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Hyper BL-Algebras

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Abstract. We put forth the concept of hyper BL-algebras which is a generalization of BL-algebras. We give some non-trivial examples and properties of hyper BL-algebras. Moreover, we introduce weak filters and weak deductive systems of hyper BL-algebras and study the relationships between them. Then we state and prove some theorems about weak filters and weak deductive systems. In particular, we define the concept of regular compatible congruence on hyper BL-algebras and construct the quotient structure in hyper BL-algebras. Finally, we discuss the conditions in which a quotient hyper BL-algebra is an MV-algebra.

1. Introduction

Residuated lattices, introduced by Ward and Dilworth [28], are a common structure among algebras associated with substructure logic. The main examples of residuated lattices are MV-algebras introduced by Chang [8] and BL-algebras [16]. BL-algebras were introduced in the 1990s by Hájek as the equivalent algebraic semantics for its basic fuzzy logic. Moreover, MV-algebras can be seen as a particular BL-algebras which satisfies $x^{--} = x$.

The hyper structure theory (called also multialgebra) was introduced in 1934 by F. Marty [23] at the 8th congress of Scandinavian Mathematicians. Nowadays, hyperstructures have a lot of applications in several domains of mathematics and computer science. In [26] Mittas et al. applied the hyperstructures to lattices and introduced the concepts of hyperlattice and superlattice. Many authors studied different aspects of semihypergroups, for instance, Bonansinga and Corsini [6], Corsini [7], Davvaz [11], Davvaz and Poursalavati [12]. Borzooei et al. [4] introduced and studied hyper K-algebras and Ghorbani et al.[15], applied the hyperstructures to MV-algebras named hyper MV-algebras which are a generalization of MV-algebras. Rasouli and Davvaz, [27], considered their results and studied homomorphisms, dual homomorphisms, strong homomorphisms and fundamental relations on hyper MV-algebras and gave some results about the connections between hyper MV-algebras and fundamental MV-algebras. Moreover, Jun et al. [23, 24] introduced some new types of deductive systems on hyper MV-algebras and investigated their relations. Omid Zahiri et al.[32] applied the the hyperstructures to residuated lattices and introduced the concept of hyper residuated lattices. They defined the concept of regular compatible congruence on hyper residuated lattices and construct the quotient structure in hyper residuated lattices. Also, they stated

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and proved some theorem with appropriate results such as the isomorphism theorems. A. Karimi Feizabadi et al.[13] studied fundamental relations in universal hyperalgebras. We have already done some work on hyper structure, for instance, in [17, 18, 29]. In particular, in [30, 31] we introduced the concepts of states, measures, state operators and state-morphism operators on hyper BCK-algebras.

At present, hyper MV-algebras and hyper residuated lattices have been introduced and studied. But research papers about hyper BL-algebras have not been proposed. Hence, as a bridge of from hyper MV-algebras to hyper residuated lattices, it is meaningful to construct hyper BL-algebras as a generalization of the concept of BL-algebras. By the way, the divisibility and the prelinearity are basic differences between residuated lattices and BL-algebras. How to choose the axiom corresponding to divisibility in the axiomatic system of hyper BL-algebras is a key problem.

This paper is organized as follows: in Section 2, we recall some basic notions and some results of some hyperstructures. In Section 3, we reasonably design the axiomatic system of hyper BL-algebras, give some non-trivial examples and discuss some basic properties. Section 4, we introduce and prove some propositions about weak filters and weak deductive systems on hyper BL-algebras. In section 5, we construct the quotient structures in hyper BL-algebras and we get some results on the quotient structures.

2. Preliminaries

Let H be a non-empty set with a binary hyperoperation "$\circ$". For any two subsets A and B of H, denote the set $\bigcup_{a\in A,b\in B} a\circ b$ by $A\circ B$. We use $x\circ y$ instead of $x\circ\{y\}$, $\{x\}\circ y$, or $\{x\}\circ\{y\}$.

First, we recall some definitions and properties about residuated lattices.

Definition 2.1. [16] *An algebra* $(L,\wedge,\vee,\odot,\rightarrow,0,1)$ *of type* $(2,2,2,2,0,0)$ *is called a* $BL-algebra$ *if it satisfies the following conditions:*

(1) $(L,\wedge,\vee,0,1)$ *is a bounded lattice,*
(2) $(L,\odot,1)$ *is a commutative monoid,*
(3) $x\odot y\le z$ *if and only if* $x\le y\rightarrow z$,
(4) $x\wedge y=x\odot(x\rightarrow y)$,
(5) $(x\rightarrow y)\vee(y\rightarrow x)=1$, *for all* $x,y,z\in L$.

Proposition 2.2. [16] *Let* A *be a BL-algebra. For all* $x,y,z\in A$, *the following are valid:*

(1) $x\otimes(x\rightarrow y)\le y$,
(2) $x\le y\rightarrow(x\otimes y)$,
(3) $x\le y\iff x\rightarrow y=1$,
(4) $x\rightarrow(y\rightarrow z)=(x\otimes y)\rightarrow z=y\rightarrow(x\rightarrow z)$,
(5) $x\le y$ *implies* $z\rightarrow x\le z\rightarrow y,\ y\rightarrow z\le x\rightarrow z$ *and* $x\otimes z\le y\otimes z$,
(6) $y\le(y\rightarrow x)\rightarrow x$,
(7) $(x\rightarrow y)\otimes(y\rightarrow z)\le x\rightarrow z$,
(8) $y\rightarrow x\le(z\rightarrow y)\rightarrow(z\rightarrow x)$,
(9) $x\rightarrow y\le(y\rightarrow z)\rightarrow(x\rightarrow z)$,
(10) $x\vee y=((x\rightarrow y)\rightarrow y)\wedge((y\rightarrow x)\rightarrow x)$,
(11) $x\le y$ *implies* $y^-\le x^-$,
(12) $1\rightarrow x=x,\ x\rightarrow x=1,\ x\rightarrow 1=1$,
(13) $x\le y\rightarrow x$, *or equivalently,* $x\rightarrow(y\rightarrow x)=1$,
(14) $((x\rightarrow y)\rightarrow y)\rightarrow y=x\rightarrow y$,
(15) $x\rightarrow y\le(x\otimes z)\rightarrow(y\otimes z)$,
(16) $x\rightarrow y\le(x\wedge z)\rightarrow(y\wedge z), x\rightarrow y\le(x\vee z)\rightarrow(y\vee z)$,
(17) $x\otimes x^-=0$,
where $x^-=x\rightarrow 0$.

Definition 2.3. [26] *A super lattice is a partially ordered set* $(S, \leq)$ *endowed with two binary hyperoperations* $\vee$ *and* $\wedge$ *satisfying the following properties: for all* $a, b, c \in S$,

(SL1) $a \in (a \vee a) \cap (a \wedge a)$,

(SL2) $a \vee b = b \vee a$, $a \wedge b = b \wedge a$,

(SL3) $(a \vee b) \vee c = a \vee (b \vee c)$, $(a \wedge b) \wedge c = a \wedge (b \wedge c)$,

(SL4) $a \in ((a \vee b) \wedge a) \cap ((a \wedge b) \vee a)$,

(SL5) $a \leq b$ *implies* $b \in a \vee b$ *and* $a \in a \wedge b$,

(SL6) if $a \in a \wedge b$ *or* $b \in a \vee b$ *then* $a \leq b$.

Definition 2.4. [6] *Let* A *be a set,* $\odot$ *be a binary hyperoperation on* A *and* $1 \in A$. $(A, \odot, 1)$ *is called a commutative semihypergroup with* 1 *as an identity if it satisfies the following properties: for all* $x, y, z \in A$,

(CSHG1) $x \odot (y \odot z) = (x \odot y) \odot z$,

(CSHG2) $x \odot y = y \odot x$,

(CSHG3) $x \in 1 \odot x$.

Moreover if $a \in A$ *such that* $|a \odot x| = 1$ *for all* $x \in A$, *we call* a *a scalar element of* A, *simply a scalar of* A.

