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Existence, Uniqueness and Stability Results for Semilinear Integrodifferential Non-local Evolution Equations with Random Impulse

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Abstract. In this paper, authors investigated the existence and uniqueness of random impulsive semilinear integrodifferential evolution equations with non-local conditions in Hilbert spaces. Also the stability results for the same evolution equation has been studied. The results are derived by using the semigroup theory and fixed point approach. An application is provided to illustrate the theory.

1. Introduction

The theory of nonlinear differential and integral equations in abstract spaces is a fascinating field with many important applications to a number of areas of analysis as well as other branches of sciences. Popular models essentially fall into two categories: the differential models and the integrodifferential models. Our work centers around the problems described by the integrodifferential models. A large class of scientific and engineering problems modelled by partial differential equations can be expressed in various forms of differential or integrodifferential equations in abstract spaces. Several authors [9–11] have investigated the integrodifferential equations in abstract spaces. The problem of existence and controllability of abstract functional differential and integrodifferential systems has been studied by fixed point principles [3].

Nonlocal Cauchy problem, namely, the differential equation with a nonlocal initial condition $x(t_0) + g(t_1, \ldots, t_p, x) = x_0$ $(0 \leq t_0 < t_1 < \ldots < t_p \leq t_0 + a$ and g is a given function) is one of the important topics in the study of analysis. Interest in such a problem stems mainly from the better effect of the nonlocal initial condition than the usual one in treating physical problems. Actually the nonlocal initial condition $x(t_0) + g(t_1, \ldots, t_p, x) = x_0$ models many interesting natural phenomena in which the normal initial condition $x(0) = x_0$ may not fit in. For instance, the function $g(t_1, \ldots, t_p, x)$ may be given by $g(t_1, \ldots, t_p, x) = \sum_{i=1}^{p} c_i x(t_i)$

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where c_i, $i = 1, \dots, p$ are constants. In this case, we are permitted to have the measurements at $t = 0, t_1, \dots, t_p$, rather than just at $t = 0$. Thus more information is available. More specially, letting $g(t_1, \dots, t_p, x) = -x(t_p)$ and $x_0 = 0$ yields a periodic problem and letting $g(t_1, \dots, t_p, x) = -x(t_0) + x(t_p)$ gives a backward problem. Using the method of semigroups, various solutions of nonlinear and semilinear evolution equations have been discussed by Pazy [18] and the nonlocal problem for the same equations has been first studied by Byszewskii [6]. Byszewski and Acka [7] established the existence and uniqueness and continuous dependence of mild solution of semilinear functional differential equation with nonlocal condition of the form

$$\frac{du(t)}{dt} + Au(t) = f(t, u_t), \qquad t \in [0, a],$$
$$u(s) + [g(u_{t_1}, ..., u_{t_p})](s) = \varphi(s), \quad s \in [-r, 0],$$

where $0 < t_1 < \dots < t_p \leq a$, $-A$ is the infinitesimal generator of a C_0 semigroup of operators on a Banach space.

To describe mathematically evolution of a real process with a short perturbation, consider these perturbations to be "instantaneous". For such an idealization, it becomes necessary to study dynamical systems with discontinuous trajectories or, as they might be called, differential equations with impulses. Many evolution processes are distinguished by the fact that at certain moments of time they abruptly experience a change of state. These processes are subject to short-term perturbations whose duration is negligible when comparing to the duration of the process. Impulses may exists as fixed-time or random-time. The action that causes change in the state of the system rapidly(fixed) or randomly and in many other factors which enables us to represent a dynamical system as a certain transformer of (deterministic or) random inputs into (deterministic or) random outputs. Randomness is notified to the mathematical formulation of many physical, biological, engineering phenomena: such as, fluctuations in the stock market, noise in population systems, etc.

Many researchers have investigated the qualitative properties of fixed-type impulses [1, 12, 22] and [23]. Radhakrishnan and Balachandran [19] studied the impulsive neutral functional evolution integrodifferential systems with infinite delay. Luo et al. [15–17] investigated the stability of impulsive functional differential equations via Liapunov functional. Zhou Yong et al. [24] studied existence and uniqueness of solutions to stochastic differential equations with random impulsive under Lipchitz conditions. There are only a few researchers have studied random-type impulses. Wu et al. [25–30] introduced existence, uniqueness and stability of random impulsive ordinary differential equations and also investigated the boundedness of solutions by Liapunovs direct method. Anguraj et al.[2] studied about the semilinear differential equations under non-uniqueness and recently, Radhakrishnan and Tamilarasi [20, 21] discussed the quasilinear random impulsive neutral differential equations and inclusions. A useful tool to study the existence and uniqueness of random differential equations is the random fixed point theorems (see [4, 5, 8, 13]). Since real world system and natural phenomena will almost invariably be affected by random factors, the translation from a real world phenomenon to a set of mathematical equations is never perfect. This is due to a combination of uncertainties, complexities and ignorance on our part which inevitably cloud our mathematical modeling process. From the above, it should be noted that there are several contributions on the existence and stability of differential equations with and without randomness using one or more parameter families. Till now, existence and stability of semilinear integrodifferential equation with random impulse is untreated in the literature. Motivated by this fact, in this paper we make a first attempt to fill the gap by studying existence, uniqueness and stability results for random impulsive semilinear integrodifferential evolution equations with non-local conditions by using the Banach contraction principle.

The paper is organized as follows. The first section gives a brief overview of our work. In section 2, some preliminaries are presented. In section 3, we investigate the existence and uniqueness of mild solutions of semilinear integrodifferential evolution equations with random impulses by using the Banach contraction principle. In section 4, we study the stability of mild solutions of semilinear integrodifferential evolution equations with random impulses through the same Banach fixed point theorem, and finally, in section 5, we construct an example to illustrate our results.

