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Schur-Convexity for a Mean of Two Variables with Three Parameters

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Abstract. Schur-convexity, Schur-geometric convexity and Schur-harmonic convexity for a mean of two variables with three parameters are investigated, and some mean value inequalities of two variables are established.

1. Introduction

Throughout the paper we assume that the set of n-dimensional row vector on the real number field by $\mathbb{R}^n$.

$$\mathbb{R}^n_+ = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_i > 0, i = 1, \dots, n\},$$

In particular, $\mathbb{R}^1$ and $\mathbb{R}^1_+$ denoted by $\mathbb{R}$ and $\mathbb{R}_+$ respectively.

In 2009, Kuang [1]defined a mean of two variables with three parameters as follows:

$$K(\omega_1, \omega_2, p; a, b) = \left[\frac{\omega_1 A(a^p, b^p) + \omega_2 G(a^p, b^p)}{\omega_1 + \omega_2}\right]^{\frac{1}{p}} \tag{1}$$

where $A(a, b) = \frac{a+b}{2}$ and $G(a, b) = \sqrt{ab}$ respectively is the arithmetic mean and geometric mean of two positive numbers a and b, parameters $p \neq 0$, $\omega_1, \omega_2 \geq 0$ with $\omega_1 + \omega_2 \neq 0$.

For simplicity, sometimes we will be $K(\omega_1, \omega_2, p; a, b)$ for $K(\omega_1, \omega_2, p)$ or $K(a, b)$.

In particular,

$$K\left(1, \frac{\omega}{2}, 1\right) = \frac{a + \omega\sqrt{ab} + b}{\omega + 2}$$

is the generalized Heron mean, which was introduced by Janous [2] in 2001.

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$$K\left(1, \frac{\omega}{2}, p\right) = \frac{a^p + \omega(ab)^{p/2} + b^p}{\omega + 2}$$

is the generalized Heron mean with parameter.

In recent years, the study on the properties of the mean with two variables by using theory of majorization is unusually active (see references [10-35]).

In this paper, Schur-convexity, Schur-geometric convexity and Schur-harmonic convexity of $K(\omega_1, \omega_2, p)$ are discussed. As consequences, some interesting inequalities are obtained.

Our main results are as follows:

Theorem 1.1. *(i) When $\omega_1\omega_2 \neq 0$, if $p \geq 2$ and $p(\omega_1 - \frac{\omega_2}{2}) - \omega_1 \geq 0$, then $K(\omega_1, \omega_2, p)$ is Schur-convex with $(a, b) \in \mathbb{R}^2_+$; if $1 \leq p < 2$ and $p(\omega_1 - \frac{\omega_2}{2}) - \omega_1 \leq 0$, then $K(\omega_1, \omega_2, p)$ is Schur-concave with $(a, b) \in \mathbb{R}^2_+$; if $p < 1$, then $K(\omega_1, w\omega_2, p)$ is Schur-concave with $(a, b) \in \mathbb{R}^2_+$.*

(ii) When $\omega_1 = 0, \omega_2 \neq 0$, $K(\omega_1, \omega_2, p)$ is Schur-concave with $(a, b) \in \mathbb{R}^2_+$.

(iii) When $\omega_1 \neq 0, \omega_2 = 0$, if $p \geq 2$, then $K(\omega_1, \omega_2, p)$ is the Schur-convex with $(a, b) \in \mathbb{R}^2_+$; if $p < 2$, then $K(\omega_1, \omega_2, p)$ is the Schur-concave with $(a, b) \in \mathbb{R}^2_+$.

