by $u_0 \le Tu_0$,

$$u_0 \le u_1 \le u_2 \le \cdots \le u_n \le u_{n+1}, \ \forall\, n. \tag{5}$$

Moreover by (1),

$$\theta \le u_{n+1} - u_n \le A(u_n - u_{n-1}) \le \cdots \le A^n(u_1 - u_0), \ \forall\, n,$$

and so by (3),

$$\begin{aligned} \theta \le u_m - u_n &= \sum_{i=n}^{m-1}(u_{i+1} - u_i) \le \sum_{i=n}^{m-1} A^i(u_1 - u_0) \\ &= A^n \sum_{i=0}^{m-n-1} A^i(u_1 - u_0) \le A^n(I-A)^{-1}(u_1 - u_0), \ \forall\, m > n. \end{aligned}$$

It follows from (4) that $A^n(I-A)^{-1}(u_1 - u_0) \overset{\|\cdot\|}{\to} \theta$ since $\|A^n(I-A)^{-1}(u_1 - u_0)\| \le \|A^n\|\|(I-A)^{-1}(u_1 - u_0)\| \le \beta^n\|(I-A)^{-1}(u_1 - u_0)\|$ for each $n \ge n_0$, and hence by Lemma 2,

$$A^n(I-A)^{-1}(u_1 - u_0) \overset{w}{\to} \theta,$$

which together with (4) and (iii) of Lemma 1 implies that

$$u_m - u_n \overset{w}{\to} \theta (m > n \to \infty), \tag{6}$$

i.e., $\{u_n\}$ is a w-Cauchy sequence. Since T is Picard-complete at u_0, there exists $u^* \in E$ such that

$$u_n \overset{w}{\to} u^*. \tag{7}$$

Note that $u_m \in [u_n, +\infty)$ for each $m \ge n$ by (5), then by (ii) of Lemma 1,

$$u_n \le u^*, \ \forall\, n. \tag{8}$$

Moreover by the nondecreasing property of T on $[u_0, +\infty)$,

$$u_{n+1} \le Tu^*, \ \forall\, n, \tag{9}$$

which together with (7) and (ii) of Lemma 1 implies that

$$u^* \le Tu^*. \tag{10}$$

Thus it follows from (5), (8) and (9) that

$$\theta \le Tu^* - u_{n+1} = Tu^* - Tu_n \le A(u^* - u_n), \ \forall\, n. \tag{11}$$

Letting $n \to \infty$ in (11), by (7), (iii) of Lemma 1 and Lemma 3, we obtain $u_{n+1} \overset{w}{\to} Tu^*$. Moreover by (i) of Lemma 1, we get $u^* = Tu^*$. Let $u^* \in [u_0, +\infty)$ be another fixed point of T such that it is comparable to u^*. We may assume that $u^* \le x^*$ (the proof of the other case $x \le u^*$ is similar). Then by (5), we get $\theta \le x^* - u^* = T^n x^* - T^n u^* \le A^n(u - u^*)$ for each n. This together with (4), (iii) of Lemma 1 and Lemma 2 implies $x^* = u^*$. $\square$

In analogy to Theorem 2, we have the following fixed point result.

Theorem 3 *Let P be a solid cone of a normed vector space $(E, \|\cdot\|)$, $v_0 \in E$ and $T : D = (-\infty, v_0] \to E$ an order-Lipschitz mapping restricted with linear bounded mappings $A : P \to P$ and $B = \hat{\theta}$. Assume that $Tv_0 \preceq v_0$, $r(A) < 1$ and T is Picard-complete at v_0. Then T has a fixed point $u^* \in (-\infty, v_0]$. Moreover, let $v \in (-\infty, v_0]$ be a fixed point of T such that it is comparable to v^*, then $v = v^*$.*