2. Preliminaries

Let $\mathbb{X}$ be a real separable Hilbert space and Ω be a non-empty set. Assume that τ_k is a random variable defined from Ω to $D_k \equiv (0, d_k)$ for $k = 1, 2, ...$ where $0 < d_k < +\infty$. Also assume that τ_i and τ_j are independent from each other as $i \neq j$ for $i, j = 1, 2, ...$ Let $\eta, T \in \mathbb{R}$ be two constants satisfying $\eta < T$. Further, we denote $\mathbb{R}^+ = [0, +\infty)$; $\mathbb{R}_\eta = [\eta, +\infty)$. Consider the random impulsive semilinear integrodifferential evolution equation with nonlocal conditions described as follows

$$\left.\begin{array}{l} u'(t) = A(t)u(t) + f(t, u(t)) + \displaystyle\int_0^t e(t, s, u(s))ds, \ t \neq \xi_k, \ t \geq \eta, \\ u(\xi_k) = \alpha_k(\tau_k)u(\xi_k^-), \ k = 1, 2, ... \\ u(t_0) = u_0 + g(u), \end{array}\right\} \tag{1}$$

where $A(t)$ is a family of linear operators which generates an evolution operator $\{U(t, s) : 0 \leq s \leq t \leq b\}$, the functionals $f : \mathbb{R}_\eta \times \mathbb{X} \to \mathbb{X}$, $e : \mathbb{R}_\eta \times \mathbb{R}_\eta \times \mathbb{X} \to \mathbb{X}$, $\alpha_k : D_k \to \mathbb{R}$, for each $k = 1, 2, ...$; $\xi_0 = t_0$ and $\xi_k = \xi_{k-1} + \tau_k$ for $k = 1, 2, ...$; here $t_0 \in \mathbb{R}_\eta$ is an arbitrary real number and $g : \mathbb{X} \to \mathbb{X}$ is a given function.

Obviously, $t_0 = \xi_0 < \xi_1 < ... < \lim_{k\to\infty} \xi_k = \infty$; $u(\xi_k^-) = \lim_{t\to\xi_k} u(t)$ according to their paths with the norm $\|u\| = \sup_{\eta\leq s\leq T} |u(s)|$, for each t satisfying $\eta \leq t \leq T$, $\|.\|$ is any given norm in $\mathbb{X}$.

Let us denote $\{\mathcal{B}_t, t \geq 0\}$ the simple counting process generated by $\{\xi_n\}$, that is, $\{\mathcal{B}_t \geq n\} = \{\xi_n \leq t\}$, and denote $\mathcal{F}_t$ the σ-algebra generated by $\{\mathcal{B}_t, \ t \geq 0\}$. Then $(\Omega, P, \{\mathcal{F}_t\})$ is a probability space. Let $\mathbb{L}_2 = \mathbb{L}_2(\Omega, \mathcal{F}_t, \mathbb{X})$ denote the Hilbert space of all $\mathcal{F}_t$-measurable square integrable random variable with values in $\mathbb{X}$ and $\mathcal{Z}$ denotes Banach space $\mathcal{Z}([\eta, T], \mathbb{L}_2)$, the family of all $\{\mathcal{F}_t\}$-measurable random variables ψ with the norm

$$\|\Psi\|^2 = \sup_{t\in[\eta,T]} \mathbb{E}\|\Psi(t)\|^2.$$

For the family $\{A(t) : 0 \leq t \leq b\}$ of linear operators, we assume the following hypotheses:

(A1) $A(t)$ is a closed linear operator and the domain $D(A)$ of $\{A(t) : 0 \leq t \leq b\}$ is dense in the Banach space $\mathbb{X}$ and independent of t.

(A2) For each $t \in [0, b]$, the resolvent $R(\lambda, A(t)) = (\lambda I - A(t))^{-1}$ of $A(t)$ exists for all λ with $Re\ \lambda \leq 0$ and $\|R(\lambda, A(t))\| \leq C(|\lambda| + 1)^{-1}$.

(A3) For any $t, s, \tau \in [0, b]$, there exists a $0 < \delta < 1$ and $L > 0$ so that

$$\|(A(t) - A(\tau))A^{-1}(s)\| \leq L|t - \tau|^\delta.$$

Statements $(A1) - (A2)$ implies that there exists a family of evolution operator $U(t, s)$, see [18].

The family of two parameter linear evolution system $\{U(t, s) : 0 \leq s \leq t \leq b\}$ satisfying the following properties:

(a) $U(t, s) \in L(\mathbb{X})$ the space of bounded linear transformation on $\mathbb{X}$, whenever $0 \leq s \leq t \leq b$ and for each $x \in \mathbb{X}$, the mapping $(t, s) \to U(t, s)x$ is continuous.

(b) $U(t, s)U(s, \tau) = U(t, \tau)$ for $0 \leq \tau \leq s \leq t \leq b$.

(c) $U(t, t) = I$.

Definition 2.1. *For a given $T \in (\eta, +\infty)$, a stochastic process $\{u(t) \in \mathcal{Z}, \eta \leq t \leq T\}$ is said to be a mild solution to the equation* (2.1) *in* $(\Omega, P, \{\mathcal{F}_t\})$, *if* *(i)* $u(t) \in \mathbb{X}$ *is* $\mathcal{F}_t$*-adapted;*

$$\begin{aligned}(ii)\ u(t) &= \sum_{k=0}^{+\infty}\Big[\prod_{i=1}^{k}\alpha_i(\tau_i)U(t,t_0)[u_0+g(u)]+\sum_{i=1}^{k}\prod_{j=i}^{k}\alpha_j(\tau_j)\int_{\xi_i-1}^{\xi_i}U(t,s)f(s,u(s))ds \\ &+\int_{\xi_k}^{t}U(t,s)f(s,u(s))ds+\sum_{i=1}^{k}\prod_{j=i}^{k}\alpha_j(\tau_j)\int_{\xi_{i-1}}^{\xi_i}U(t,s)\left(\int_0^s e(s,\theta,u(\theta))d\theta\right)ds \\ &+\int_{\xi_k}^{t}U(t,s)\left(\int_0^s e(s,\theta,u(\theta))d\theta\right)ds\Big]I_{[\xi_k,\xi_{k+1})}(t),\quad t\in[\eta,T]\end{aligned}$$

where $\prod_{j=m}^{n}(.)=1$ as $m>n$, $\prod_{j=i}^{k}\alpha_j(\tau_j)=\alpha_k(\tau_k)\alpha_{k-1}(\tau_{k-1})...\alpha_i(\tau_i)$ and $I_{\mathcal{A}}(.)$ is the index function, that is,

$$I_{\mathcal{A}}(t)=\begin{cases}1 & if\ \ t\in\mathcal{A}\\ 0 & if\ \ t\notin\mathcal{A}.\end{cases}$$

The following hypotheses are needed to prove the existence, uniqueness and stability results.