Theorem 1.2. *If $p \geq 0$, then $K(\omega_1, \omega_2, p)$ is Schur-geometrically convex with $(a, b) \in \mathbb{R}^2_+$. If $p < 0$, then $K(\omega_1, \omega_2, p)$ is Schur-geometrically concave with $(a, b) \in \mathbb{R}^2_+$.*

Theorem 1.3. *If $p \geq -1$, then $K(\omega_1, \omega_2, p)$ is Schur-harmonically convex with $(a, b) \in \mathbb{R}^2_+$. If $-2 < p < -1$ and $\omega_1(p + 1) + \omega_2(\frac{p}{2} + 1) \geq 0$, then $K(\omega_1, \omega_2, p)$ is Schur-harmonically convex with $(a, b) \in \mathbb{R}^2_+$. If $p \leq -2$ and $\omega_1(\frac{p}{2} + 1) + \omega_2 = 0$, then $K(\omega_1, \omega_2, p)$ is Schur-harmonically concave with $(a, b) \in \mathbb{R}^2_+$.*

2. Definitions and Lemmas

We need the following definitions and lemmas.

Definition 2.1 ([3, 4]). *Let $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$.*

(i) $\mathbf{x}$ is said to be majorized by $\mathbf{y}$ (in symbols $\mathbf{x} \prec \mathbf{y}$) if $\sum_{i=1}^k x_{[i]} \leq \sum_{i=1}^k y_{[i]}$ for $k = 1, 2, \ldots, n-1$ and $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$, where $x_{[1]} \geq \cdots \geq x_{[n]}$ and $y_{[1]} \geq \cdots \geq y_{[n]}$ are rearrangements of $\mathbf{x}$ and $\mathbf{y}$ in a descending order.

(ii) $\Omega \subset \mathbb{R}^n$ is called a convex set if $(\alpha x_1 + \beta y_1, \ldots, \alpha x_n + \beta y_n) \in \Omega$ for any $\mathbf{x}$ and $\mathbf{y} \in \Omega$, where α and $\beta \in [0, 1]$ with $\alpha + \beta = 1$.

(iii) let $\Omega \subset \mathbb{R}^n$, φ: $\Omega \to \mathbb{R}$ is said to be a Schur-convex function on Ω if $\mathbf{x} \prec \mathbf{y}$ on Ω implies $\varphi(\mathbf{x}) \leq \varphi(\mathbf{y})$. φ is said to be a Schur-concave function on Ω if and only if $-\varphi$ is Schur-convex function.

Definition 2.2 ([5, 6]). *Let $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n_+$.*

(i) $\Omega \subset \mathbb{R}^n_+$ is called a geometrically convex set if $(x_1^\alpha y_1^\beta, \ldots, x_n^\alpha y_n^\beta) \in \Omega$ for any $\mathbf{x}$ and $\mathbf{y} \in \Omega$, where α and $\beta \in [0, 1]$ with $\alpha + \beta = 1$.

(ii) let $\Omega \subset \mathbb{R}^n_+$, φ: $\Omega \to \mathbb{R}_+$ is said to be a Schur-geometrically convex function on Ω if $(\ln x_1, \ldots, \ln x_n) \prec (\ln y_1, \ldots, \ln y_n)$ on Ω implies $\varphi(\mathbf{x}) \leq \varphi(\mathbf{y})$. φ is said to be a Schur-geometrically concave function on Ω if and only if $-\varphi$ is Schur-geometrically convex function.

Definition 2.3 ([7, 8]). *Let $\Omega \subset \mathbb{R}^n_+$.*

(i) A set Ω is said to be a harmonically convex set if $\frac{\boldsymbol{xy}}{\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y}} \in \Omega$ for every $\boldsymbol{x}, \boldsymbol{y} \in \Omega$ and $\lambda \in [0, 1]$, where $\boldsymbol{xy} = \sum_{i=1}^n x_i y_i$ and $\frac{1}{\boldsymbol{x}} = \left(\frac{1}{x_1}, \cdots, \frac{1}{x_n}\right)$.