Proof. Note that (1) holds for $D = (-\infty, v_0]$, then $B = \hat{\theta}$ implies that T is nondecreasing on $(-\infty, v_0]$. Set $\{v_n\} = O(T, v_0)$, from $Tv_0 \preceq v_0$ we get

$$v_{n+1} \preceq v_n \preceq \cdots \preceq v_1 \preceq v_0,\ \forall\, n. \tag{12}$$

In analogy to the proof of Theorem 2, by (1), (3) and (12), we get

$$\begin{aligned} \theta \preceq v_n - v_m &= \sum_{i=n}^{m-1}(v_i - u_{i+1}) \preceq \sum_{i=n}^{m-1} A^i(v_0 - v_1) \\ &= A^n \sum_{i=0}^{m-n-1} A^i(v_0 - v_1) \preceq A^n(I - A)^{-1}(v_0 - v_1),\ \forall\, m > n, \end{aligned}$$

which together with (4), (iii) of Lemma 1 and Lemma 3 implies that

$$v_n - v_m \xrightarrow{w} \theta (m > n \to \infty). \tag{13}$$

Thus by the Picard-completeness of T at v_0, there exists $v^* \in E$ such that

$$v_n \xrightarrow{w} v^*. \tag{14}$$

In analogy to (8)-(10), by (12) and (ii) of Lemma 1 we get

$$Tv^* \preceq v^* \preceq v_n,\ \forall\, n. \tag{15}$$

Thus by (12) and (15), we get

$$\theta \preceq v_{n+1} - Tv^* = Tv_n - Tv^* \preceq A(v_n - v^*),\ \forall\, n. \tag{16}$$

Letting $n \to \infty$ in (16), by (iii) of Lemma 1 and Lemma 3, we obtain $v_{n+1} \xrightarrow{w} Tv^*$. Moreover by (i) of Lemma 1, we get $u^* = Tu^*$. The rest proof is totally similar to that of Theorem 2, we omit it here. □

In the case that T has a pair of lower and upper solutions, we obtain the unique existence theorem of fixed points as follows.

Theorem 4 *Let P be a solid cone of a normed vector space $(E, \|\cdot\|)$, $u_0, v_0 \in E$ with $u_0 \preceq v_0$ and $T : D = [u_0, v_0] \to E$ an order-Lipschitz mapping restricted with linear bounded mappings $A : P \to P$ and $B = \hat{\theta}$. Assume that $u_0 \preceq Tu_0, Tv_0 \preceq v_0$, $r(A) < 1$ and T is Picard-complete at u_0 and v_0. Then T has a unique fixed point $u^* \in [u_0, v_0]$, and for each $x_0 \in [u_0, v_0]$, $x_n \xrightarrow{w} u^*$, where $\{x_n\} = O(T, x_0)$.*

Proof. The existence of fixed points immediately follows from Theorems 2 and 3. Thus it suffices to show the uniqueness of fixed point. Following the proof of Theorems 2 and 3, we know that $T : [u_0, v_0] \to [u_0, v_0]$ is nondecreasing, and

$$u_0 \preceq u_1 \preceq \cdots \preceq u_n \preceq \cdots \preceq v_n \preceq \cdots \preceq v_1 \preceq v_0,\ \forall\, n. \tag{17}$$

By (1) and (17),

$$\theta \preceq v_{n+1} - u_{n+1} \preceq A(v_n - u_n),\ \forall\, n. \tag{18}$$

Letting $n \to \infty$ in (18), by (7), (14), (iii) of Lemma 1 and Lemma 3, we obtain $u^* = v^*$.

For each $x_0 \in [u_0, v_0]$, set $\{x_n\} = O(T, x_0)$. By the nondecreasing property of T on $[u_0, v_0]$,

$$u_n \preceq x_n \preceq v_n. \tag{19}$$

Letting $n \to \infty$ in (19), by $u^* = v^*$, (7), (14) and (iii) of Lemma 1, we get

$$x_n \xrightarrow{w} u^*. \tag{20}$$

Let $x^* \in [u_0, v_0]$ be another fixed point of T. Set $\{y_n\} = O(T, x^*)$, then $y_n \equiv x^*$ and hence $y_n \xrightarrow{w} x^*$. On the other hand, in analogy to (20) we obtain $y_n \xrightarrow{w} u^*$. Thus by (i) of Lemma 1, we have $x^* = u^*$. This shows that u^* is the unique fixed point of T. $\square$

Remark 2 It is clear that Theorem 4 is still valid in the case that E is a Banach algebra. Thus Theorem 1 is a particular case of Theorem 4 in Banach algebras with $A \in P$.