(H1) $A(t)$ generates a family of evolution operators $U(t,s)$ in $\mathbb{X}$ and there exists a constant $M>0$ such that

$$\|U(t,s)\|\le M,\ \text{ for }\ 0\le s\le t\le T.$$

(H2) The continuous function $f:\mathbb{R}_\eta\times\mathbb{X}\to\mathbb{X}$, satisfy the Lipchitz condition, that is, for $u_1,u_2\in\mathbb{X}$ and $\eta\le t\le T$ there exists arbitrary constants $L_f,C_f\ge 0$ such that

$$\begin{aligned}\mathbb{E}\|f(t,u_1(s))-f(t,u_2(s))\|^2 &\le L_f\mathbb{E}\|u_1-u_2\|^2,\\ \mathbb{E}\|f(t,0)\|^2 &\le C_f.\end{aligned}$$

(H3) The continuous function $e:\mathbb{R}_\eta\times\mathbb{R}_\eta\times\mathbb{X}\to\mathbb{X}$, satisfy the Lipchitz condition, that is, for $u_1,u_2\in\mathbb{X}$ and $\eta\le t\le T$ there exists arbitrary constants $L_e,C_e\ge 0$ such that

$$\begin{aligned}\mathbb{E}\|\int_0^t[e(t,s,u_1(s))-e(t,s,u_2(s))]\,ds\|^2 &\le L_e\mathbb{E}\|u_1-u_2\|^2,\\ \mathbb{E}\|\int_0^t e(t,s,0)ds\|^2 &\le C_e.\end{aligned}$$

(H4) The condition $\max_{i,k}\{\prod_{j=i}^{k}\|\alpha_j(\tau_j)\|\}$ is uniformly bounded if there is a constant $N>0$ such that for all $\tau_j\in D_j;\ j=1,2,...$

$$\max_{i,k}\{\prod_{j=i}^{k}\|\alpha_j(\tau_j)\|\}\le N.$$

(H5) $g:\mathbb{X}\to\mathbb{X}$ is Lipchitz continuous in the following sense; for $u_1,u_2\in\mathbb{X}$ there exists a constants $L_g,C_g\ge 0$ such that

$$\begin{aligned}\mathbb{E}\|g(u_1)-g(u_2)\|^2 &\le L_g\mathbb{E}\|u_1-u_2\|^2,\\ \mathbb{E}\|g(0)\|^2 &\le C_g.\end{aligned}$$

3. Existence and Uniqueness

Theorem 3.1. *If the conditions* $(H1)-(H5)$ *are satisfied then the equation (2.1) has a unique mild solution in* $\mathcal{Z}$*, provided the following inequality holds,*

$$\Lambda = M^2\max(1,N^2)(T-\eta)^2L<1.$$

Proof. Let T be an arbitrary number $\eta<T<+\infty$. First we define the nonlinear operator $\Phi:\mathcal{Z}\to\mathcal{Z}$ as follows

$$\begin{aligned}(\Phi u)(t) &= \sum_{k=0}^{+\infty}\Big[\prod_{i=1}^{k}\alpha_i(\tau_i)U(t,t_0)[u_0+g(u)]+\sum_{i=1}^{k}\prod_{j=i}^{k}\alpha_j(\tau_j)\int_{\xi_i-1}^{\xi_i}U(t,s)f(s,u(s))ds\\
&+\int_{\xi_k}^{t}U(t,s)f(s,u(s))ds+\sum_{i=1}^{k}\prod_{j=i}^{k}\alpha_j(\tau_j)\int_{\xi_{i-1}}^{\xi_i}U(t,s)\left(\int_0^s e(s,\theta,u(\theta))d\theta\right)ds\\
&+\int_{\xi_k}^{t}U(t,s)\left(\int_0^s e(s,\theta,u(\theta))d\theta\right)ds\Big]I_{[\xi_k,\xi_{k+1})}(t),\qquad t\in[\eta,T].\end{aligned}$$

It is easy to verify the continuity of Φ. Now we have to show that Φ maps $\mathcal{Z}$ into itself.