(ii) A function $\varphi : \Omega \to \mathbb{R}_+$ is said to be a Schur harmonically convex function on Ω if $\frac{1}{\boldsymbol{x}} \prec \frac{1}{\boldsymbol{y}}$ implies $\varphi(\boldsymbol{x}) \leq \varphi(\boldsymbol{y})$. A function φ is said to be a Schur harmonically concave function on Ω if and only if $-\varphi$ is a Schur harmonically convex function.

Lemma 2.4 ([3, 4]). *Let $\Omega \subset \mathbb{R}^n$ is convex set, and has a nonempty interior set Ω^0 . Let $\varphi : \Omega \to \mathbb{R}$ is continuous on Ω and differentiable in Ω^0. Then φ is the $Schur - convex(Schur - concave) function$, if and only if it is symmetric on Ω and if*

$$(x_1 - x_2)\left(\frac{\partial\varphi}{\partial x_1} - \frac{\partial\varphi}{\partial x_2}\right) \geq 0(\leq 0)$$

holds for any $\mathbf{x} = (x_1, x_2, \cdots, x_n) \in \Omega^0$.

Lemma 2.5 ([5, 6]). *Let $\Omega \subset \mathbb{R}_+^n$ be a symmetric geometrically convex set with a nonempty interior Ω^0. Let $\varphi : \Omega \to \mathbb{R}_+$ be continuous on Ω and differentiable on Ω^0. Then φ is a Schur geometrically convex (Schur geometrically concave) function if and only if φ is symmetric on Ω and*

$$(x_1 - x_2)\left(x_1\frac{\partial\varphi}{\partial x_1} - x_2\frac{\partial\varphi}{\partial x_2}\right) \geq 0 \quad (\leq 0) \tag{2}$$

holds for any $\mathbf{x} = (x_1, \cdots, x_n) \in \Omega^0$.

Lemma 2.6 ([7, 8]). *Let $\Omega \subset \mathbb{R}_+^n$ be a symmetric harmonically convex set with a nonempty interior Ω^0. Let $\varphi : \Omega \to \mathbb{R}_+$ be continuous on Ω and differentiable on Ω^0. Then φ is a Schur harmonically convex (Schur harmonically concave) function if and only if φ is symmetric on Ω and*

$$(x_1 - x_2)\left(x_1^2\frac{\partial\varphi}{\partial x_1} - x_2^2\frac{\partial\varphi}{\partial x_2}\right) \geq 0 \quad (\leq 0) \tag{3}$$

holds for any $\mathbf{x} = (x_1, \cdots, x_n) \in \Omega^0$.

Lemma 2.7 ([9]). *Let $a \leq b, u(t) = tb + (1-t)a, v(t) = ta + (1-t)b$. If $\frac{1}{2} \leq t_2 \leq t_1 \leq 1$ or $0 \leq t_1 \leq t_2 \leq \frac{1}{2}$, then*

$$(u(t_2), v(t_2)) \prec (u(t_1), v(t_1)) \prec (a, b). \tag{4}$$

Lemma 2.8. *Let*

$$f(x) = \omega_1(p+1)x^{\frac{p}{2}+1} + \omega_2(\frac{p}{2}+1)x - \omega_2\frac{p}{2}, \ \ x \in [1, \infty)$$

where $\omega_1, \omega_2 \geq 0, \omega_1^2 + \omega_2^2 \neq 0$.

If $-2 < p < -1$ and $\omega_1(p+1) + \omega_2(\frac{p}{2}+1) \geq 0$, then $f(x) \geq 0$, if $p \leq -2$ and $\omega_1(p+1) + \omega_2 = 0$, then $f(x) \leq 0$.

Proof. If $-2 < p < -1$ and $\omega_1(p+1) + \omega_2(\frac{p}{2}+1) \geq 0$, then

$$\begin{aligned} g(x) :&= \omega_1(p+1)x^{\frac{p}{2}+1} + \omega_2(\frac{p}{2}+1)x \\ &\geq \omega_1(p+1)x^{\frac{p}{2}+1} + \omega_2(\frac{p}{2}+1)x^{\frac{p}{2}+1} \\ &= x^{\frac{p}{2}+1}[\omega_1(p+1) + \omega_2(\frac{p}{2}+1)] \\ &\geq 0, \end{aligned}$$

and then $f(x) = g(x) - \omega_2\frac{p}{2} \geq 0$.

