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Dynamical Behavior of a Stochastic Ratio-Dependent Predator-Prey System with Holling Type IV Functional Response

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Abstract. In this paper, we investigate the dynamical properties of a stochastic ratio-dependent predator-prey system with Holling type IV functional response. The existence of the globally positive solutions to the system with positive initial value is shown employing comparison theorem of stochastic equation and Itô's formula. We derived some sufficient conditions for the persistence in mean and extinction. This system has a stable stationary distribution which is ergodic. Numerical simulations are carried out for further support of present research.

1. Introduction

May [1] proposed the classic predator-prey model with deterministic environment. This model can be expressible in the form

$$\begin{aligned} \frac{dx}{dt} &= rx(1-\frac{x}{K}) - y\phi(x), \\ \frac{dy}{dt} &= y[s(1-\frac{hy}{x})], \end{aligned} \tag{1}$$

in which $x = x(t)$ and $y = y(t)$ represent population densities of the prey and predator respectively at time t. All parameters r, K, s and h are positive constants and the prey population grows logistically with carrying capacity K. Parameters r and s stand for the intrinsic growth rate of prey and predator species respectively, and h is the number of prey required to feed one predator at equilibrium conditions. The term $\frac{hy}{x}$ involved with the predator growth equation is known as Leslie-Gower term, for example, [2], [3]. The consumption

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of prey by the predators is governed by the prey dependent functional response $\phi(x)$, denoted by Holling type I, II, III and IV function. Include Holling-Tanner model namely Leslie type with Holling type II by

$$\phi(x) = \frac{cx}{a + x}. \tag{2}$$

In above equation the parameters c and a are all positive constants. Many researchers analyzed this model in detail. Hsu and Hwang [4], Saez and Gonzalez-Olivares [5] and Gasull and Kooij [6] discussed the relationship between asymptotic stability of positive equilibrium and global stability. The system (1) of Leslie type with generalized Holling type III

$$\phi(x) = \frac{cx^2}{x^2 + bx + a} \tag{3}$$

subjected to the same conditions as given above and an arbitrary constant b was studied. Under the condition of $b > 0$, the system was investigated and the global stability was showed. The unique positive equilibrium of this system is globally asymptotically stable for some parameter values if it was local asymptotically stable [4]. Recently Huang [7] considered this model in more general situation of $b > -2\sqrt{a}$, and exhibited two non-hyperbolic positive equilibria for some values of parameters and subcritical Hopf bifurcation and Bogdanov-Takens bifurcation simultaneously in the corresponding small neighborhoods of the two degenerate equilibria respectively. Collings [8] focused on the model systems (1) with the type IV functional responses

$$\phi(x) = \frac{cx}{x^2 + bx + a} \tag{4}$$

subjected to the same conditions as given above, and found qualitatively similar bifurcation and stability behavior at low levels of prey interference to the types I, II and III functional responses. In this context, some studies [9]-[11] have drawn attention on the other properties of Leslie system with Holling type IV.

In [12], Arditi and Ginzburg based on numerous ecological fields data and laboratory experiments [13]-[15], have suggested that a ratio dependent model is more suitable one for predator-prey interactions where predation involves searching process. This model (1) is obtained by replacing prey-dependent functional response term in (2) and (3) by ratio-dependent functional response $\phi(\frac{x}{y}) = \frac{cx}{ay+x}$ and $\phi(\frac{x}{y}) = \frac{cx^2}{x^2+bxy+ay^2}$ subjected to the same conditions as given above. Recently several researchers have considered the ratio-dependent predator-prey models of Holling type II and III, and discussed the stability of deterministic models and their equilibrium solutions, for example, [4], [16], [17], [18] and [19]. For the stochastic model with Beddington-DeAngelis type functional response and logistic growth for predators, Mandal and Banerjee [20] showed that the system admits unique positive global solution starting from the positive initial value. They proved that the system is strongly persistent in mean. Zhang [21] studied dynamics of a stochastic Holling II one-predator two-prey system with jumps. Liu [22] and Ji [23] also investigated the effect of environment on dynamical behaviors of stochastic models. An increasing number of researchers revealed the effect of white noise forcing on the prey-predator model (see [24]-[29]).