$$\begin{aligned}\|(\Phi u)(t)\|^2 &\le \Big(\sum_{k=0}^{+\infty}\Big[\|\prod_{i=1}^{k}\alpha_i(\tau_i)\|\|U(t,t_0)\|\|u_0+g(u)\|+\sum_{i=1}^{k}\|\prod_{j=i}^{k}\alpha_j(\tau_j)\|\int_{\xi_i-1}^{\xi_i}\|U(t,s)f(s,u(s))\|ds\\
&+\int_{\xi_k}^{t}\|U(t,s)f(s,u(s))\|ds+\sum_{i=1}^{k}\|\prod_{j=i}^{k}\alpha_j(\tau_j)\|\int_{\xi_{i-1}}^{\xi_i}\|U(t,s)\left(\int_0^s e(s,\theta,u(\theta))d\theta\right)\|ds\\
&+\int_{\xi_k}^{t}\|U(t,s)\left(\int_0^s e(s,\theta,u(\theta))d\theta\right)\|ds\Big]I_{[\xi_k,\xi_{k+1})}(t)\Big)^2\\
&\le 2\Big(\sum_{k=0}^{+\infty}\Big[\|\prod_{i=1}^{k}\alpha_i(\tau_i)\|^2\|U(t,t_0)\|^2\|u_0+g(u)\|^2I_{[\xi_k,\xi_{k+1})}(t)\Big]\\
&+\Big[\sum_{k=0}^{+\infty}\Big(\sum_{i=1}^{k}\|\prod_{j=i}^{k}\alpha_j(\tau_j)\|\int_{\xi_i-1}^{\xi_i}\|U(t,s)f(s,u(s))\|ds\\
&+\int_{\xi_k}^{t}\|U(t,s)f(s,u(s))\|ds\Big)I_{[\xi_k,\xi_{k+1})}(t)\Big]^2\\
&+\Big[\sum_{k=0}^{+\infty}\Big(\sum_{i=1}^{k}\|\prod_{j=i}^{k}\alpha_j(\tau_j)\|\int_{\xi_{i-1}}^{\xi_i}\|U(t,s)\left(\int_0^s e(s,\theta,u(\theta))d\theta\right)\|ds\\
&+\int_{\xi_k}^{t}\|U(t,s)\left(\int_0^s e(s,\theta,u(\theta))d\theta\right)\|ds\Big)I_{[\xi_k,\xi_{k+1})}(t)\Big]^2\Big)\\
&\le 2M^2\max\left(\|\prod_{i=1}^{k}\alpha_i(\tau_i)\|^2\right)\|u_0+g(u)\|^2+2M^2\left[\max\left(1,\|\prod_{j=i}^{k}\alpha_j(\tau_j)\|\right)\right]^2\left(\int_{t_0}^{t}\|f(s,u(s))\|ds.I_{[\xi_k,\xi_{k+1}]}(t)\right)^2\\
&+2M^2\left[\max\left(1,\|\prod_{j=i}^{k}\alpha_j(\tau_j)\|\right)\right]^2\left(\int_{t_0}^{t}\|\int_0^s e(s,\theta,u(\theta))d\theta\|ds.I_{[\xi_k,\xi_{k+1}]}(t)\right)^2\end{aligned}$$

$$\leq 2M^2B^2\|u_0+g(u)\|^2 + 2M^2\max(1,N^2)(t-t_0)\int_{t_0}^{t}\|f(s,u(s))\|^2ds$$
$$+2M^2\max(1,N^2)(t-t_0)\int_{t_0}^{t}\|\int_0^s e(s,\theta,u(\theta))d\theta\|^2ds.$$

Thus, by Hypothesis $(H1)$ - $(H5)$, we have

$$\begin{aligned}\mathbb{E}\|(\Phi u)(t)\|^2 &\leq 2M^2N^2\|u_0+g(u)\|^2 + 2M^2\max(1,N^2)(T-\eta)\int_{t_0}^{t}\mathbb{E}\|f(s,u(s))\|^2ds\\ &\quad +2M^2\max(1,N^2)(T-\eta)\int_{t_0}^{t}\mathbb{E}\|\int_0^s e(s,\theta,u(\theta))d\theta\|^2ds.\end{aligned}$$

Because the last two terms of the right hand side of the above inequality also increases in t, we have

$$\begin{aligned}\mathbb{E}\|(\Phi u)(t)\|^2 &\leq 2M^2N^2\|u_0+g(u)\|^2 + 4M^2\max(1,N^2)(T-\eta)L_f\int_{t_0}^{t}\mathbb{E}\|u(s)\|^2ds\\ &\quad +4M^2\max(1,N^2)(T-\eta)^2C_f + 4M^2\max(1,N^2)(T-\eta)^2C_e\\ &\quad +4M^2\max(1,N^2)(T-\eta)L_k\int_{t_0}^{t}\mathbb{E}\|u(s)\|^2ds.\end{aligned}$$

Thus

$$\begin{aligned}\sup_{t\in[\eta,T]}\mathbb{E}\|(\Phi u)(t)\|^2 &\leq 2M^2N^2\|u_0+g(u)\|^2 + 4M^2\max(1,N^2)(T-\eta)L_f\int_{t_0}^{t}\sup_{t\in[\eta,T]}\mathbb{E}\|u(s)\|^2ds\\ &\quad +4M^2\max(1,N^2)(T-\eta)^2C_f + 4M^2\max(1,N^2)(T-\eta)^2C_e\\ &\quad +4M^2\max(1,N^2)(T-\eta)L_e\int_{t_0}^{t}\sup_{t\in[\eta,T]}\mathbb{E}\|u(s)\|^2ds,\end{aligned}$$

for all $t\in[\eta,T]$, therefore Φ maps $\mathcal{Z}$ into itself.

In the next step, we will show that Φ is a contraction mapping.

$$\begin{aligned}&\|(\Phi u_1)(t)-(\Phi u_2)(t)\|^2\\ &\leq \Big[\sum_{k=0}^{+\infty}\prod_{i=1}^{k}\|\alpha_j(\tau_j)\|\|U(t,s)\|\|g(u_1)-g(u_2)\|I_{[\xi_k,\xi_{k+1})}\Big]^2\\ &\quad +\Big[\sum_{k=0}^{+\infty}\Big(\sum_{i=1}^{k}\prod_{j=i}^{k}\|\alpha_j(\tau_j)\|\int_{\xi_{i-1}}^{\xi_i}\|U(t,s)\|\|f(s,u_1(s))-f(s,u_2(s))\|ds\\ &\quad +\int_{\xi_k}^{t}\|U(t,s)\|\|f(s,u_1(s))-f(s,u_2(s))\|ds\Big)I_{[\xi_k,\xi_{k+1})}\Big]^2\\ &\quad +\Big[\sum_{k=0}^{+\infty}\Big(\sum_{i=1}^{k}\prod_{j=i}^{k}\|\alpha_j(\tau_j)\|\int_{\xi_{i-1}}^{\xi_i}\|U(t,s)\|\|\int_0^s[e(s,\theta,u_1(\theta))-e(s,\theta,u_2(\theta))]d\theta\|ds\\ &\quad +\int_{\xi_k}^{t}\|U(t,s)\|\|\int_0^s[e(s,\theta,u_1(\theta))-e(s,\theta,u_2(\theta))]d\theta\|ds\Big)I_{[\xi_k,\xi_{k+1})}\Big]^2\\ &\leq M^2\Big[\max\{\prod_{i=1}^{k}\|\alpha_j(\tau_j)\|\}^2\|g(u_1)-g(u_2)\|^2\Big]\end{aligned}$$