If $p \leq -2$ and $\omega_1(p+1) + \omega_2 = 0$, then

$$f'(x) = \omega_1(p+1)(\frac{p}{2}+1)x^{\frac{p}{2}} + \omega_2(\frac{p}{2}+1),$$

and

$$f''(x) = \omega_1 \frac{p}{2}(p+1)(\frac{p}{2}+1)x^{\frac{p}{2}-1} \le 0,$$

so $f'(x)$ is decreasing, but

$$f'(1) = \omega_1(p+1)(\frac{p}{2}+1) + \omega_2(\frac{p}{2}+1) = (\frac{p}{2}+1)[\omega_1(p+1)+\omega_2] = 0,$$

then $f'(x) \le 0$, so $f(x)$ is decreasing, furthermore

$$f(1) = \omega_1(p+1) + \omega_2(\frac{p}{2}+1) - \omega_2\frac{p}{2} = \omega_1(p+1) + \omega_2 = 0,$$

thus $f(x) \le 0$. □

3. Proofs of Main results

From the definition of $K(\omega_1, \omega_2, p)$, we have

$$K(\omega_1,\omega_2,p) = \left(\frac{\omega_1 \frac{a^p+b^p}{2} + \omega_2 a^{\frac{p}{2}} b^{\frac{p}{2}}}{\omega_1+\omega_2} \right)^{\frac{1}{p}}.$$

It is clear that $K(\omega_1, \omega_2, p)$ is symmetric with $(a, b) \in \mathbb{R}^2_+$.

Write

$$m(a,b) := \left[\frac{\omega_1(a^p+b^p) + 2\omega_2 a^{\frac{p}{2}} b^{\frac{p}{2}}}{2(\omega_1+\omega_2)} \right]^{\frac{1}{p}-1}.$$

Proof. [Proof of Theorem 1.1] (*i*) When $\omega_1\omega_2 \ne 0$,

$$\frac{\partial K}{\partial a} = m(a,b)\left(\frac{\omega_1 a^{p-1} + \omega_2 a^{\frac{p}{2}-1} b^{\frac{p}{2}}}{\omega_1+\omega_2} \right),$$

$$\frac{\partial K}{\partial b} = m(a,b)\left(\frac{\omega_1 b^{p-1} + \omega_2 a^{\frac{p}{2}} b^{\frac{p}{2}-1}}{\omega_1+\omega_2} \right),$$

and then

$$\begin{aligned} \Delta_1 :&= (a-b)\left(\frac{\partial K}{\partial a} - \frac{\partial K}{\partial b} \right) \\ &= \frac{a-b}{2(\omega_1+\omega_2)} m(a,b) \left[\omega_1(a^{p-1}-b^{p-1}) - \omega_2(a-b)a^{\frac{p}{2}-1}b^{\frac{p}{2}-1} \right]. \end{aligned}$$

Without loss of generality, we may assume that $a \ge b$, then $z := \frac{a}{b} \ge 1$, and then

$$\Delta_1 = \frac{a-b}{2(\omega_1+\omega_2)} m(a,b) b^{p-1} f(z),$$

where

$$f(z) = \omega_1(z^{p-1}-1) - \omega_2(z-1)z^{\frac{p}{2}-1}, \ \ z \ge 1.$$

$$\begin{aligned} f'(z) &= \omega_1(p-1)z^{p-2} - \omega_2 z^{\frac{p}{2}-1} - \omega_2(\frac{p}{2}-1)(z-1)z^{\frac{p}{2}-2} \\ &= z^{\frac{p}{2}-2}\left[\omega_1(p-1)z^{\frac{p}{2}} - \omega_2\frac{p}{2}z + \omega_2(\frac{p}{2}-1) \right]. \end{aligned}$$