Example 2.5. *Let* $(L, \leq)$ *be a* $\wedge-$*semilattice with the least element* 0 *and the largest element* 1. *Define the binary hyperoperations* $\bar{\vee}$ *on* L *as follows:* $a \bar{\vee} b = \{c | a \leq c, b \leq c\}$, $a \bar{\wedge} b = \{a \wedge b\}$ *for all* $a, b \in L$. *Then* $(L, \bar{\vee}, \bar{\wedge})$ *is a bounded super lattice.*

Proposition 2.6. [26] *Let* $(L, \leq)$ *be a partially ordered set with the least element* 0 *and the largest element* 1. *Define the binary hyperoperations* $\vee$ *and* $\wedge$ *on* L *as follows:* $a \vee b = \{c | a \leq c, b \leq c\}$ *and* $a \wedge b = \{c | c \leq a, c \leq b\}$, *for all* $a, b \in L$. *Then* $(L, \vee, \wedge)$ *is a bounded super lattice.*

Definition 2.7. [32] *Let* $(P, \leq)$ *be a partially ordered set and* γ *be an equivalence relation on* P. *Then* γ *is called regular if the set* $P/\gamma = \{[x] | x \in P\}$ *can be ordered in such a way that the natural map* $\pi : P \to P/\gamma$ *is order preserving.*

Definition 2.8. [32] *Let* γ *be a regular equivalence relation on partially ordered set* $(P, \leq)$.

(1) By a γ*-fence we shall mean an ordered subset of* P *having the following diagram (Figure 1), where* $a_i \leq b_{i+1}$ *and three vertical lines indicate the equivalence modulo* γ. *We often denote this* γ*-fence by* $\langle a_1, b_n \rangle_\gamma$, *and say that a* γ*-fence* $\langle a_1, b_n \rangle_\gamma$ *joins* a_1 *to* b_n.

(2) By a γ*-crown we shall mean an ordered subset of* P *having the following diagram (Figure 2), where* $a_i \leq b_{i+1}, a_n \leq b_1$ *and three vertical lines indicate the equivalence modulo* γ. *We often denote this -crown by* $\langle\langle a_1, b_n \rangle\rangle_\gamma$.

(3) A γ*-crown* $\langle\langle a_1, b_n \rangle\rangle_\gamma$ *is called* γ*-closed, when* $a_i \gamma b_j$, *for all* $i, j \in \{1, 2, ..., n\}$.

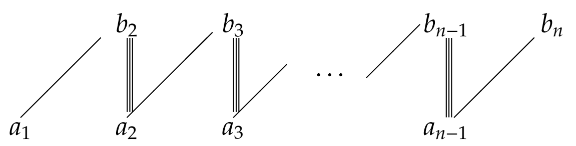

Figure 1 $\gamma - fence$

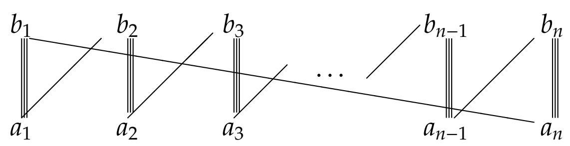

Figure 2 $\gamma - crown$

Theorem 2.9. [5] *Let γ be an equivalence relation on ordered set $(P, \leq)$ and $\leq_\gamma$ be the relation on $P/\gamma = \{[x] \mid x \in P\}$ defined by $[x] \leq_\gamma [y]$ if and only if there is a $\gamma - fence$ that joins x to y. Then the following statements are equivalent:*

(1) $\leq_\gamma$ is an order on P/γ,

(2) γ is regular,

(3) every $\gamma - crown$ is $\gamma - closed$.

Definition 2.10. [32] *By a hyper residuated lattice we mean a nonempty set L endowed with four binary hyperoperations $\vee, \wedge, \odot, \rightarrow$ and two constants 0 and 1 satisfying the following conditions:*

(HRL1) $(L; \vee, \wedge, 0, 1)$ is a bounded super lattice,

(HRL2) $(L; \odot, 1)$ is commutative semihypergroup with 1 as an identity,

(HRL3) $a \odot c \ll b$ if and only if $c \ll a \rightarrow b$,

where $A \ll B$ means that there exist $a \in A$ and $b \in B$ such that $a \leq b$, for all nonempty subsets A and B of L.

Definition 2.11. [15] *A hyper MV-algebra is a nonempty set M endowed with a hyper-operation $\oplus$, a unary operation $*$ and a constant 0 satisfying the following axioms:*

(a1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$,

(a2) $x \oplus y = y \oplus x$,

(a3) $(x^*)^* = x$,

(a4) $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$,

(a5) $0^* \in x \oplus 0^*$,

(a6) $0^* \in x \oplus x^*$,

(a7) $x \ll y, y \ll x \Rightarrow x = y$,

for all $x, y, z \in M$, where $x \ll y$ is defined by $0^* \in x^* \oplus y$.

For every subsets A and B of M, we define $A \ll B \Leftrightarrow (\exists a \in A)(\exists b \in B)(a \ll b)$, $A \oplus B = \bigcup_{a\in A, b\in B} a \oplus b$. We also define $0^* = 1$ and $A^* = \{a^* | a \in A\}$.

3. Hyper BL-algebras

In this section, we will introduce the notion of hyper BL-algebras, and study some properties of it.

Definition 3.1. *By a hyper BL-algebra we mean a nonempty set L endowed with four binary hyperoperations $\vee, \wedge, \odot, \rightarrow$ and two constants 0 and 1 satisfying the following conditions:*

(HBL1) $(L, \vee, \wedge, 0, 1)$ is a bounded super lattice,

(HBL2) $(L, \odot, 1)$ is commutative semihypergroup with 1 as an identity,

(HBL3) $a \odot c \ll b$ if and only if $c \ll a \rightarrow b$,

(HBL4) $x \wedge y \ll x \odot (x \rightarrow y)$,

(HBL5) $1 \in (x \rightarrow y) \vee (y \rightarrow x)$,

where $A \ll B$ means that there exist $a \in A$ and $b \in B$ such that $a \leq b$, for all nonempty subsets A and B of L.

Proposition 3.2. *(1) Let $(L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ be a BL-algebra. We define $x \bar{\circ} y = \{x \circ y\}$, for any $\circ \in \{\vee, \wedge, \odot, \rightarrow\}$. Then $(L, \bar{\vee}, \bar{\wedge}, \bar{\odot}, \bar{\rightarrow}, 0, 1)$ is a hyper BL-algebra.*

(2) Let $(L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ be a hyper BL-algebra satisfying that four binary hyperoperations $\vee, \wedge, \odot, \rightarrow$ are all binary operations. Then L is a BL-algebra.

(3) Any hyper BL-algebra is a hyper residuated lattice.

Proof. (1) Straightforward.

(2) Let L be a hyper BL-algebra satisfying that all binary hyperoperations are binary operations. Then by (HBL1-3), we have L is a residuated lattice. It follows from (1) and (5) of Proposition 2.2 that $x \odot (x \rightarrow y) \leq y$ and $x \odot (x \rightarrow y) \leq x \odot 1 = x$ and thus $x \odot (x \rightarrow y) \leq x \wedge y$. By (HBL4), we get $x \wedge y \ll x \odot (x \rightarrow y)$ and hence $x \wedge y = x \odot (x \rightarrow y)$. It follows from (HBL5) that $(x \rightarrow y) \vee (y \rightarrow x) = 1$.

(3) Straightforward. □

Remark 3.3. *(1) From Proposition 3.2(1) we have that any BL-algebra can be seen as a hyper BL-algebra. And from Proposition 3.2(2), we know that the notion of hyper BL-algebras is a generalization of the notion of BL-algebras.*

(2) From Proposition 3.2(3), we know that any hyper BL-algebra is a hyper residuated lattice. When all hyper-operations of a hyper BL-algebra L are binary operations, then L is a residuated lattice. Since in every residuated lattice, $x \odot (x \to y) \leq x \wedge y$ *holds. By (HBL4) we can get* $x \odot (x \to y) = x \wedge y$ *in L. Combining (HBL5) we get that L is a BL-algebra. By the way when we choose (HBL4) as one of the axioms of hyper BL-algebras, we can find many examples for hyper BL-algebra. Based on the above two points we choose (HBL4) as one of the axioms of hyper BL-algebras, instead of* $x \wedge y = x \odot (x \to y)$.