Remark 3 It follows from Theorem 4 and Remark 1 that T is Picard-complete on $[u_0, v_0]$ provided that T is Picard complete at u_0 and v_0 since for each $x_0 \in [u_0, v_0]$, the Picard iteration sequence $\{x_n\}$ is weakly convergent. Thus we have the following characterization of Picard-completeness of order-Lipschitz mappings.

Proposition 2 *Let P be a solid cone of a normed vector space $(E, \|\cdot\|)$, $u_0, v_0 \in E$ with $u_0 \preceq v_0$ and $T : [u_0, v_0] \to E$ an order-Lipschitz mapping restricted with linear bounded mappings $A : P \to P$ and $B = \hat{\theta}$. Assume that $u_0 \preceq Tu_0, Tv_0 \preceq v_0$ and $r(A) < 1$. Then the following two statements are equivalent:*

(i) T is Picard-complete on $[u_0, v_0]$;

(ii) T has a unique fixed point $u^* \in [u_0, v_0]$, and for each $x_0 \in [u_0, v_0]$, the Picard iteration sequence $\{x_n\}$ weakly converges to u^*.

Example 2 Let E and P be the same ones as those in Example 1. Let $u_0 = \theta$, $v_0(t) \equiv 1$ and $(Tx)(t) = \int_0^t x^2(s)ds$ for each $x \in E$ and each $t \in [0, 1]$.

Clearly, $u_0 \preceq Tu_0$, $(Tv_0)(t) = t \leq 1 = v_0(t)$ for each $t \in [0, 1]$, and $u_n = u_0$ for each n. By Example 1, T is Picard-complete at u_0 and v_0. For each $x, y \in [u_0, v_0]$ with $y \preceq x$ and each $t \in [0, 1]$, we have

$$0 \leq (Tx)(t) - (Ty)(t) = \int_0^t (x(s) - y(s))(x(s) + y(s))ds \leq 2(A(x - y))(t),$$

where $(Ax)(t) = \int_0^t x(s)ds$ for each $x \in P$ and $t \in [0, 1]$. For each $x \in E$ and $t \in [0, 1]$, by induction we get $(A^n x)(t) \leq \frac{\|x\|_\infty t^n}{n!} \leq \frac{\|x\|}{n!}$, and so $\|A^n x\|_\infty \leq \frac{\|x\|}{n!}$. On the other hand, we have $\|(A^n x)'\|_\infty = \|A^{n-1}x\|_\infty \leq \frac{\|x\|}{(n-1)!}$ since $(A^n x)'(t) = (A^{n-1}x)(t)$. Thus $\|A^n x\| = \|A^n x\|_\infty + \|(A^n x)'\|_\infty \leq \frac{\|x\|}{n!} + \frac{\|x\|}{(n-1)!}$ and $\|A^n\| \leq \frac{1}{n!} + \frac{1}{(n-1)!}$. By Gelfand's formula, we obtain $0 \leq r(A) = \lim_{n\to\infty} \sqrt[n]{\frac{1}{n!} + \frac{1}{(n-1)!}} \leq \lim_{n\to\infty} \frac{1}{\sqrt[n]{n!}} + \lim_{n\to\infty} \frac{1}{\sqrt[n]{(n-1)!}} = 0$, and hence $r(2A) = 0$. This shows that all the assumptions of Theorem 5 are satisfied, and hence T has a unique fixed point $u_0 = \theta$.

However, none of the results in [1-4] is applicable here since the cone P is non-normal and there does not exist $A \in P$ such that (1) is satisfied.