If $p \geq 2$ and $p(\omega_1 - \frac{\omega_2}{2}) - \omega_1 \geq 0$, then

$$\omega_1(p-1)z^{\frac{p}{2}} - \omega_2\frac{p}{2}z = z\left[\omega_1(p-1)z^{\frac{p}{2}-1} - \omega_2\frac{p}{2}\right] \geq z\left[\omega_1(p-1) - \omega_2\frac{p}{2}\right].$$

Notice that

$$\omega_1(p-1) - \omega_2\frac{p}{2} \geq 0 \Leftrightarrow p(\omega_1 - \frac{\omega_2}{2}) - \omega_1 \geq 0,$$

we have $f'(z) \geq 0$, for $z \in [1, \infty)$, but $f(1) = 0$, so $f(z) \geq 0$, further $\Delta_1 \geq 0$. By Lemma 1, it follows that $K(\omega_1, \omega_2, p)$ is Schur-convex with $(a, b) \in \mathbb{R}^2_+$.

If $1 \leq p < 2$ and $p(\omega_1 - \frac{\omega_2}{2}) - \omega_1 \leq 0$, then

$$\omega_1(p-1)z^{\frac{p}{2}} - \omega_2\frac{p}{2}z = z\left[\omega_1(p-1)z^{\frac{p}{2}-1} - \omega_2\frac{p}{2}\right] \leq z\left[\omega_1(p-1) - \omega_2\frac{p}{2}\right].$$

Notice that

$$\omega_1(p-1) - \omega_2\frac{p}{2} \geq 0 \Leftrightarrow p(\omega_1 - \frac{\omega_2}{2}) - \omega_1 \leq 0,$$

we have $f'(z) \leq 0$, for $z \in [1, \infty)$, but $f(1) = 0$, so $f(z) \leq 0$, further $\Delta_1 \leq 0$. By Lemma 2.4, it follows that $K(w_1, w_2, p)$ is Schur-concave with $(a, b) \in \mathbb{R}^2_+$.

If $p < 1$, then

$$f(z) = \omega_1(z^{p-1} - 1) - \omega_2(z-1)z^{\frac{p}{2}-1} \leq \omega_1(1-1) - \omega_2(z-1)z^{\frac{p}{2}-1} \leq 0,$$

and then $\Delta_1 \leq 0$. By Lemma 2.4, it follows that $K(\omega_1, \omega_2, p)$ is Schur-concave with $(a, b) \in \mathbb{R}^2_+$.

(*ii*) When $\omega_1 = 0, \omega_2 \neq 0, K(\omega_1, \omega_2, p) = \sqrt{ab}$, then

$$\Delta_1 := (a-b)\left(\frac{\partial K}{\partial a} - \frac{\partial K}{\partial b}\right) = -\frac{1}{2}\frac{(a-b)^2}{\sqrt{ab}} \leq 0.$$

By Lemma 2.4, it follows that $K(\omega_1, \omega_2, p)$ is Schur-concave with $(a, b) \in \mathbb{R}^2_+$.

(*iii*) When $\omega_1 \neq 0, \omega_2 = 0, K(w_1, w_2, p) = \left(\frac{a^p+b^p}{2}\right)^{\frac{1}{p}}$, then

$$\Delta_1 := (a-b)\left(\frac{\partial K}{\partial a} - \frac{\partial K}{\partial b}\right) = \frac{1}{2}(a-b)(a^p + b^p)^{\frac{1}{p}-1}(a^{p-1} + b^{p-1}).$$

If $p \geq 2$, then $a - b$ and $a^{p-1} - b^{p-1}$ has the same sign, so $\Delta_1 \geq 0$. By Lemma 2.4, it follows that $K(\omega_1, \omega_2, p)$ is Schur-convex with $(a, b) \in \mathbb{R}^2_+$. If $p < 2$, then $a - b$ and $a^{p-1} - b^{p-1}$ has the opposite sign, so $\Delta_1 \leq 0$. By Lemma 2.4, it follows that $K(\omega_1, \omega_2, p)$ is Schur-concave with $(a, b) \in \mathbb{R}^2_+$.