Proposition 3.4. *Let* $(L, \vee, \wedge)$ *be a bounded super lattice given in Example 2.5. Moreover define* $a \odot b = a \wedge b$ *and* $a \to b = \{c | a \wedge c \leq b\}$. *Then* $(L, \vee, \wedge, \odot, \to, 0, 1)$ *is a hyper BL-algebra.*

Proof. (HBL1) and (HBL5) follow from Example 2.5. It is clear that (HBL2) hold. Now we prove that (HBL3) holds. Let $a \odot c \leq b$. Then $c \in a \to b$ and hence $c \ll a \to b$. Conversely let $c \ll a \to b$. Then there is $c_1 \in a \to b$ such that $c \leq c_1$. Therefore $a \wedge c \leq a \wedge c_1 \leq b$. This shows that $a \wedge c \leq b$, or $a \odot c \leq b$. Finally we prove that (HBL4) holds. Let $x, y \in L$ and $c = x \wedge y$. Then $c \leq x, y$ and hence $x \wedge c \leq y$. Thus $c \in x \to y$. This shows that $c = x \odot c \in x \odot (x \to y)$. Therefore $x \wedge y \subseteq x \odot (x \to y)$. This shows that (HBL4) holds. □

Example 3.5. *let* $L = [0, 1]$, *we define the binary hyperoperations as* $a \bar{\wedge} b = min\{a, b\}$, $a \bar{\vee} b = \{x \in L \mid a \leq x, b \leq x\}$, *for all* $a, b \in L$. *Then* $(L, \bar{\vee}, \bar{\wedge})$ *is a bounded super lattice. Moreover define* $a \bar{\odot} b = a \bar{\wedge} b$ *and* $a \bar{\to} b = \{c | a \wedge c \leq b\}$, *for all* $a, b \in L$. *Then* $(L, \bar{\vee}, \bar{\wedge}, \bar{\odot}, \bar{\to}, 0, 1)$ *is a hyper BL-algebra.*

Example 3.6. *Let* $L = \{0, a, b, c, 1\}$ *and* $(L, \leq)$ *be a partially ordered set such that* $0 < a < b < c < 1$. *Define the binary hyperoperations* $\vee$ *and* $\wedge$ *by* $x \vee y = \{u \mid x \leq u, y \leq u\}$ *and* $x \wedge y = \{u \mid u \leq x, u \leq y\}$ *for all* $x, y \in L$. *Then by Proposition 2.6,* $(L, \vee, \wedge, 0, 1)$ *is a bounded super lattice. Moreover define* $\odot$ *by*

$$x \odot y = \begin{cases} \{0\}, & if\ x = 0\ or\ y = 0; \\ \{y\}, & if\ x = 1; \\ \{x\}, & if\ y = 1; \\ (x \wedge y) \setminus \{0\}, & if\ x, y \in L \setminus \{0, 1\}. \end{cases}$$

Now, consider the following table:

$\to$	0	a	b	c	1
0	{1}	{1}	{1}	{1}	{1}
a	{0}	{b,1}	{b,1}	{c,1}	{1}
b	{0}	{b,c}	{b,1}	{b,1}	{1}
c	{0}	{a,c}	{b,c}	{c,1}	{1}
1	{0}	{a}	{b}	{c}	{1}

It is easy to verify that $(L, \vee, \wedge, \odot, \to, 0, 1)$ *is a hyper BL-algebra.*

Example 3.7. *Let* $(\{0, a, b, c, 1\}, \leq)$ *be a partially ordered set such that* $0 < a < b < c < 1$. *Consider the following tables:*

$\vee$	0	a	b	c	1
0	{0, a, c, 1}	{a, c, 1}	{b, c, 1}	{c, 1}	{1}
a	{a, c, 1}	{a, c, 1}	{b, c, 1}	{c, 1}	{1}
b	{b, c, 1}	{b, c, 1}	{b, c, 1}	{c, 1}	{1}
c	{c, 1}	{c, 1}	{c, 1}	{c,1}	{1}
1	{1}	{1}	{1}	{1}	{1}

$\wedge$	0	a	b	c	1
0	{0}	{0}	{0}	{0}	{0}
a	{0}	{a, 0}	{a, 0}	{a, 0}	{a, 0}
b	{0}	{a, 0}	{b, 0}	{a, b, 0}	{a, b, 0}
c	{0}	{a, 0}	{a, b, 0}	{a, c, 0}	{a, c, 0}
1	{0}	{a, 0}	{a, b, 0}	{a, c, 0}	{0, 1, a, c}

→	0	a	b	c	1
0	{1}	{1}	{1}	{1}	{1}
a	{a, c, 1}	{a, 1}	{b,1}	{c, 1}	{1}
b	{b, c, 1}	{b, 1}	{b, 1}	{c, 1}	{1}
c	{c, 1}	{c,1}	{c,1}	{c,1}	{1}
1	{1, c}	{1, c}	{1, c}	{1, c}	{1}

Let $\odot = \wedge$. It is easy to verify that $(L; \vee, \wedge, \odot, \rightarrow, 0, 1)$ is a hyper BL-algebra.

Example 3.8. *Let $L = \{0, a, b, c, 1\}$ and $(L, \leq)$ be a partially ordered set such that $0 < c < a < b < 1$. Define the binary hyperoperations $\vee, \wedge, \odot$ on L as follows: $x \vee y = \{z \mid x \leq z, y \leq z\}$ and $x \odot y = x \wedge y = \{z \mid z \leq x, z \leq y\}$, for all $x, y \in L$. Now, let $\rightarrow$ be a hyperoperation on L defined by the following table.*

→	0	a	b	c	1
0	{1}	{1}	{1}	{1}	{1}
a	{0, 1}	{1}	{1}	{c, 1}	{1}
b	{0, 1}	{a, b, 1}	{1}	{c, 1}	{1}
c	{1}	{1}	{1}	{1}	{1}
1	{0, 1}	{a, b, 1}	{b, 1}	{1, c}	{1}

It is easy to check that $(L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ is a hyper residuated lattice([32]). We also can check that it is a hyper BL-algebra.

In the following we give some basic properties of hyper BL-algebras.

Proposition 3.9. *In any hyper BL-algebra $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$, the following properties hold: for any $x, y, z \in L$ and for any non-empty subsets A, B, C of L,*

(1) $1 \ll A$ *implies* $1 \in A$, $A \ll 0$ *implies* $0 \in A$;

(2) $A \odot B \ll C$ *if and only if* $A \ll B \rightarrow C$;

(3) $A \ll x \ll B$ *implies* $A \ll B$. *If* $A \cap B \neq \emptyset$, *then* $A \ll B$ *and* $B \ll A$,

(4) $x \odot (x \rightarrow y) \ll y$, $x \ll y \rightarrow (x \odot y)$,

(5) $x \ll (x \rightarrow y) \rightarrow y$, $A \ll (A \rightarrow B) \rightarrow B$;

(6) $x \rightarrow y \ll ((x \rightarrow y) \rightarrow y) \rightarrow y$;

(7) $x \leq y$ *implies* $1 \in x \rightarrow y$, *and if* 1 *is a scalar element of* L, *the converse holds. Moreover,* $A \ll B$ *implies* $1 \in A \rightarrow B$;

(8) $1 \in x \rightarrow x$, $1 \in x \rightarrow 1$, *and* $1 \in 0 \rightarrow x$;

(9) *if* 1 *is a scalar element, for all* $x \in L$, $x \in 1 \rightarrow x$ *and* x *is the greatest element of* $1 \rightarrow x$.

(10) $(x \odot y) \rightarrow z \ll x \rightarrow (y \rightarrow z)$,
$(x \odot y) \rightarrow z \ll y \rightarrow (x \rightarrow z)$;

(11) $x \rightarrow (y \rightarrow z) \ll (x \odot y) \rightarrow z$,
$y \rightarrow (x \rightarrow z) \ll (x \odot y) \rightarrow z$,

(12) $x \rightarrow (y \rightarrow z) \ll y \rightarrow (x \rightarrow z)$

(13) $x \leq y$ *implies* $x \odot z \ll y \odot z$, $z \rightarrow x \ll z \rightarrow y$, *and* $y \rightarrow z \ll x \rightarrow z$,

(14) $x \leq y$ *implies* $y^- \ll x^-$, *where* $x^- = x \rightarrow 0$;

(15) $x \odot y \ll x$, $x \odot y \ll y$. *Particularly,* $0 \in x \odot 0$;

(16) $x \ll y \rightarrow x$, $1 \in x \rightarrow (y \rightarrow x)$;

(17) $y \ll (x \rightarrow y) \rightarrow y$, $B \ll (A \rightarrow B) \rightarrow B$;

(18) $0 \in x \odot x^-$;

(19) $x \ll x^{--}$;

(20) $x \leq y$ *and* $x \leq z$ *imply* $x \ll y \wedge z$,
$y \leq x$ *and* $z \leq x$ *imply* $y \vee z \ll x$;

(21) $x \wedge y \ll x$, $x \ll x \vee y$, $A \wedge B \ll A$, $A \ll A \vee B$;

(22) $x \rightarrow (x \wedge y) \ll x \rightarrow y$;

(23) $x \ll y^- \Leftrightarrow 0 \in x \odot y$.