$$+M^2\Big[\max\left(1,\prod_{j=i}^{k}\|\alpha_j(\tau_j)\|\right)\Big]^2\left[\int_{t_0}^{t}\|f(s,u_1(s))-f(s,u_2)(s)\|ds.I_{[\xi_k,\xi_{k+1})}(t)\right]^2$$

$$+M^2\Big[\max\left(1,\prod_{j=i}^{k}\|\alpha_j(\tau_j)\|\right)\Big]^2\Big[\int_{t_0}^{t}\|\int_0^s[e(s,\theta,u_1(\theta))-e(s,\theta,u_2(\theta))]\,d\theta\|dsI_{[\xi_k,\xi_{k+1})}(t)\Big]^2$$

$$\leq\ M^2N^2\|g(u_1)-g(u_2)\|^2+M^2\left[\max\left(1,N^2\right)\right](t-t_0)\int_{t_0}^{t}\|f(s,u_1(s))-f(s,u_2(s))\|^2ds$$

$$+M^2\left[\max\left(1,N^2\right)\right](t-t_0)\int_{t_0}^{t}\|\int_0^s[e(s,\theta,u_2(\theta))-e(s,\theta,u_2(\theta))]\,d\theta\|^2ds.$$

This implies that

$$\begin{aligned}
\mathbb{E}\|(\Phi u_1)(t)-(\Phi u_2)(t)\|^2 \leq\ & M^2N^2\mathbb{E}\|g(u_1)-g(u_2)\|^2\\
&+M^2\left[\max\left(1,N^2\right)\right](t-t_0)\int_{t_0}^{t}\mathbb{E}\|f(s,u_1(s))-f(s,u_2(s))\|^2ds\\
&+M^2\left[\max\left(1,N^2\right)\right](t-t_0).\\
&\int_{t_0}^{t}\mathbb{E}\|\int_0^s[e(s,\theta,u_1(\theta))-e(s,\theta,u_2(\theta))]\,d\theta\|^2ds\\
\leq\ & M^2N^2\mathbb{E}\|g(u_1)-g(u_2)\|^2\\
&+M^2\left[\max\left(1,N^2\right)\right](T-\eta)L_f\int_{t_0}^{t}\mathbb{E}\|u_1(s)-u_2(s)\|^2ds\\
&+M^2\left[\max\left(1,N^2\right)\right](T-\eta)L_e\int_{t_0}^{t}\mathbb{E}\|u_1(s)-u_2(s)\|^2ds.
\end{aligned}$$

Taking supremum over t, we have

$$\begin{aligned}
\|\Phi u_1-\Phi u_2\|^2 \leq\ & M^2N^2L_g\|u_1-u_2\|^2+M^2\left[\max\left(1,N^2\right)\right](T-\eta)^2L_f\|u_1-u_2\|^2\\
&+M^2\left[\max\left(1,N^2\right)\right](T-\eta)^2L_e\|u_1(s)-u_2\|^2\\
\leq\ & M^2L\left[\max\left(1,N^2\right)\right](T-\eta)^2\|u_1-u_2\|^2\\
\leq\ & \Lambda\,\|u_1-u_2\|^2,
\end{aligned}$$

where $L=L_f+L_e+\frac{L_g}{(T-\eta)^2}$, a constant. Since $0<\Lambda<1$, the nonlinear operator Φ satisfies the Banach contraction mapping principle and therefore, Φ has a unique fixed point which is the mild solution of the equation (2.1). Thus the proof is completed. □

4. Stability

In this section, we study the stability of the equation (2.1) through the continuous dependence of solutions on initial condition.

Definition 4.1. *A solution $u(t)$ of the system (2.1) with initial value $u_0+g(u)$ is said to be stable in mean square if for all $\epsilon>0$, there exists $\delta>0$ such that*

$$\mathbb{E}\|u(t)-v(t)\|^2\leq\epsilon \quad \textit{whenever} \quad \mathbb{E}\|u_0-v_0\|^2<\delta,$$

for all $t\in[\eta,T]$ where $v(t)$ is another solution of the system (2.1) with initial value $v_0+g(v)$.

Theorem 4.2. *Let $u(t)$ and $v(t)$ be solutions of the equation (2.1) with initial values $u_0 + g(u)$ and $v_0 + g(v) \in \mathcal{R}$ respectively. If the assumptions of Theorem 3.1 are satisfied, then the system (2.1) is stable in the mean square.*

Proof. By the assumptions, $u(t)$ and $v(t)$ are two solutions of the equation (2.1), for $t \in [\eta, T]$. Then

$$\begin{aligned} u(t) - v(t) &= \sum_{k=0}^{+\infty} \Big[\prod_{i=1}^{k} \alpha_i(\tau_i) U(t, t_0)(u_0 - v_0) + \sum_{k=0}^{+\infty} \Big[\prod_{i=1}^{k} \alpha_i(\tau_i) U(t, t_0)(g(u) - g(v)) \\ &\quad + \sum_{i=1}^{k} \prod_{j=i}^{k} \alpha_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} U(t,s)\left[f(s,u(s)) - f(s,v(s))\right] ds \\ &\quad + \int_{\xi_k}^{k} U(t,s)\left[f(s,u(s)) - f(s,v(s))\right] ds \\ &\quad + \sum_{i=1}^{k} \prod_{j=i}^{k} \alpha_j(\tau_j) \int_{\xi_{i-1}}^{\xi_i} U(t,s)\left(\int_0^s [e(s,\theta,u(\theta)) - e(s,\theta,v(\theta))]\, d\theta\right) ds \\ &\quad + \int_{\xi_k}^{k} U(t,s)\left(\int_0^s [e(s,\theta,u(\theta)) - e(s,\theta,v(\theta)))\, d\theta\right) ds\Big] I_{[\xi_k,\xi_k+1)}(t). \end{aligned}$$