The proof of Theorem 1.1 is complete. □

Proof. [Proof of Theorem 1.2] It is easy to see that

$$a\frac{\partial K}{\partial a} = m(a, b)\left(\frac{\omega_1 a^p + \omega_2 a^{\frac{p}{2}} b^{\frac{p}{2}}}{\omega_1 + \omega_2}\right),$$

$$b\frac{\partial K}{\partial b} = m(a, b)\left(\frac{\omega_1 b^p + \omega_2 a^{\frac{p}{2}} b^{\frac{p}{2}}}{\omega_1 + \omega_2}\right),$$

and then

$$\Delta_2 := (a-b)\left(a\frac{\partial K}{\partial a} - b\frac{\partial K}{\partial b}\right) = \frac{(a-b)m(a, b)\omega_1(a^p - b^p)}{2(\omega_1 + \omega_2)}.$$

If $p \geq 0$, then $a-b$ and $a^p - b^p$ has the same sign, so $\Delta_2 \geq 0$. By Lemma 2.5, it follows that $K(\omega_1, \omega_2, p)$ is Schur-geometrically convex with $(a,b) \in \mathbb{R}^2_+$. If $p < 0$, then $a-b$ and $a^p - b^p$ has the opposite sign, so $\Delta_2 \leq 0$. By Lemma 2.5, it follows that $K(\omega_1, \omega_2, p)$ is Schur-geometrically concave with $(a,b) \in \mathbb{R}^2_+$.

The proof of Theorem 1.2 is complete. □

Proof. [Proof of Theorem 1.3] It is easy to see that

$$a^2 \frac{\partial K}{\partial a} = \frac{m(a,b)}{2(\omega_1+\omega_2)} \left(\omega_1 a^{p+1} + \omega_2 a^{\frac{p}{2}+1} b^{\frac{p}{2}} \right),$$

$$b^2 \frac{\partial K}{\partial b} = \frac{m(a,b)}{2(\omega_1+\omega_2)} \left(\omega_1 b^{p+1} + \omega_2 a^{\frac{p}{2}} b^{\frac{p}{2}+1} \right),$$

and then

$$\Delta_3 := (a-b)\left(a^2 \frac{\partial K}{\partial a} - b^2 \frac{\partial K}{\partial b} \right) = \frac{m(a,b)}{2(\omega_1+\omega_2)} f(x,y).$$

where

$$f(a,b) := \omega_1 (a-b)(a^{p+1} - b^{p+1}) + \omega_2 a^{\frac{p}{2}} b^{\frac{p}{2}} (a-b)^2.$$

If $p \geq -1$, then $a-b$ and $a^{p+1} - b^{p+1}$ has the same sign, so $f(a,b) \geq 0$, and then $\Delta_3 \geq 0$. By Lemma 2.6, it follows that $K(\omega_1, \omega_2, p)$ is Schur-harmonically convex with $(a,b) \in \mathbb{R}^2_+$.