Proof. (1) If $1 \ll A$, then $1 \leq a$, for some $a \in A$ whence $1 = a \in A$. If $A \ll 0$, then $a \leq 0$, for some $a \in A$ whence $0 = a \in A$.

(2) Let $A \ll B \to C$. Then there are $a \in A$, $b \in B$, and $c \in C$ such that $a \ll b \to c$. Therefore $a \odot b \ll c$ for $a \in A$, $b \in B$, and $c \in C$. This means $A \odot B \ll C$. Conversely let $A \odot B \ll C$. Then $a \odot b \ll c$ for $a \in A$, $b \in B$, and $c \in C$. Therefore $a \ll b \to c$. It follows that $A \ll B \to C$.

(3) Straightforward.

(4) It follows from $x \to y \ll x \to y$ and $x \odot y \ll x \odot y$ that $x \odot (x \to y) \ll y$ and $x \ll y \to (x \odot y)$

(5) Clearly by (2) and (4).

(6) Clearly by (5).

(7) Let $x \leq y$. Note that $x \in 1 \odot x$ and hence $1 \odot x \ll x$. Since $x \leq y$, then $1 \odot x \ll y$. Thus $1 \ll x \to y$ and hence $1 \in x \to y$. Let $1 \in x \to y$. Then $1 \ll x \to y$ and so $1 \odot x \ll y$ by (HBL3). By (CSHG3), $x \in 1 \odot x$, and hence $x = 1 \odot x$ since 1 is scalar. Therefore $x \leq y$. Moreover, let $A \ll B$. Then $a \leq b$ for some $a \in A, b \in B$ and hence $1 \in a \to b \subseteq A \to B$.

(8) Clearly by (7).

(9) For any $u \in 1 \to x$, we have $u \ll 1 \to x$ and so $\{u\} = u \odot 1 \ll x$. Thus $u \leq x$. Since $x \in x \odot 1$, then we get $1 \odot x \ll x$ it follows that $x \ll 1 \to x$. Hence there exists $u \in 1 \to x$ such that $x \leq u$. So $x \leq u \leq x$. Therefore, $x \in 1 \to x$ and x is the greatest element of $1 \to x$.

(10) Let $u \in (x \odot y) \to z$, then $u \ll (x \odot y) \to z \Leftrightarrow u \odot (x \odot y) \ll z \Leftrightarrow (u \odot x) \odot y \ll z \Leftrightarrow u \odot x \ll y \to z \Leftrightarrow u \ll x \to (y \to z)$, so $(x \odot y) \to z \ll x \to (y \to z)$. Moreover we have $(x \odot y) \to z = (y \odot x) \to z \ll y \to (x \to z)$.

(11) Let $t \in x \to (y \to z)$. Then there is $u \in y \to z$ such that $t \in x \to u$. Hence we have $t \ll x \to u$, and so $x \odot t \ll u$. Thus $x \odot t \ll y \to z$, and hence $s \ll y \to z$ for some $s \in x \odot t$. By (HBL3) $s \odot y \ll z$, for some $s \in x \odot t$. Therefore $(x \odot t) \odot y \ll z$, or $(x \odot y) \odot t \ll z$. Then $v \odot t \ll z$ for some $v \in x \odot y$. By (HBL3) again we get $t \ll v \to z$, and thus $t \ll (x \odot y) \to z$. This shows that $x \to (y \to z) \ll (x \odot y) \to z$.

Similarly, we have $y \to (x \to z) \ll (x \odot y) \to z$.

(12) Let $u \in x \to (y \to z)$, then $u \ll x \to (y \to z)$ and so $u \ll x \to a$, for some $a \in y \to z$. Hence $u \odot x \ll a$ and so $b \leq a$, for some $b \in u \odot x$. Since $a \in y \to z$, then we get $b \ll y \to z$ and so $b \odot y \ll z$. Hence $(u \odot y) \odot x = (u \odot x) \odot y \ll z$, so we get $u \odot y \ll x \to z$. Therefore, $u \ll y \to (x \to z)$ and so $x \to (y \to z) \ll y \to (x \to z)$.

(13) Since $y \ll z \to (y \odot z)$ and $x \leq y$, then $x \ll z \to (y \odot z)$. Hence we get $x \odot z \ll y \odot z$.

Let $u \in z \to x$, then $u \ll z \to x$, so $u \odot z \ll x$. Since $x \leq y$, then we get $u \odot z \ll y$ and so $u \ll z \to y$. Therefore, $z \to x \ll z \to y$,

Let $u \in y \to z$, then $u \ll y \to z$ and so then $y \ll u \to z$. Since $x \leq y$, we get that $x \ll u \to z$. Hence we get $u \ll x \to z$ and so $y \to z \ll x \to z$.

(14) Clearly by (13).

(15) Since $y \leq 1 \in x \to x$, so $x \odot y \ll x$. Similarly, it follows that $x \odot y \ll y$. Particularly, $x \odot 0 \ll 0$, so $0 \in x \odot 0$.

(16) Since $x \odot y \ll x$, we get $x \ll y \to x$, then by (7) $1 \in x \to (y \to x)$.

(17) Since $y \odot (x \to y) \ll y$, we get $y \ll (x \to y) \to y$, and $B \ll (A \to B) \to B$;

(18) Since $x^- \ll x^-$, we have $x \odot x^- \ll 0$, then $0 \in x \odot x^-$.

(19) By (18), since $0 \in x \odot x^-$, we have $x \odot x^- \ll 0$, then $x \ll x^{--}$.

(20) Let $x \leq y$ and $x \leq z$. Then $x \in x \wedge y$ and $x \in x \wedge z$ by (SL5). Thus $x \in (x \wedge y) \wedge z = x \wedge (y \wedge z)$. This shows that there is $a \in y \wedge z$ such that $x \in x \wedge a$. It follows from (SL6) that $x \leq a$. Hence $x \ll y \wedge z$. Similarly we can prove that $y \leq x$ and $z \leq x$ imply $y \vee z \ll x$.

(21) From the properties of super lattices, it is known that $x \wedge y \ll x$ and $x \ll x \vee y$. Then since $a \wedge b \ll a$ for any $a \in A, b \in B$, we have $A \wedge B \ll A$. Similarly, $a \ll a \vee b$, for any $a \in A, b \in B$, implies $A \ll A \vee B$.

(22) Clearly by (13) and (21).

(23) Note that $x \ll y^- = y \to 0 \Leftrightarrow x \odot y \ll 0 \Leftrightarrow 0 \in x \odot y$.

□

4. Hyper filters on hyper BL-algebras

In the section we set up the theory of filters on hyper BL-algebras.

Definition 4.1. *A non-empty subset F of a hyper BL-algebra L satisfying*
(F) $x \leqslant y$ and $x \in F$ imply $y \in F$,
is called
(1) an h-filter of L if it satisfies (HF): $x \odot y \subseteq F$ for all $x, y \in F$,
(2) a weak h-filter of L if it satisfies (WHF): $F \ll x \odot y$ for all $x, y \in F$.

Definition 4.2. *Let D be a non-empty subset of a hyper BL-algebra L satisfying:*
(DS) $1 \in D$,
(WDS) *$x \in D$ and $D \ll x \to y$ imply $y \in D$.*
Then D is called a weak h-deductive system.

Example 4.3. *(1) Consider the hyper BL-algebra L given in Example 3.7. Let $F = \{b, c, 1\}$. Then F is a weak h-filter of L but F is not a weak h-deductive system since $b \in F, F \ll b \to a$ but $a \notin F$.*

(2) Consider the hyper BL-algebra L given in Example 3.6. Let $D = \{1\}$. Then D is a weak h-deductive system of L.