By using the hypothesis (H1) - (H5), we get

$$\begin{aligned} \mathbb{E}\|u(t) - v(t)\|^2 &\leq 2\sum_{k=0}^{+\infty} \Big[\prod_{i=1}^{k} \|\alpha_i(\tau_i)\|^2 \|U(t,t_0)\|^2 \mathbb{E}\|u_0 - v_0\|^2 . I_{[\xi_k,\xi_{k+1})}(t)\Big] \\ &\quad +2\sum_{k=0}^{+\infty} \Big[\prod_{i=1}^{k} \|\alpha_i(\tau_i)\|^2 \|U(t,t_0)\|^2 \mathbb{E}\|g(u) - g(v)\|^2 . I_{[\xi_k,\xi_{k+1})}(t)\Big] \\ &\quad +2\mathbb{E}\Big[\sum_{k=0}^{+\infty} \Big[\sum_{i=1}^{k} \prod_{j=i}^{k} \|\alpha_j(\tau_j)\| \int_{\xi_{i-1}}^{\xi_i} \|U(t,s)\|\|f(s,u(s)) - f(s,v(s))\| ds \\ &\quad + \int_{\xi_k}^{t} \|U(t,s)\|\|f(s,u(s)) - f(s,v(s))\| ds\Big] I_{[\xi,\xi_{k+1})}(t)\Big]^2 \\ &\quad +2\mathbb{E}\Big[\sum_{k=0}^{+\infty}\Big[\sum_{i=1}^{k} \prod_{j=i}^{k} \|\alpha_j(\tau_j)\| \int_{\xi_{i-1}}^{\xi_i} \|U(t,s)\| \\ &\quad \times\Big\| \int_0^s [e(s,\theta,u(\theta)) - e(s,\theta,v(\theta))] d\theta\Big\| ds \\ &\quad + \int_{\xi_k}^{t} \|U(t,s)\|\Big\| \int_0^s [e(s,\theta,u(\theta)) - e(s,\theta,v(\theta))] d\theta\Big\| ds\Big] I_{[\xi,\xi_{k+1})}(t)\Big]^2 \\ &\leq 2M^2 \max\left(\Big\|\prod_{i=1}^{k} \alpha_i(\tau_i)\Big\|^2\right) \mathbb{E}\|u_0 - v_0\|^2 + 2M^2 \max\left(\Big\|\prod_{i=1}^{k} \alpha_i(\tau_i)\Big\|^2\right) \mathbb{E}\|g(u) - g(v)\|^2 \\ &\quad +2M^2\Big[\max\left(1, \Big\|\prod_{j=i}^{k} \alpha_j(\tau_j)\Big\|\right)\Big]^2 \mathbb{E}\left(\int_{t_0}^{t} \|f(s,u(s)) - f(s,v(s))\| ds . I_{[\xi,\xi_{k+1})}(t)\right)^2 \\ &\quad +2M^2\Big[\max\left(1, \Big\|\prod_{j=i}^{k} \alpha_j(\tau_j)\Big\|\right)\Big]^2 \mathbb{E}\left(\int_{t_0}^{t} \Big\| \int_0^s [e(s,\theta,u(\theta)) - e(s,\theta,v(\theta))]\, d\theta\Big\| ds\right)^2 . \end{aligned}$$

$$\begin{aligned}\sup_{t\in[\eta,T]}\mathbb{E}\|u(t)-v(t)\|^2 &\leq 2M^2N^2\mathbb{E}\|u_0-v_0\|^2+2M^2N^2\mathbb{E}\|g(u)-g(v)\|^2\\ &+2M^2\max(1,N^2)(T-\eta)^2L_f\int_{t_0}^{t}sup_{s\in[\eta,T]}\mathbb{E}\|u(s)-v(s)\|^2ds\\ &+2M^2\max(1,N^2)(T-\eta)^2L_e\int_{t_0}^{t}sup_{s\in[\eta,T]}\mathbb{E}\|u(s)-v(s)\|^2ds.\end{aligned}$$

By applying Grownwall's inequality, we have

$$\begin{aligned}\sup_{t\in[\eta,T]}\mathbb{E}\|u(t)-v(t)\|^2 &\leq 2M^2N^2\mathbb{E}\|u_0-v_0\|^2exp(2M^2\max(1,N^2)(T-\eta)^2)L\\ &\leq \rho\mathbb{E}\|u_0-v_0\|^2,\end{aligned}$$

where $\rho=2M^2N^2[exp(2M^2\max(1,N^2)(T-\eta)^2)L]$ and $L=L_f+L_e+\frac{L_g}{(T-\eta)^2}$.

Now given $\epsilon>0$, choose $\delta=\frac{\epsilon}{\rho}$ such that $\mathbb{E}\|u_0-v_0\|^2<\delta$. Then

$$\sup_{t\in[\eta,T]}\mathbb{E}\|u(t)-v(t)\|^2\leq\epsilon.$$

Thus, it is apparent that the difference between the solutions $u(t)$ and $v(t)$ in the interval $[\eta,T]$ is small provided the change in the initial point (t_0,u_0) as well as in the functions $g(u), f(t,u(t))$ and $\int_0^t e(t,s,u(s))ds$ do not exceed prescribed amounts. Thus, the proof is completed. □

5. Application

The purpose of this section is to provide an example to show applications of our obtained results.