Sun [30] studied the ratio-dependent predator-prey models of Holling type IV ($\phi(\frac{x}{y}) = \frac{cxy}{x^2+bxy+ay^2}$) with human-controlled biological system, and derived the explicit formulas determining the stability, direction and other properties of bifurcation. We also assume that there are neither significant time lags nor human-controlled in the system which is given as follows

$$\begin{aligned} \frac{dx(t)}{dt} &= rx(t) - fx^2(t) - \frac{cx(t)y^2(t)}{x^2(t) + bx(t)y(t) + ay^2(t)}, \\ \frac{dy(t)}{dt} &= sy(t) - \frac{my^2(t)}{x(t)}, \end{aligned} \tag{5}$$

where $x(t)$ and $y(t)$ depict population densities respectively at time t. Parameters r, f, s, m, c, a and b are positive constants in which r/f is the carrying capacity of the prey and m/s is the number of prey required to feed one predator at equilibrium conditions. Here r, s, a and c represent the growth rate of prey and predator species, capturing rate and half capturing saturation constant respectively. To the best of our knowledge, the model (5) was acquired little attention for the dynamic behavior with deterministic or stochastic system. In the present work, we will analyze model (5) with the aspect of white noise due to the effect of environmental fluctuation in reality. To capture this effect it is necessary to establish the stochastic equation according to the deterministic model (5).

Suppose that the environmental noise affect the intrinsic growth rate of the prey and the predator population, specifically,

$$r \to r + \alpha \dot{B}_1(t), \qquad s \to s + \beta \dot{B}_2(t),$$

where $B_1(t)$ and $B_2(t)$ are independent Brownian motions and α and β are positive constants. Then we obtain the following stochastic system

$$\begin{aligned} \dot{x}(t) &= rx(t) - fx^2(t) - \frac{cx(t)y^2(t)}{x^2(t) + bx(t)y(t) + ay^2(t)} + \alpha x(t)\dot{B}_1(t), \\ \dot{y}(t) &= sy(t) - \frac{my^2(t)}{x(t)} + \beta y(t)\dot{B}_2(t). \end{aligned} \tag{6}$$

The parameters are subjected to the same conditions as mentioned before. This paper is organized as follows. In Section 2, by Itô's formula and the comparison theorem of stochastic equations, we show that system (6) has a unique positive solution $(x(t), y(t))$ a.s. with any initial value $(x_0, y_0) \in R^2_+$. In section 3, assuming

$$r - \frac{c}{a} - \frac{\alpha^2}{2} > 0, \ \ s - \frac{\beta^2}{2} > 0, \tag{7}$$

one has

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \frac{y(z)}{x(z)} dz = \frac{s - \frac{\beta^2}{2}}{m} \ a.s.$$

and system (6) is peresistent in mean under condition (7). If $r - \frac{\alpha^2}{2} < 0$ the prey population $x(t)$ is going to be extinct. On the other hand for $s - \frac{\beta^2}{2} < 0$, the predator population $y(t)$ will die out. Meanwhile system (6) in section 3 holding condition

$$r - \frac{c}{a} - \alpha^2 > 0, \ \ s - \beta^2 > 0$$

has a stationary distribution which is ergodicity. Finally numerical simulations for a hypothetical set of parametric values are presented to illustrate the analytical findings.

Throughout this paper, unless otherwise specified, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e. it is right continuous and $\mathcal{F}_0$ contains all P-null sets). Let $B_i(t)$ $(i = 1, 2)$ denote the independent standard Brownian motions defined on this probability space.

2. Existence of the positive solution

It is known that if the coefficients of equation satisfy the linear growth condition and local Lipschitz condition, then the stochastic differential equation has a unique global (i.e.no explosion in a finite time) solution for any given initial value (see [31]-[34]). However, the coefficients of system (6) neither satisfy the linear growth condition, nor local Lipschitz continuous. In this section by making the change of variables and the comparison theorem of stochastic equation (see [35]), we will show there is a unique global positive solution with positive initial value of system (6).

Lemma 2.1. *There is a unique positive local solution $(x(t), y(t))$ for $t \in [0, \tau_e)$ of system (6) a.s. for any initial value $(x_0, y_0) \in R^2_+$.*

Proof. First we transform the system (6) into the following equivalent form Consider the following system

$$\begin{aligned} du(t) &= \left(r - \frac{\alpha^2}{2} - fe^{u(t)} - \frac{ce^{2v(t)}}{ae^{2v(t)} + be^{u(t)+v(t)} + e^{2u(t)}}\right)dt + \alpha dB_1(t), \\ dv(t) &= \left(s - \frac{\beta^2}{2} + \frac{me^{v(t)}}{e^{u(t)}}\right)dt + \beta dB_2(t) \end{aligned} \tag{8}$$

for $t \geq 0$ with initial value $u(0) = lnx_0$, $v(0) = lny_0$. Obviously the coefficients of Eq. (8) satisfy the local Lipschitz condition then there is a unique local solution $(u(t), v(t))$ on $t \in [0, \tau_e)$, where τ_e is the explosion time (see [32], [36]). By Itô's formula, it is easy to see $(x(t) = e^{u(t)}, y(t) = e^{v(t)})$ is the unique positive local solution to equation (6) with initial value $(x_0, y_0) \in R^2_+$ accordingly. □