Remark 4.4. *It is easy to see that any h-filter of a hyper BL-algebra L is a weak h-filter of L. Moreover $1 \in F$ for any (weak) h-filter of L.*

Proposition 4.5. *Let L be a hyper BL-algebra, then*
(1) every weak h-deductive system satisfies (F);
(2) if D is a non-empty subset of L containing 1, then D is a weak h-deductive system of L if and only if D satisfies the following condition:
(D) *$(x \to y) \cap D \neq \emptyset$ and $x \in D$ imply $y \in D$,*
(3) every weak h-deductive system is a weak h-filter,
(4) $\{1\}$ is a weak h-filter of L,
(5) if 1 is scalar, then $\{1\}$ is a weak h-deductive system of L.

Proof. (1) Let F be a weak h-deductive system of L, $x \leq y$ and $x \in F$ for $x, y \in L$. Then by Proposition 3.9(7), $1 \in x \to y$, and so $F \ll x \to y$. Now, from (WDS) it follows that $y \in F$. Thus, (F) holds.

(2) ($\Rightarrow$) Let D be a weak h-deductive system of L. Then $1 \in D$ by (DS). Let $(x \to y) \cap D \neq \emptyset$ and $x \in D$. Then there is $a \in (x \to y) \cap D$ and so $D \ll x \to y$. By (WDS) we have $y \in D$.

($\Leftarrow$) Let D be a non-empty subset of L containing 1 and satisfying (D). First we prove that D satisfies (F). Let $x \leq y$ and $x \in D$. Then $1 \in x \to y$ by Proposition 3.9(7). Thus $(x \to y) \cap D \neq \emptyset$. Therefore $y \in D$ by (D). This means that (F) holds. Now, let $x \in D$, and $D \ll x \to y$. Then there exist $d \in D$ and $u \in x \to y$ such that $d \leq u$ and so by (F), $u \in D$. Hence $(x \to y) \cap D \neq \emptyset$ and so $y \in D$. Therefore D is a weak h-deductive system of L.

(3) Let F be a weak h-deductive system of L. Then by (1), (F) holds. Now, let $x, y \in F$. By Proposition 3.9(4), $y \ll x \to (x \odot y)$ and so $y \leq u$ for some $u \in x \to (x \odot y)$. Hence $u \in F$ and so $F \ll x \to v$ for some $v \in x \odot y$. Since $x \in F$, so $v \in F$ and hence $F \ll x \odot y$.

(4) Denote $F = \{1\}$. Let $x \leq y$ and $x \in F$. Then $x = 1$ and $1 \leq y$. Therefore $y = 1$ and hence $y \in F$. This shows that F satisfies (F). Moreover let $x, y \in F$. Then $x = y = 1$ and hence $1 \in x \odot y = 1 \odot 1$. It follows that $F \ll x \odot y$, that is (WHF).

(5) Let 1 be scalar and $D = \{1\}$. Clearly (DS) is true. Assume $x \in D$ and $D \ll x \to y$. Then $x = 1$ and $1 \ll x \to y$. Therefore $1 \in x \to y$. By Proposition 3.9(7), we have $x \leq y$ and hence $y = 1$. This shows that $y \in D$, that is, (WDS) holds.
□

In general, if 1 is not a scalar element of a hyper BL-algebra L, then $\{1\}$ need not be a weak h-deductive system of L. We give the following counter example.

Example 4.6. *Consider the hyper BL-algebra L given in Example 3.7, in which 1 is not a scalar element of L. Then $D = \{1\}$ is not a weak h-deductive system of L since $1 \in D$ and $D \ll 1 \to b$ but $b \notin D$.*

Proposition 4.7. *Let $\{F_i \mid i \in I\}$ be a family of non-empty subsets of a hyper BL-algebra L.*

(1) If F_i is a weak h-deductive system, for all $i \in I$, then $\cap F_i$ is a weak h-deductive system of L.

(2) Assume that $\{F_i \mid i \in I\}$ be a chain, if F_i is a weak h-filter (weak h-deductive system), for all $i \in I$,then $\cup F_i$ is a weak h-filter (weak h-deductive system) of L

Proof. Obviously.
□

Recall that a weak h-filter F (weak h-deductive system D) is called proper if $F \neq L$ ($D \neq L$).

Definition 4.8. *Let F be a proper weak h-filter (weak h-deductive system) of a hyper BL-algebra L. Then F is said to be maximal if $F \subseteq J \subseteq L$, implies $F = J$ or $J = L$, for all weak h-filters (weak h-deductive systems) J of L.*

Example 4.9. *Consider the hyper BL-algebra L given in Example 3.6. Let $D = \{a, b, c, 1\}$. Then D is a maximal weak h-filter and maximal weak h-deductive system of L.*

Proposition 4.10. *Let L be a hyper BL-algebra. Then every proper weak h-filter (weak h-deductive system) of L is contained in a maximal weak h-filter (weak h-deductive system) of L.*

Proof. Let F be a proper weak h-filter of L and S be the collection of all proper weak h-filter of L containing F. Then $F \in S$ and $(S, \subseteq)$ is a poset. Let $\{F_i \mid i \in I\}$ be a chain in S, Then $\cup F_i$ is a weak h-filter of L containing F. If $0 \in \cup F_i$, then there exists $i \in I$ such that $0 \in F_i$, which is impossible. Hence then $\cup F_i$ is a proper weak h-filter of L containing F and so $\cup F_i \in S$. Hence every chain of elements of S has an upper bounded in S. By Zorn's lemma, S has a maximal element such as M. We shall show that M is a maximal weak h-filter of L. Let $M \subseteq J \subseteq L$, for some weak h-filter J of L. If $J \neq L$, then $J \in S$. Since M is a maximal element of S we get $M = J$. Therefore, M is a maximal weak h-filter of L. □

Recall that a hyper BL-algebra L is called nontrivial, if $L \neq \{1\}$. Using Proposition 4.5 and 4.10, we have the following corollaries.

Corollary 4.11. *Every nontrivial hyper BL-algebra has a maximal weak h-filter.*

Corollary 4.12. *If 1 is a scalar element, every nontrivial hyper BL-algebra has a maximal weak h-deductive system.*

Definition 4.13. *Let $(L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ be a hyper BL-algebra and D be a non-empty subset of L containing 1. Then D is called*

(1) an implicative weak h-deductive system or simply IWDS if $(x \rightarrow (y \rightarrow z)) \cap D \neq \emptyset$ and $(x \rightarrow y) \cap D \neq \emptyset$ imply $(x \rightarrow z) \cap D \neq \emptyset$, for all $x, y, z \in L$,

(2) a positive implicative weak h-deductive system or simply PIWDS if $(x \rightarrow ((y \rightarrow z) \rightarrow y)) \cap D \neq \emptyset$ and $x \in D$ imply $y \in D$ for all $x, y, z \in L$.

Example 4.14. *Consider the hyper BL-algebra L given in Example 3.6. Let $D_1 = \{a, b, c, 1\}$ and $D_2 = \{b, c, 1\}$. Then D_1 is an IWDS and a PIWDS of L. Note that D_2 is not a IWDS since $(1 \rightarrow (b \rightarrow a)) \cap D_2 \neq \emptyset$ and $(1 \rightarrow b) \cap D_2 \neq \emptyset$ but $(1 \rightarrow a) \cap D_2 = \emptyset$, and it is not a PIWDS since $b \in D_2$ and $(b \rightarrow ((a \rightarrow 0) \rightarrow a)) \cap D_2 = \{1\} \cap D_2 \neq \emptyset$ but $a \notin D_2$.*

Proposition 4.15. *Let D be a non-empty subset of a hyper BL-algebra L. Then*

(1) if D is a PIWDS of L and 1 is scalar, then D is a weak h-deductive system,

(2) if D is a IWDS of L and an upset, and 1 is scalar, then D is a weak h-deductive system.

Proof. (1) Assume that D is a *PIWDS* of L. Clearly, (DS) holds. Let $(x \rightarrow y) \cap D \neq \emptyset$ and $x \in D$. Then by Proposition 3.9(9), $x \rightarrow y \subseteq x \rightarrow (1 \rightarrow y) \subseteq x \rightarrow ((y \rightarrow 1) \rightarrow y)$, and so $(x \rightarrow ((y \rightarrow 1) \rightarrow y)) \cap D \neq \emptyset$. Since $x \in D$ and D is a *PIWDS* of L, we conclude that $y \in D$. Therefore D is a weak h-deductive system by Proposition 4.5(2).