Without loss of generality, we may assume that $a \geq b$, then $z := \frac{a}{b} \geq 1$, and then

$$f(a,b) = b^{p+2}(z-1)g(z),$$

where

$$g(z) = \omega_1(z^{p+1} - 1) + \omega_2 z^{\frac{p}{2}}(z-1).$$

$$g'(z) = z^{\frac{p}{2}-1} s(z),$$

where

$$s(z) = \omega_1(p+1)(z^{\frac{p}{2}+1} + \omega_2(\frac{p}{2}+1)z - \omega_2 \frac{p}{2}.$$

If $-2 < p < -1$ and $\omega_1(p+1) + \omega_2(\frac{p}{2}+1) \geq 0$, from Lemma 2.8, it follows $s(z) \geq 0$, and then $g'(z) \geq 0$, but $g(1) = 0$, so $g(z) \geq 0$ and $f(a,b) \geq 0$. Thus $\Delta_3 \geq 0$, by Lemma 2.6, it follows that $K(\omega_1, \omega_2, p)$ is Schur-harmonically convex with $(a,b) \in \mathbb{R}^2_+$.

If f $p \leq -2$ and $\omega_1(\frac{p}{2}+1) + \omega_2 = 0$, from Lemma 2.8, it follows $s(z) \leq 0$, and then $g'(z) \leq 0$, but $g(1) = 0$, so $g(z) \leq 0$ and $f(a,b) \leq 0$. Thus $\Delta_3 \leq 0$, by Lemma 2.6, it follows that $K(\omega_1, \omega_2, p)$ is Schur-harmonically concave with $(a,b) \in \mathbb{R}^2_+$.

The proof of Theorem 1.3 is complete. □

4. Applications

Theorem 4.1. *Let $(a,b) \in \mathbb{R}^2_+$, $u(t) = tb + (1-t)a, v(t) = ta + (1-t)b$. Assume also that $\frac{1}{2} \leq t_2 \leq t_1 \leq 1$ or $0 \leq t_1 \leq t_2 \leq \frac{1}{2}$.*

If $\omega_1\omega_2 \neq 0$, $p \geq 2$ and $p(\omega_1 - \frac{\omega_2}{2}) - \omega_1 \geq 0$, then we have we have

$$K\left(\omega_1, \omega_2, p; \frac{a+b}{2}, \frac{a+b}{2}\right) \leq K(\omega_1, \omega_2, p; u(t_2), v(t_2))$$
$$\leq K(\omega_1, \omega_2, p; u(t_1), v(t_1)) \leq K(\omega_1, \omega_2, p; a, b) \leq G(\omega_1, \omega_2, p; a+b, 0). \tag{5}$$

If $\omega_1\omega_2 \neq 0$, $1 \leq p < 2$ and $p(\omega_1 - \frac{\omega_2}{2}) - \omega_1 \leq 0$, then inequalities in (5) *are all reversed.*

Proof. From Lemma 2.7, we have

$$\left(\frac{a+b}{2}, \frac{a+b}{2}\right) \prec (u(t_2), v(t_2)) \prec (u(t_1), v(t_1)) \prec (a, b),$$

and it is clear that $(a, b) \prec (a + b - \varepsilon, \varepsilon)$, where ε is enough small positive number.

If $\omega_1\omega_2 \neq 0$, $p \geq 2$ and $p(\omega_1 - \frac{\omega_2}{2}) - \omega_1 \geq 0$, by Theorem 1, and let $\varepsilon \to 0$, it follows that (5) are holds. If $\omega_1\omega_2 \neq 0$, $1 \leq p < 2$ and $p(\omega_1 - \frac{\omega_2}{2}) - \omega_1 \leq 0$, then inequalities in (5) are all reversed.

The proof is complete. □

Theorem 4.1 enable us to obtain a large number of refined inequalities by assigning appropriate values to the parameters ω_1, ω_2, p, t_1 and t_2.