Lemma 2.1 only tells us there is a unique positive local solution of system (6). Next we show this solution is global, i.e., $\tau_e = \infty$. We consider the following auxiliary stochastic differential equations:

$$\begin{aligned} d\Phi(t) &= (r\Phi(t) - f\Phi^2(t))dt + \alpha\Phi(t)dB_1(t), \\ d\phi(t) &= (r\phi(t) - \frac{c\phi(t)}{a} - f\phi^2(t))dt + \alpha\phi(t)dB_1(t), \end{aligned} \tag{9}$$

and

$$\begin{aligned} d\Psi(t) &= \left(s\Psi(t) - \frac{m}{\Phi(t)}\Psi^2(t)\right)dt + \beta\Psi(t)dB_2(t), \\ d\psi(t) &= \left(s\psi(t) - \frac{m}{\phi(t)}\psi^2(t)\right)dt + \beta\psi(t)dB_2(t), \end{aligned} \tag{10}$$

with the initial value $\Phi(0) = \phi(0) = x_0$ and $\Psi(0) = \psi(0) = y_0$. The solutions of the equations (9) and (10) are (see [34]):

$$\begin{aligned} \Phi(t) &= \frac{e^{(r-\frac{\alpha^2}{2})t+\alpha B_1(t)}}{\frac{1}{x_0} + f\int_0^t e^{(r-\frac{\alpha^2}{2})z+\alpha B_1(z)}dz}, \\ \phi(t) &= \frac{e^{(r-\frac{c}{a}-\frac{\alpha^2}{2})t+\alpha B_1(t)}}{\frac{1}{x_0} + f\int_0^t e^{(r-\frac{c}{a}-\frac{\alpha^2}{2})z+\alpha B_1(z)}dz}, \end{aligned} \tag{11}$$

and

$$\begin{aligned} \Psi(t) &= \frac{e^{(s-\frac{\beta^2}{2})t+\beta B_2(t)}}{\frac{1}{y_0} + m\int_0^t \frac{1}{\Phi(z)}e^{(s-\frac{\beta^2}{2})z+\beta B_2(z)}dz}, \\ \psi(t) &= \frac{e^{(s-\frac{\beta^2}{2})t+\beta B_2(t)}}{\frac{1}{y_0} + m\int_0^t \frac{1}{\phi(z)}e^{(s-\frac{\beta^2}{2})z+\beta B_2(z)}dz}. \end{aligned} \tag{12}$$

Using the comparison principle of stochastic equation, one can write

$$\phi(t) \leq x(t) \leq \Phi(t), \psi(t) \leq y(t) \leq \Psi(t) \ \ a.s. \tag{13}$$

From the process of solutions $\Phi(t), \phi(t), \Psi(t), \psi(t)$, it is clear that these solutions are well defined for all $t \in [0, \infty)$ *a.s.* Thus we obtain:

Theorem 2.2. *There is a unique positive global solution $(x(t), y(t))$ for $t \geq 0$ a.s. of equation (6) for any given initial value $(x_0, y_0) \in R^2_+$. Meanwhile there exists the functions $\Phi(t), \phi(t), \Psi(t)$ and $\psi(t)$ such that $\phi(t) \leq x(t) \leq \Phi(t)$ and $\psi(t) \leq y(t) \leq \Psi(t)$, $t \geq 0$.*

3. The long time behavior of system (6)

3.1. Persistence

Assuming

$(H)\ r-\frac{c}{a}-\frac{\alpha^2}{2}>0,\ \ s-\frac{\beta^2}{2}>0,$

the solution $\Phi(t)$, $\phi(t)$ of the system (9) is similar to the Lemma A.1 of [23], when $r-\frac{c}{a}-\frac{\alpha^2}{2}>0$. Thus

$$\lim_{t\to\infty}\frac{\log\Phi(t)}{t}=0,\quad \lim_{t\to\infty}\frac{1}{t}\int_0^t\Phi(s)ds=\frac{r-\frac{\alpha^2}{2}}{f}\ a.s.$$

and

$$\lim_{t\to\infty}\frac{\log\phi(t)}{t}=0,\quad \lim_{t\to\infty}\frac{1}{t}\int_0^t\phi(s)ds=\frac{r-\frac{c}{a}-\frac{\alpha^2}{2}}{f}\ a.s.$$