(2) Assume that D is a *IWDS* of L. Clearly, (DS) holds. Let $(x \rightarrow y) \cap D \neq \emptyset$ and $x \in D$. Then $(1 \rightarrow (x \rightarrow y)) \cap D \neq \emptyset$ and $(1 \rightarrow x) \cap D \neq \emptyset$ by Proposition 3.9(9). Since D is a *IWDS* of L, then

$(1 \to y) \cap D \neq \emptyset$. Hence there is $a \in D$ and $a \in 1 \to y$. Thus $a \ll 1 \to y$ and so $1 \odot a \ll y$. Since 1 is scalar, $a = 1 \odot a \ll y$. Note that D is an upset, then we get $y \in D$. Therefore D satisfies (D) and so D is a weak h-deductive system.

□

Proposition 4.16. *Let D be a non-empty subset of a hyper BL-algebra L. Then*

(1) if 1 is scalar, then D is a PIWDS of L if and only if D is a weak h-deductive system such that $((x \to y) \to x) \cap D \neq \emptyset$ implies $x \in D$, for all $x, y \in L$;

(2) D is a IWDS of L if and only if $1 \in D$ and $D_x = \{u \in L \mid (x \to u) \cap D \neq \emptyset\}$ is a weak h-deductive system of L, for all $x \in L$.

Proof. (1) Let D be a PIWDS. Then by Proposition 4.15, D is a weak h-deductive system. Now, let $((x \to y) \to x) \cap D \neq \emptyset$, then there exist $u \in ((x \to y) \to x) \cap D$, and so $u \in 1 \to u \subseteq (1 \to ((x \to y) \to x)) \cap D$. Since $1 \in D$ and D is a *PIWDS*, then we get $x \in D$.

Conversely, let D be a weak h-deductive system such that $((x \to y) \to x) \cap D \neq \emptyset$ implies $x \in D$, for all $x, y \in L$. Let $(x \to ((y \to z) \to y)) \cap D \neq \emptyset$ and $x \in D$. Since D is a weak h-deductive system and $x \in D$, then $(y \to z) \to y)) \cap D \neq \emptyset$ and so $y \in D$. Therefore D is a PIWDS.

(2) Let D be an IWDS of L and $x \in L$. By Proposition 3.9(8), $1 \in D_x$. Now, let $(a \to b) \cap D_x \neq \emptyset$ and $a \in D_x$, for some $a, b \in L$. Then $(x \to a) \cap D \neq \emptyset$ and $(x \to (a \to b)) \cap D \neq \emptyset$. Since D is a *IWDS*, we get $(x \to b) \cap D \neq \emptyset$ and so $b \in D_x$. Hence D_x satisfies (D) and so D_x is a weak h-deductive system.

Conversely, let $1 \in D$ and $D_x = \{u \in L \mid (x \to u) \cap D \neq \emptyset\}$ be a weak h-deductive system of L, for all $x \in L$. If $(x \to (y \to z)) \cap D \neq \emptyset$ and $(x \to y) \cap D \neq \emptyset$, for $x, y, z \in L$, then $y \in D_x$ and $(y \to z) \cap D_x \neq \emptyset$. Since D_x is a weak h-deductive system of L, then we conclude that $z \in D_x$ and so $(x \to z) \cap D \neq \emptyset$. Therefore, D is a IWDS of L.

□

Theorem 4.17. *Let D be a subset of a hyper BL-algebra L. Then the following are equivalent:*

(1) D is an IWDS and a maximal weak h-deductive system of L,

(2) D is a weak h-deductive system and $(x \to y) \cap D \neq \emptyset$, and $(y \to x) \cap D \neq \emptyset$, for all $x, y \in L - D$.

Proof. (1)⇒(2) Suppose that D is an IWDS and a maximal weak h-deductive system and $x, y \in L - D$. We get $x \in D_x$, $y \in D_y$, $D \subset D_x \subseteq L$ and $D \subset D_y \subseteq L$. Moreover, by Proposition 4.16(2) we get D_x and D_y are weak h-deductive systems of L. Hence by assumption $D_x = L = D_y$ and so $y \in D_x, x \in D_y$. Therefore, $(x \to y) \cap D \neq \emptyset$, and $(y \to x) \cap D \neq \emptyset$. Clearly D is a weak h-deductive system.

(2)⇒(1) Let D be a weak h-deductive system such that $(x \to y) \cap D \neq \emptyset$, and $(y \to x) \cap D \neq \emptyset$, for all $x, y \in L - D$. If there exists $a \in L$ such that D_a is not a weak h-deductive system of L, then there are $x, y \in L$ such that $(x \to y) \cap D_a \neq \emptyset$, $x \in D_a$ and $\{y\} \cap D_a = \emptyset$. Hence $(a \to x) \cap D \neq \emptyset$ and $(a \to u) \cap D \neq \emptyset$, for some $u \in x \to y$. But $(a \to y) \cap D = \emptyset$ and so $\{y\} \cap D = \emptyset$. Hence by assumption $a \in D$. Since $(a \to x) \cap D \neq \emptyset$ and $(a \to u) \cap D \neq \emptyset$, then we get $x \in D$ and $u \in D$. It follows that $(x \to y) \cap D \neq \emptyset$. That is, by (WDS) $y \in D$, which is a contradiction. Hence D_a is a weak h-deductive system of L, for any $a \in L$. By Proposition 4.16(2), D is an IWDS. Now, we show that, D_a is the least weak h-deductive system of L containing $D \cup \{a\}$, for any $a \in L - D$. Let $a \in L - D$ and D' be a weak h-deductive system of L containing $D \cup \{a\}$ and u be an arbitrary element of D_a. Then $(a \to u) \cap D \neq \emptyset$ and so $(a \to u) \cap D' \neq \emptyset$. Since $a \in D'$, then $u \in D'$. Therefore $D_a \subseteq D'$. That is D_a is the least weak h-deductive system of L containing $D \cup \{a\}$. Assume that $D \subsetneq E \subseteq L$, for some weak h-deductive system E of L. Then there exists $a \in E - D$. It follows that $D_a \subseteq E$. Since $a \in L - D$, by assumption we get $D_a = L$ and so $E = L$. Therefore, D is a maximal weak h-deductive system of L. □

5. Hyper congruences on hyper BL-algebras

Similar to S-homomorphisms on residuated lattices, we introduce S-homomorphisms on BL-algebras.

Definition 5.1. *Let $(L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ and $(L', \vee', \wedge', \odot', \rightarrow', 0', 1')$ be two hyper BL-algebras. A function $f : L \rightarrow L'$ is called an S-homomorphism if it satisfies the following conditions: for all $x, y \in L$,*

(1) $f(x \vee y) = f(x) \vee' f(y)$,
(2) $f(x \wedge y) = f(x) \wedge' f(y)$,
(3) $f(x \odot y) = f(x) \odot' f(y)$,
(4) $f(x \rightarrow y) = f(x) \rightarrow' f(y)$,
(5) $f(1) = 1'$ and $f(0) = 0'$.
Where $f(A) = \{f(t) \mid t \in A\}$.

Example 5.2. *Let L be the hyper BL-algebra in Example 3.7. Let $L' = \{0', 1'\}$ and $(L', \leq')$ be a partially ordered set such that $0' < 1'$. Define the binary hyperoperations $\vee', \wedge', \odot'$ and $\rightarrow'$ by the following tables:*

$\vee'$	$0'$	$1'$
$0'$	$\{0', 1'\}$	$\{1'\}$
$1'$	$\{1'\}$	$\{1'\}$

$\wedge' = \odot'$	$0'$	$1'$
$0'$	$\{0'\}$	$\{0'\}$
$1'$	$\{0'\}$	$\{0', 1'\}$

$\rightarrow'$	$0'$	$1'$
$0'$	$\{1'\}$	$\{1'\}$
$1'$	$\{1'\}$	$\{1'\}$

It is easy to check that $(L', \vee', \wedge', \odot', \rightarrow', 0', 1')$ is a hyper BL-algebra. Define $f : L \rightarrow L'$ by $f(0) = 0'$, $f(1) = 1'$ and $f(a) = f(b) = f(c) = 1'$. Then f is an S-homomorphism.