For example, putting $\omega_1 = \omega_2 = 1$ in (5), we can get

Corollary 4.2. *Let $p \geq 2$. Then for $(a, b) \in \mathbb{R}^2_+$, we have*

$$A(a^p, b^p) + G(a^p, b^p) \geq 2(A(a, b))^p. \tag{6}$$

Putting $p = \frac{1}{2}, \omega_1 = 2, \omega_2 = 1$ and $t_1 = \frac{3}{4}, t_2 = \frac{1}{2}$ in (5), we can get

Corollary 4.3. *Let $(a, b) \in \mathbb{R}^2_+$. Then*

$$\frac{a+b}{2} \geq \frac{1}{36}\left[\sqrt{a+3b} + \sqrt[4]{(a+3b)(3a+b)} + \sqrt{3a+b}\right]^2 \geq \frac{1}{9}\left(\sqrt{a} + \sqrt[4]{ab} + \sqrt{b}\right)^2. \tag{7}$$

Theorem 4.4. *Let $(a, b) \in \mathbb{R}^2_+$. If $p \geq 0 (< 0)$, we have*

$$G(a, b) \leq (\geq) K(\omega_1, \omega_2, p; a, b). \tag{8}$$

Proof. Since $(\log\sqrt{ab}, \log\sqrt{ab}) \prec (\log a, \log b)$, if $p \geq 0 (< 0)$, by Theorem 1.2, it follows

$$G(a, b) = K\left(\omega_1, \omega_2, p; \sqrt{ab}, \sqrt{ab}\right) \leq (\geq) K(\omega_1, \omega_2, p; a, b).$$

The proof is complete. □

For example, putting $\omega_1 = \omega_2 = 1$ in (8), we can get

Corollary 4.5. *Let $(a, b) \in \mathbb{R}^2_+$. If $p \geq 0 (< 0)$, then*

$$A(a^p, b^p) + G(a^p, b^p) \leq (\geq) 2(G(a, b))^p. \tag{9}$$

Theorem 4.6. *Let $(a, b) \in \mathbb{R}^2_+$. If $p \geq -1$ or if $-2 < p < -1$ and $\omega_1(p + 1) + \omega_2(\frac{p}{2} + 1) \geq 0$, then*

$$H(a, b) \leq K\left(\omega_1, \omega_2, p; \frac{ab}{tb + (1-t)a}, \frac{ab}{ta + (1-t)b}\right) \leq K(\omega_1, \omega_2, p; a, b). \tag{10}$$

where $H(a, b) = \frac{2}{a^{-1}+b^{-1}}$ is the harmonic mean.

If $p \leq -2$ and $\omega_1(\frac{p}{2} + 1) + \omega_2 = 0$, then inequalities in (10) are all reversed.

Proof. By Lemma 2.7, we have

$$\left(\frac{a^{-1}+b^{-1}}{2}, \frac{a^{-1}+b^{-1}}{2}\right) \prec \left(ta^{-1} + (1-t)b^{-1}, tb^{-1} + (1-t)a^{-1}\right) \prec (a^{-1}, b^{-1}).$$

If $p \geq -1$ or if $-2 < p < -1$ and $\omega_1(p+1) + \omega_2(\frac{p}{2}+1) \geq 0$, then by Theorem 1.3, it follows

$$H(a,b) = K\left(\omega_1, \omega_2, p; \frac{2}{a^{-1}+b^{-1}}, \frac{2}{a^{-1}+b^{-1}}\right)$$
$$\leq K\left(\omega_1, \omega_2, p; \frac{ab}{tb+(1-t)a}, \frac{ab}{ta+(1-t)b}\right)$$
$$\leq K(\omega_1, \omega_2, p; a, b).$$

If $p \leq -2$ and $\omega_1(\frac{p}{2}+1) + \omega_2 = 0$, then inequalities in (10) are all reversed.

The proof is complete. □

Putting $\omega_1 = \omega_2 = 1$ in (10), we can get

Corollary 4.7. *Let* $(a,b) \in \mathbb{R}^2_+$. *If* $p \geq -1$ *or* $-\frac{4}{3} < p < -1$, *then*

$$A(a^p, b^p) + G(a^p, b^p) \geq 2(H(a,b))^p. \quad (11)$$

If $p = -4$, *then the inequality in* (11) *is reversed.*

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