In view of (13)

$$\lim_{t\to\infty}\frac{\log x(t)}{t}=0\ a.s.,$$

and

$$\frac{r-\frac{c}{a}-\frac{\alpha^2}{2}}{f}\le\liminf_{t\to\infty}\frac{1}{t}\int_0^t x(s)ds\le\limsup_{t\to\infty}\frac{1}{t}\int_0^t x(s)ds\le\frac{r-\frac{\alpha^2}{2}}{f}\ a.s. \tag{14}$$

Theorem 3.1. *The solution $x(t)$ of the system (6) for any initial value $(x_0, y_0)\in R_+^2$ is peresistent in mean if condition (H) is satisfied. That is*

$$\frac{r-\frac{c}{a}-\frac{\alpha^2}{2}}{f}\le\liminf_{t\to\infty}\frac{1}{t}\int_0^t x(s)ds\le\limsup_{t\to\infty}\frac{1}{t}\int_0^t x(s)ds\le\frac{r-\frac{\alpha^2}{2}}{f}\ a.s.$$

Next we get the following result (consulting Ji [23] (page 1332-1333)):

$$\liminf_{t\to\infty}\frac{\log\psi(t)}{t}\ge 0\ a.s.$$

Accordingly

$$\liminf_{t\to\infty}\frac{\log y(t)}{t}\ge 0\ a.s. \tag{15}$$

Through [23] (page 1331) we know

$$\frac{1}{\Phi(z)}\ge\frac{b}{2(r-\frac{\alpha^2}{2})}e^{\alpha(\min\limits_{0\le u\le z}\{B_1(u)\}-B_1(z))},$$

therefore,

$$\begin{aligned}\frac{1}{\Psi(t)}&=\frac{1}{y_0}e^{-(s-\frac{\beta^2}{2})t-\beta B_2(t)}+m\int_0^t\frac{1}{\Phi(z)}e^{-(s-\frac{\beta^2}{2})(t-z)-\beta(B_2(t)-B_2(z))}dz\\&\ge\frac{1}{y_0}e^{-(s-\frac{\beta^2}{2})t-\beta B_2(t)}+\frac{mb}{2(r-\frac{\alpha^2}{2})}e^{\alpha(\min\limits_{0\le z\le t}\{B_1(z)-\max\limits_{0\le z\le t}\{B_1(z)\}\})+\beta(\min\limits_{0\le z\le t}\{B_2(z)\}-B_2(t))}\int_0^t e^{-(s-\frac{\beta^2}{2})(t-z)}dz.\end{aligned}$$

Subsequent proof is similar to Lemma A.1 (page 1339) of Ji [23] i.e.

$$\limsup_{t\to\infty} \frac{\log \Psi(t)}{t} \leq 0 \; a.s.$$

Consequently

$$\limsup_{t\to\infty} \frac{\log y(t)}{t} \leq 0 \; a.s. \tag{16}$$

Combining of (15) and (16) yields

$$\limsup_{t\to\infty} \frac{\log y(t)}{t} = 0 \; a.s.$$

Besides by Itô's formula, system (6) can be expressed as follows:

$$d\log y(t) = \left(s - \frac{\beta^2}{2} - \frac{my(t)}{x(t)}\right)dt - \beta dB_2(t). \tag{17}$$

Integrating from 0 to t, and dividing by t on both sides of (3.4), we have

$$\frac{\log y(t) - \log y(0)}{t} = s - \frac{\beta^2}{2} - \frac{1}{t}\int_0^t \frac{my(z)}{x(z)}dz - \beta\frac{B_2(t)}{t}.$$

Letting $t \to \infty$, one has

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \frac{y(z)}{x(z)}dz = \frac{s - \frac{\beta^2}{2}}{m} \; a.s.,$$

Theorem 3.2. *Supposing (H) is satisfied then the solution $(x(t), y(t))$ of system (6) for any initial value $(x_0, y_0) \in R_+^2$ has*

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \frac{y(z)}{x(z)}dz = \frac{s - \frac{\beta^2}{2}}{m} \; a.s. \tag{18}$$

Definition 3.3. *System (6) is said to be peresistent in mean if*

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t x(s)ds > 0, \quad \liminf_{t\to\infty} \frac{1}{t}\int_0^t \frac{y(s)}{x(s)}ds > 0 \;\; a.s.$$

From (14) and (18), we directly write that:

Theorem 3.4. *System (6) is said to be peresistent in mean if condition (H) is satisfied.*

3.2. Extinction

We have known whether the establishment of the persistence of system (6) depend on condition (H). We suppose condition (H) is not satisfied and get the following results.