Example 5.1. *Consider the following semilinear partial integrodifferential equation with random impulse:*

$$\left.\begin{aligned}\frac{\partial}{\partial t}z(t,x) &= a(t,x)\frac{\partial^2}{\partial x^2}z(t,x)+b(t,z(t,x))+\int_0^t d(t,s,z(t,x))ds,\ t\neq\xi_k,\ t\geq\eta\\ z(t,0) &= z(t,\pi)=0,\quad t\geq0,\ t\in J=[0,T],\\ z(\xi_k,x) &= q_k(\tau_k)z(\xi_k^-,x),\ k=1,2,...\\ z(t_0,x) &= z_0(t,x)+\sum_{i=1}^{n}\gamma_i z(s_i,x),\ \gamma_i\in\mathbb{X},\ 0\leq x\leq\pi,\end{aligned}\right\}\quad(2)$$

where $z_t-a(t,x)z_{xx}$ is a uniform parabolic differential operator with $a(t,x)$ continuous on $0\leq x\leq\pi, 0\leq t\leq T$ and is uniformly Holder continuous in t, and constants $\gamma_i, i=1,2,\ldots n$ are small and b,d are continuous.

Assume that τ_k is the random variable defined on $D_k\equiv(0,d_k)$ for $k=1,2,...$, where $0<d_k<+\infty$. Further, assume that τ_i and τ_j are independent of each other as $i\neq j$ for $i,j=1,2,...$; $\xi_0=t_0$; $\xi_k=\xi_{k-1}+\tau_k$, for $k=1,2,...$ and $\max_{i,k}\prod_{j=i}^{k}\|q_k(\tau_j)\|^2<\infty$ Here $t_0\in\mathbb{R}_\eta$ is an arbitrarily given real number.

Take $u(t)=z(t,x)$ is continuous and define the operators $f,\ e$ and g by

$$\begin{aligned}f(t,u(t)) &= b(t,z(t,x));\\ e(t,s,u(s)) &= d(t,s,z(t,x));\\ \text{and } g(u) &= \sum_{i=1}^{n}\gamma_i z(s_i,x).\end{aligned}$$

In particular, set $\mathbb{X} = \mathbb{L}_2([0,\pi])$ and $t \in J = (0,1]$,

$$\begin{aligned} A(t) &= \frac{e^{-t}}{2}; \\ f(t,u(t)) &= \frac{e^{-t}}{e^{-t}+2}\left(\frac{|u(t)|}{1+|u(t)|}\right); \\ e(t,s,u(s)) &= e^{\frac{-u(s)}{4}}, \quad \text{and} \quad g(u) = \frac{1}{2}\sin u. \end{aligned}$$

Define the operator $A : D(A) \subset X \to X$ by $Au = u''$ with the domain

$$D(A) = \{u \in X : u,\ u' \text{ are absolutely continuous, } u'' \in X,\ u(0) = u_0\}.$$

Then

$$Au = \sum_{n=1}^{\infty} n^2 (u,u_n)u_n, \quad u \in D(A),$$

where $\lambda_n = n^2,\ n = 1,2,\ldots$ are the eigen values of A and $\{u_n(s) = \sqrt{2/\pi}\sin ns\}_{n\geq 1}$ is the orthogonal set of eigenvectors of A. It is well known that A is the infinitesimal generator of an analytic semigroup $S(t), t \geq 0$ in $\mathbb{X}$ and is given by

$$S(t)u = \sum_{n=1}^{\infty} e^{-n^2 t}(u,u_n)u_n, \quad u \in \mathbb{X}.$$

Now, we define the operator $A(t) : D(A) \subset \mathbb{X} \to \mathbb{X}$, such that $A(t)$ generates an evolution system $U(t,s)$ satisfying assumptions $(A1)-(A3)$.

With this choice of $A(t)$, $f(t,u(t))$, $e(t,s,u(s))$ and $g(u)$, we see that equation (5.1) can be written in the abstract form as

$$\left.\begin{aligned} u'(t) &= A(t)u(t) + f(t,u(t)) + \int_0^t e(t,s,u(s))ds,\ t \neq \xi_k,\ t \geq \eta, \\ u(\xi_k) &= q_k(\tau_k)u(\xi_k^-),\ k = 1,2,... \\ u(t_0) &= u_0 + g(u). \end{aligned}\right\} \qquad (3)$$

Let $u, v \in \mathbb{X}$ and $t \in (0,1]$. Then

$$\begin{aligned} \mathbb{E}\left\|f(t,u(t)) - f(t,v(t))\right\|^2 &= \mathbb{E}\left\|\frac{e^{-t}}{e^{-t}+2}\left(\frac{|u(t)|}{1+|u(t)|}\right) - \frac{e^{-t}}{e^{-t}+2}\left(\frac{|v(t)|}{1+|v(t)|}\right)\right\|^2 \\ &\leq \frac{e^{-t}}{e^{-t}+2}\mathbb{E}\left\|\frac{|u(t)|}{1+|u(t)|} - \frac{|v(t)|}{1+|v(t)|}\right\|^2 \\ &\leq \frac{1}{3}\mathbb{E}\|u-v\|^2. \end{aligned}$$

From the conditions of Hypothesis (H2), $L_f = \frac{1}{3}$ and $C_f = \frac{1}{3}$. Similarly,

$$\begin{aligned} \mathbb{E}\|\int_0^t e(s,\theta,u(\theta))d\theta - \int_0^t e(s,\theta,v(\theta))d\theta\|^2 &= \mathbb{E}\left\|\int_0^t e^{\frac{-u(s)}{4}}ds - \int_0^t e^{\frac{-v(s)}{4}}\right\|^2 \\ &\leq \frac{1}{4}\mathbb{E}\|u-v\|^2. \end{aligned}$$

Hence the condition for (H3) holds with $L_e = \frac{1}{4}$ and $C_e = \frac{1}{4}$. Also,

$$\begin{aligned} \mathbb{E}\|g(u) - g(v)\|^2 &= \mathbb{E}\|\frac{1}{2}sinu - \frac{1}{2}sinv\|^2 \\ &\leq \frac{1}{2}\mathbb{E}\|u-v\|^2. \end{aligned}$$

From the conditions of the hypothesis (H5), $L_g = \frac{1}{2}$ and $C_g = \frac{1}{2}$.
Choose $B, M = 1$ in such a way that

$$M^2 \max(1, N^2)(T-\eta)^2 L = \frac{7}{12}(T-\eta)^2 < 1,$$

where $L = L_f + L_e + \frac{L_g}{(T-\eta)^2}$.