Example 5.3. *Let L be the hyper BL-algebra in Example 3.7. Let $L' = \{0', e, 1'\}$ and $(L', \leq')$ be a partially ordered set such that $0' < e < 1'$. Define the binary hyperoperations $\vee', \wedge'$ by $x \vee' y = \{u \in L' \mid x \leq' u, y \leq' u\}$ and $x \wedge' y = \{u \in L' \mid u \leq' x, u \leq' y\}$, for all $x, y \in L'$. It is easy to check that $(L', \vee', \wedge', 0', 1')$ is a bounded super lattice.*
Consider the following tables:

$\odot'$	$0'$	e	$1'$
$0'$	$\{0'\}$	$\{0'\}$	$\{0'\}$
e	$\{0'\}$	$\{0', e\}$	$\{0', e\}$
$1'$	$\{0'\}$	$\{0', e\}$	$\{0', e, 1'\}$

$\rightarrow'$	$0'$	e	$1'$
$0'$	$\{1'\}$	$\{1'\}$	$\{1'\}$
e	$\{e, 1'\}$	$\{e, 1'\}$	$\{1'\}$
$1'$	$\{e, 1'\}$	$\{e, 1'\}$	$\{1'\}$

It is easy to check that $(L', \vee', \wedge', \odot', \rightarrow', 0', 1')$ is a hyper BL-algebras. Define $f : L \rightarrow L'$ by $f(0) = 0'$, $f(1) = 1'$ and $f(a) = f(b) = f(c) = e$. Then f is an S-homomorphism.

In this section, we study hyper congruences on hyper BL-algebras. From now on, L and L' denote two hyper BL-algebras.

In the following we introduce the notion of regular compatible congruence relations on a hyper BL-algebra and discuss some useful properties of these relations.

Definition 5.4. *Let θ be an equivalence relation on a hyper BL-algebra L and $A, B \subseteq L$. Then*
(i) $A\theta B$ means that there exist $a \in A$ and $b \in B$ such that $a\theta b$,
(ii) $A\overline{\theta}B$ means that for all $a \in A$, there exist $b \in B$ such that $a\theta b$ and for all $b \in B$, there exist $a \in A$ such that $a\theta b$.

Definition 5.5. *An equivalence relation θ on a hyper BL-algebra L is called a congruence relation if for all $x, y, z, w \in L$, $x\theta y$ and $z\theta w$ imply $(x * z)\overline{\theta}(y * w)$, where $* \in \{\wedge, \vee, \odot, \rightarrow\}$.*

Proposition 5.6. *Let θ be a regular congruence relation on a hyper BL-algebra L. Then $[1] = \{x \in L \mid x\theta 1\}$ is a weak h-filter of L.*

Proof. Clearly, $[1] \neq \emptyset$. Let $x, y \in [1]$. Since $(x \odot y)\overline{\theta}(1 \odot 1)$ and $1 \in 1 \odot 1$, then $x \odot y\theta 1$. Hence $(x \odot y) \cap [1] \neq \emptyset$ and so $[1] \ll x \odot y$. Now, let $x, y \in L$ be such that $x \in [1]$ and $x \leq y$. Since θ is regular, then $x \rightarrow [x]$ is order preserving. It follows from $x \leq y \leq 1$ that $[x] \leq_\theta [y] \leq_\theta [1]$. Since $[x] = [1]$, we have $[x] = [y] = [1]$ and so $y \in [1]$. Therefore, [1] is a weak hyper filter of L. □

Proposition 5.7. *Let θ be a regular congruence relation on a hyper BL-algebra L, $L/\theta = \{[x] \mid x \in L\}$ and $\leq_\theta$ be the relation on L/θ defined as in Theorem 2.9. For all $x, y \in L$, define $[x]\overline{\odot}[y] = [x \odot y]$, $[x]\overline{\vee}[y] = [x \vee y]$, $[x]\overline{\wedge}[y] = [x \wedge y]$ and $[x] \rightsquigarrow [y] = [x \to y]$, where $[A] = \{[a] \mid a \in A\}$, for all $A \subseteq L$. Then*

(i) $\overline{\odot}, \overline{\vee}, \overline{\wedge}, \rightsquigarrow$ are well defined,

(ii) $[x] \leq_\theta [y] \rightsquigarrow [z]$ if and only if $[x]\overline{\odot}[y] \leq_\theta [z]$, where $[A] \leq_\theta [B]$ if and only if $[a] \leq_\theta [b]$, for some $a \in A$ and $b \in B$.

Proof. Similar to Lemma 3.8 in paper [32]. □

Definition 5.8. *Let θ be a regular congruence relation on a hyper BL-algebra L. We say that $\leq_\theta$, $\overline{\vee}$ and $\overline{\wedge}$ are compatible if they satisfy the following conditions: for all $x, y \in L$,*

(i) $[x] \in [x]\overline{\vee}[y]$ if and only if $[y] \leq_\theta [x]$,

(ii) $[x] \in [x]\overline{\wedge}[y]$ if and only if $[x] \leq_\theta [y]$

By a regular compatible congruence relation on L we mean a regular congruence relation on L such that $\leq_\theta$, $\overline{\vee}$ and $\overline{\wedge}$ are compatible.

Example 5.9. *Let L be the hyper BL-algebra in Example 3.7. Let $\theta = \{(x, x) \mid x \in L\} \cup \{(a, b), (b, a)\}$. We can check that θ is a regular compatible congruence relation on L.*

Proposition 5.10. *Let θ be a regular compatible congruence relation on a hyper BL-algebra L. Then $(L/\theta, \overline{\vee}, \overline{\wedge}, \overline{\odot}, \rightsquigarrow, [0], [1])$ is a hyper BL-algebra.*

Proof. By Proposition 3.2(3), L is a hyper residuated lattice. From Theorem 3.10 in [32], L/θ is a hyper residuated lattice. It is only need to check (BL4) and (BL5).

For (BL4), since $x \wedge y \ll x \odot (x \to y)$, then there exist $a \in x \wedge y$ and $b \in x \odot (x \to y)$ such that $a \leq b$, so $[a] \leq_\theta [b]$, and so $[x \wedge y] \ll_\theta [x \odot (x \to y)]$, therefore, $[x]\overline{\wedge}[y] \ll_\theta [x]\overline{\odot}([x] \rightsquigarrow [y])$.

For (BL5), since $1 \in (x \to y) \vee (y \to x)$, we have $[1] \in [(x \to y) \vee (y \to x)]$, and so $[1] \in ([x] \rightsquigarrow [y])\overline{\vee}([y] \rightsquigarrow [x])$. Therefore, $(L/\theta, \overline{\vee}, \overline{\wedge}, \overline{\odot}, \rightsquigarrow, [0], [1])$ is a hyper BL-algebra. □

Example 5.11. *Let L and θ be the hyper BL-algebra and the regular compatible congruence relation in Example 5.9, repectively. Therefore, by Proposition 5.10, $(L/\theta; \overline{\vee}, \overline{\wedge}, \overline{\odot}, \rightsquigarrow, [0], [1])$ is a hyper BL-algebra. Where its hyper operations are as follows:*

$\overline{\vee}$	[0]	[a]	[c]	[1]
[0]	{[0],[a],[c],[1]}	{[a],[c],[1]}	{[c],[1]}	{[1]}
[a]	{[a],[c],[1]}	{[a],[c],[1]}	{[c],[1]}	{[1]}
[c]	{[c],[1]}	{[c],[1]}	{[c],[1]}	{[1]}
[1]	{[1]}	{[1]}	{[1]}	{[1]}

$\overline{\wedge}$	[0]	[a]	[c]	[1]
[0]	{[0]}	{[0]}	{[0]}	{[0]}
[a]	{[0]}	{[a],[0]}	{[a],[0]}	{[a],[0]}
[c]	{[0]}	{[a],[0]}	{[a],[c],[0]}	{[a],[c],[0]}
[1]	{[0]}	{[a],[0]}	{[a],[c],[0]}	{[0],[a],[c],[1]}

$\rightsquigarrow$	[0]	[a]	[c]	[1]
[0]	{[1]}	{[1]}	{[1]}	{[1]}
[a]	{[a],[c],[1]}	{[a],[1]}	{[c],[1]}	{[1]}
[c]	{[c],[1]}	{[c],[1]}	{[c],[1]}	{[1]}
[1]	{[c],[1]}	{[c],[1]}	{[c],[1]}	{[1]}

And $\overline{\odot} = \overline{\wedge}$.