Note that in Theorems 2-4 it is assumed that $B = \hat{\theta}$, which together with (1) implies the nondecreasing property of T. In what follows, we shall consider the case that $B \neq \hat{\theta}$.

Corollary 1 *Let P be a solid cone of a normed vector space $(E, \|\cdot\|)$, $u_0, v_0 \in E$ with $u_0 \preceq v_0$ and $T : D = [u_0, v_0] \to E$ an order-Lipschitz mapping restricted with linear bounded mappings $A, B : P \to P$. Assume that $u_0 \preceq Tu_0, Tv_0 \preceq v_0$, A, B are commutative (i.e., $AB = BA$), $(I + B)$ is invertible (i.e., $(I + B)^{-1}$ exists), $r(\widetilde{A}) < 1$, and $\widetilde{T}$ is Picard-complete at u_0 and v_0, where $\widetilde{A} = (I + B)^{-1}(A + B)$ and $\widetilde{T} = (I + B)^{-1}(T + B)$, then T has a unique fixed point in $[u_0, v_0]$.*

Proof. By $u_0 \preceq Tu_0, Tv_0 \preceq v_0$ and (1),

$$u_0 \preceq \widetilde{T}u_0, \ \ \widetilde{T}v_0 \preceq v_0,$$
$$\theta \preceq \widetilde{T}x - \widetilde{T}y \preceq \widetilde{A}(x-y), \ \forall\, x, y \in [u_0, v_0], \ y \preceq x.$$

Note that $r(\widetilde{A}) < 1$ and $\widetilde{T}$ is Picard-complete at u_0 and v_0, then by Theorem 4, $\widetilde{T}$ has a unique fixed point $u^* \in [u_0, v_0]$. Thus we have $Tu^* + Bu^* = u^* + Bu^*$ and so $Tu^* = u^*$. Let $x \in [u_0, v_0]$ be another fixed point of T, then $Tx = x$ and hence $\widetilde{T}x = x$. Moreover by the uniqueness of fixed point of $\widetilde{T}$ in $[u_0, v_0]$, we get $x = u^*$. Hence u^* is the unique fixed point of T in $[u_0, v_0]$. □

In analogy to Corollary 1, we obtain the following fixed point result corresponding to Theorems 2 and 3.

Corollary 2 *Let P be a solid cone of a normed vector space $(E, \|\cdot\|)$, $u_0 \in E$ and $T : D = [u_0, +\infty) \to E$ (resp. $T : D = (-\infty, u_0] \to E$) an order-Lipschitz mapping restricted with linear bounded mappings$A, B : P \to P$. Assume that $u_0 \preceq Tu_0$ (resp. $Tu_0 \preceq u_0$), A, B are commutative, $I + B$ is invertible, $r(\widetilde{A}) < 1$, and $\widetilde{T}$ is Picard-complete at u_0. Then T has a fixed point in $[u_0, +\infty)$ (resp. $(-\infty, u_0]$).*

In particular when T is an order-Lipschitz mapping restricted with nonnegative real numbers, we have the following fixed point result by Corollary 1.

Corollary 3 *Let P be a solid cone of a normed vector space $(E, \|\cdot\|)$, $u_0, v_0 \in E$ with $u_0 \preceq v_0$ and $T : [u_0, v_0] \to E$ an order-Lipschitz mapping restricted with nonnegative real numbers $A \in [0, 1)$ and $B \in [0, +\infty)$. Assume that $u_0 \preceq Tu_0, Tv_0 \preceq v_0$, $\widetilde{T}$ is Picard-complete at u_0 and v_0, where $\widetilde{T}u = \frac{Tu+Bu}{1+B}$ for each $u \in [u_0, v_0]$, then T has a unique fixed point $u^* \in [u_0, v_0]$.*

Remark 4 Note that in Theorems 2-4 and Corollaries 1-3, E is not confined to a Banach space, i.e., E needs not to be complete.

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