Hence $\Lambda < 1$, for $t \in (0,1]$. All the conditions for the Theorem (3.1) are satisfied. Therefore, there exists a unique mild solution to given equation (5.1) by using the Banach contraction mapping principle.

References

[1] M.U.Akhmetov, A.Zafer, Stability of Zero solution of Impulsive differential Equations by the Lyapunov second method, Journal of Mathematical Analysis and Applications 248 (2000) 69–82.
[2] A.Anguraj, S.Wu, A.Vinodkumar, The existence and exponential stability of semilinear functional differential equations with random impulses under non-uniqueness, Nonlinear Analaysis 74 (2011) 331–342.
[3] K.Balachandran, Controllability of nonlinear systems in Banach space: A survey, Journal of Optimization Theory and Applications 115 (2002) 7–28.
[4] A.T.Bharucha-Reid, Random Integral Equations, Academic Press, New York, 1972.
[5] A.T.Bharucha-Reid , Fixed point theorems in probabilistic analysis, Bulletin of the American Mathematical Society 82 (1976) 641–657.
[6] L.Byszewski, Theorems about the existence and uniqueness of solutions of a semilinear evolution non-local Cauchy problem, Journal of Mathematical Analysis and Applications 162 (1992) 494–505.
[7] L.Byszewski, H.Akca, Existence of solutions of a semilinear functional differential evolution nonlocal problem, Nonlinear Analaysis: Theory, Methods & Applications 34 (1998) 65–72.
[8] B.C.Dhage, A random version of Schaefers fixed point theorem with applications to functional random integral equations, Tamkang Journal of Mathematics 35 (2004) 197–205.
[9] K.Ezzinbi, H.Toure, I.Zabsonre, Existence and regularity of solutions for some partial functional integrodifferential equations in Banach spaces, Nonlinear Analysis: Theory, Methods & Applications 70 (2009) 2761–2771.
[10] H.R.Henriquez, E.Hernandez, J.P.C.dos Santos, Existence results for abstract partial neutral integrodifferential equation with unbounded delay, Electronic Journal of Qualitative Theory Differential Equations 29 (2009) 1–23.
[11] E. Hernandez, Existence results for partial neutral functional integrodifferential equations with unbounded delay, Journal of Mathematical Analysis and Applications 292 (2004) 194–210.
[12] E.Hernndez, M.Rabello, H.R.Henriquez, Existence of solutions for impulsive partial neutral functional differential equations, Journal of Mathematical Analysis and Applications 331 (2007) 1135–1158.
[13] S.Itho, Random fixed point theorems with an application to random differential equations in Banach spaces, Journal of Mathematical Analysis and Applications 67 (1979) 261–273.
[14] C.Li, J.Sun, R.Sun, Stability analysis of a class of stochastic differential delay equations with nonlinear impulsive effects, Journal of the Franklin Institute Engineering and Applied Mathematics 347 (2010) 1186–1198.
[15] Z.Luo, J.Shen, Stability results for impulsive functional differential equations with infinite delays, Journal of Computational & Applied Mathematics 131 (2001) 55–64.
[16] Z.Luo and J.Shen, Impulsive stabilization of Functional differential equations with infinite delays, Applied Mathematics Letters 16 (2003) 695–701.
[17] Z.Luo, J.Shen, Stability of impulsive functional differential equations via the Liapunov functional, Applied Mathematics Letters 22 (2009) 163–169.
[18] A.Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer-Verlag, New York, 1983.
[19] B.Radhakrishnan, K.Balachandran, Controllability of impulsive neutral functional evolution integrodifferential systems with infinite delay, Nonlinear Analysis: Hybrid Systems 5 (2011) 655–670.
[20] B.Radhakrishnan, M.Tamilarasi, Existence of solutions for quasilinear random impulsive neutral differential evolution equation, Arab Journal of Mathematical Sciences 24 (2018) 235–246.
[21] B.Radhakrishnan, M.Tamilarasi, Existence of random impulsive quasilinear differential inclusions, Journal of Analysis (2018) https://doi.org/10.1007/s41478-018-0132-3.
[22] Yu.V. Rogovchenko, Impusive evolution systems: main results and new trends, Dynamics of Continuous, Discrete and Impulsive Systems 3 (1997) 57–88.
[23] A.M.Samoilenko, N.A.Perestyuk, Impulsive Differential Equations, World Scientific, Singapore, 1995.
[24] Z.Yong, S.J.Wu, Existence and Uniqueness of solutions to Stochastic differential equations with random impulsive under Lipchitz conditions, Chinese Journal of Applied Probability and Statistic 28 (2010) 347–356.
[25] S.J.Wu, X.Z.Meng, Boundedness of nonlinear differential systems with impulsive effect on random moments, Acta Mathematica Applicatae Sinica 20 (2004) 147–154.
[26] S.J. Wu, D.Han, Exponential stability of functional differential systems with impulsive effect on Random Moments, Computers & Mathematics with Applications 50 (2005) 321–328.
[27] S.J.Wu, Y.R.Duan, Oscillation, stability, and boundedness of second-order differential systems with random impulses, Computers & Mathematics with Applications 49 (2005) 1375–1386.

[28] S.J.Wu, X.L.Guo, Y.Zhou, p-moment stability of functional differential equations with random impulses, Computers & Mathematics with Applications 52 (2006) 1683–1694.

[29] S.J.Wu, X.Guo, Z.S.Lin, Existence and uniqueness of solutions to random impulsive differential systems, Acta Mathematica Applicatae Sinica 4 (2006) 627–632.

[30] S.J. Wu, X.Guo, R. Zhai, Almost sure stability of functional differential equations with random impulses, Dynamics of Continuous, Discrete and Impulsive Systems 15 (2008) 403–415.