Proposition 5.12. *Let θ be a regular compatible congruence relation on a hyper BL-algebra L. Then*
(1) $[1]$ *is an h-filter of L if and only if* $\{[1]\}$ *is an h-filter of* L/θ,
(2) if $[1]$ *is a maximal weak h-filter of L, then* L/θ *is simple.*

Proof. Similar to Proposition 3.12 in paper [32]. □

Example 5.13. *Let L be the hyper BL-algebra in Example 3.8. Let*
$\theta = \{(x,x) \mid x \in L\} \cup \{(1,a),(a,1),(1,b),(b,1),(a,b),(b,a),(c,0),(0,c)\}$.
We can check that θ is a regular compatible congruence relation of L and so by Proposition 5.10, $(L/\theta; \overline{\vee}, \overline{\wedge}, \overline{\odot}, \rightsquigarrow, [0], [1])$ *is a hyper BL-algebra. Since* $L/\theta = \{[0],[1]\}$, L/θ *is simple. Moreover,* $F = \{1,a,b,c\}$ *is a weak h-filter of L and* $[1] \subset F \subset L$ *and so* $[1] = \{1,a,b\}$ *is not a maximal weak h-filter of L. Therefore, the converse of the above Proposition (2) is not true.*

Let L and L' be two hyper BL-algebras and $f : L \to L'$ be an S-homomorphism. It is straightforward to check that $ker(f) = \{(x,y) \in L \times L \mid f(x) = f(y)\}$ is an equivalence relation on L.

Theorem 5.14. *Let L and L' be two BL-algebras, $f : L \to L'$ be an S-homomorphism and $\theta = ker(f)$. If $f(x) \leq f(y)$ implies there is a $\theta - fence$ that joins x to y, for all $x, y \in L$, then*
(i) *θ is a regular compatible congruence relation on L and $L/ker(f)$ is a hyper BL-algebra.*
(ii) *f induces a unique S-homomorphism $\overline{f} : L/ker(f) \to L'$ by $\overline{f}([x]) = f(x)$, for all $x \in L$ such that $Im(\overline{f}) = Im(f)$ and $\overline{f}$ is a S-homomorphism.*

Proof. Similar to Theorem 3.14 in paper [32]. □

Example 5.15. *If L and L' are two BL-algebras and $f : L \to L'$ is a homomorphism (which could be treated as an S-homomorphism between L and L'), then $f(x) \leq f(y)$ implies $f(x) = f(x) \wedge f(y) = f(x \wedge y)$ and so the set $\{x, x, x \wedge y, y\}$ forms a $ker(f) - fence$ that joins x to y. Therefore, f satisfies the conditions (i) and (ii) in the above Theorem.*

Example 5.16. *Let L, L' and f be two hyper BL-algebras and an S-homomorphism given in Example 5.3. Then*
$ker(f) = \{(x,x) \mid x \in L\} \cup \{(a,b),(b,a),(a,c),(c,a),(b,c),(c,b)\}$
and $L/ker(f) = \{[0],[a],[1]\}$.
Assume $<= \{(x,x) \mid x \in L/ker(f)\} \cup \{([0],[a]),([a],[1]),([0],[1])\}$. *Then clearly, $<$ is a partially order on $L/ker(f)$. Since the map $\pi : L \to L/ker(f)$ defined by $\pi(x) = [x]$ is an order preserving map, then $ker(f)$ is regular. We can show that $f(x) \leq f(y)$ implies there exists a $\theta - fence$ that joins x to y, for any $x, y \in L$, and so by the above proposition, we have $\overline{f} : L/ker(f) \to L'$ is a one to one homomorphism.*

Example 5.17. *Let* $(L_1; \vee_1, \wedge_1, \odot_1, \to_1, 0_1, 1_1)$ *and* $(L_2; \vee_2, \wedge_2, \odot_2, \to_2, 0_2, 1_2)$ *be two hyper BL-algebras. We define the hyperoperations $\vee, \wedge, \odot, \to$ on $L = L_1 \times L_2$ as follows:*
$(x_1,x_2) \vee (y_1,y_2) = (x_1 \vee_1 y_1, x_2 \vee_2 y_2)$,
$(x_1,x_2) \wedge (y_1,y_2) = (x_1 \wedge_1 y_1, x_2 \wedge_2 y_2)$,
$(x_1,x_2) \odot (y_1,y_2) = (x_1 \odot_1 y_1, x_2 \odot_2 y_2)$,
$(x_1,x_2) \to (y_1,y_2) = (x_1 \to_1 y_1, x_2 \to_2 y_2)$,
where $(A,B) = \{(a,b) \mid a \in A, b \in B\}$, *for all subsets $A \subseteq L_1$ and $B \subseteq L_2$. Then $(L_1 \times L_2, \leq)$ satisfies $(HBL1) - (HBL5)$ in which the order $\leq$ is given by $(a,b) \leq (c,d) \Leftrightarrow a \leq_1 c, b \leq_2 d$, for any $a, c \in L_1, b, d \in L_2$. It follows that $(L; \vee, \wedge, \odot, \to, 0, 1)$ is a hyper BL-algebra, where $1 = (1_1, 1_2)$ and $0 = (0_1, 0_2)$.*

Theorem 5.18. *Let L_1 and L_2 be two BL-algebras. If θ_1 and θ_2 are two regular compatible congruence relations on L_1 and L_2, respectively, and θ is a relation on $L = L_1 \times L_2$ defined by $(a,b)\theta(u,v)$ if and only if $(a,u) \in \theta_1$ and $(b,v) \in \theta_2$. Then θ is a regular compatible congruence relation on L and $L/\theta \cong (L_1/\theta_1) \times (L_2/\theta_2)$.*

Proof. Similar to Theorem 3.20 in paper [32]. □

Definition 5.19. *Let θ be a regular compatible congruence relations on L. Then θ is called good if the following conditions hold:*

(1) $x * y \subseteq [t_*]_\theta$, for some $t_* \in L$, $* \in \{\vee, \wedge, \odot, \rightarrow\}$,

(2) $(x^{--})\overline{\theta}x$.

Example 5.20. *Let $L = (\{0, e, 1\}$ and $(L, \leq)$ be a partially ordered set such that $0 < e < 1$. Define the binary hyperoperations $\vee, \wedge, \odot, \rightarrow$ as the following tables:*

$\wedge$	0	e	1
0	{0}	{0}	{0}
e	{0}	{e}	{e}
1	{0}	{e}	{e, 1}

$\vee$	0	e	1
0	{0}	{e, 1}	{1}
e	{e, 1}	{e, 1}	{1}
1	{1}	{1}	{1}

$\rightarrow$	0	e	1
0	{1}	{1}	{1}
e	{0}	{e, 1}	{1}
1	{0}	{e, 1}	{1}

Taking $\odot = \wedge$, we can check that $(L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ is a hyper BL-algebra. Let $\theta = \{(x, x) \mid x \in L\} \cup \{(e, 1), (1, e)\}$. We can check that θ is a good regular compatible congruence relation on L.

Proposition 5.21. *Let θ be a good regular compatible congruence relation on a hyper BL-algebra L, then $(L/\theta, \overline{\vee}, \overline{\wedge}, \overline{\odot}, \rightsquigarrow, [0], [1])$ is an MV-algebra.*

Proof. Since θ is a regular compatible congruence relation, then by Proposition 5.10, we have $(L/\theta, \overline{\vee}, \overline{\wedge}, \overline{\odot}, \rightsquigarrow, [0], [1])$ is a hyper BL-algebra. By the definition of good regular compatible congruence relation, we can obtain $\overline{\vee}, \overline{\wedge}, \overline{\odot}, \rightsquigarrow$ are binary operations. So $(L/\theta, \overline{\vee}, \overline{\wedge}, \overline{\odot}, \rightsquigarrow, [0], [1])$ is a BL-algebra. Since $(x^{--})\overline{\theta}x$, then we have $[x^{--}] = [x]$, and so the operation $^-$ is involutory. Therefore, we have that $(L/\theta, \overline{\vee}, \overline{\wedge}, \overline{\odot}, \rightsquigarrow, [0], [1])$ is an MV-algebra. □

Consider L and θ given in Example 5.20. Then $L/\theta = \{[0], [1]\}$ is an MV-algebra by the above proposition.

6. Conclusions

In this paper, we establish the theory of hyper BL-algebras which are special hyper residuated lattices. However, the relations between hyper BL-algebras and hyper MV-algebras have not been proposed. Hence the next work is how to find the suitable conditions on which a hyper BL-algebra is a hyper MV-algebra.

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