Theorem 3.5. *Let $(x(t), y(t))$ be the solution of system (6) for any initial value $(x_0, y_0) \in R_+^2$,*
(i) if $s - \frac{\beta^2}{2} < 0$, $lim_{t\to\infty} y(t) = 0$ a.s.
(ii) if $r - \frac{\alpha^2}{2} < 0$, $lim_{t\to\infty} x(t) = 0$ and $lim_{t\to\infty} y(t) = 0$ a.s.

Proof. Obviously the predator is extinct if $s-\frac{\beta^2}{2}<0$; the prey is extinct if $r-\frac{\alpha^2}{2}<0$ by the similar proof as in section (3.2) of [33]. Next we prove that the predator will die out when $r-\frac{\alpha^2}{2}<0$ and $s-\frac{\beta^2}{2}\geq 0$. The solution $y(t)$ of the equation (6) is (see [34]):

$$\begin{aligned}\frac{1}{y(t)} &= \frac{1}{y_0}e^{-(s-\frac{\beta^2}{2})t-\beta B_2(t)} + m\int_0^t \frac{1}{x(z)}e^{-(s-\frac{\beta^2}{2})(t-z)-\beta(B_2(t)-B_2(z))}dz\\ &\geq me^{\beta(\min\limits_{0\leq z\leq t}\{B_2(z)\}-B_2(t))}\int_0^t \frac{1}{x(z)}e^{-(s-\frac{\beta^2}{2})(t-z)}dz.\end{aligned}$$

Since

$$\limsup_{t\to\infty}\frac{\log x(t)}{t}\leq r-\frac{\alpha^2}{2}\ a.s.,$$

for any $0<\overline{\varepsilon}<\frac{\alpha^2}{2}-r$, we have

$$x(t)\leq e^{(r-\frac{\alpha^2}{2}+\overline{\varepsilon})t},$$

where $t\geq T$ and $T=T(\overline{\varepsilon})$. Thus we get

$$\begin{aligned}\frac{1}{y(t)} &\geq me^{\beta(\min\limits_{0\leq z\leq t}\{B_2(z)\}-B_2(t))}\int_T^t e^{-(r-\frac{\alpha^2}{2}+\overline{\varepsilon})z}e^{-(s-\frac{\beta^2}{2})(t-z)}dz\\ &= me^{\beta(\min\limits_{0\leq z\leq t}\{B_2(z)\}-B_2(t))}e^{-(s-\frac{\beta^2}{2})t}\int_T^t e^{(\frac{\alpha^2}{2}-r+s-\frac{\beta^2}{2}-\overline{\varepsilon})z}dz\\ &= me^{\beta(\min\limits_{0\leq z\leq t}\{B_2(z)\}-B_2(t))}e^{-(s-\frac{\beta^2}{2})t}\frac{e^{(\frac{\alpha^2}{2}-r+s-\frac{\beta^2}{2}-\overline{\varepsilon})t}-e^{(\frac{\alpha^2}{2}-r+s-\frac{\beta^2}{2}-\overline{\varepsilon})T}}{\frac{\alpha^2}{2}-r+s-\frac{\beta^2}{2}-\overline{\varepsilon}}\\ &= me^{\beta(\min\limits_{0\leq z\leq t}\{B_2(z)\}-B_2(t))}\frac{e^{(\frac{\alpha^2}{2}-r-\overline{\varepsilon})t}-e^{(\frac{\alpha^2}{2}-r-\overline{\varepsilon})T}}{\frac{\alpha^2}{2}-r+s-\frac{\beta^2}{2}-\overline{\varepsilon}},\end{aligned}$$

therefore

$$\begin{aligned}\log\frac{1}{y(t)} \geq{} & \log m+\beta(\min_{0\leq z\leq t}\{B_2(z)\}-B_2(t))+\log(e^{(\frac{\alpha^2}{2}-r-\overline{\varepsilon})t}-e^{(\frac{\alpha^2}{2}-r-\overline{\varepsilon})T})\\ & -\log(\frac{\alpha^2}{2}-r+s-\frac{\beta^2}{2}-\overline{\varepsilon}).\end{aligned}$$

Noticing that

$$\lim_{t\to\infty}\frac{\max\limits_{0\leq z\leq t}B_2(z)}{t}=0\ a.s.$$

and

$$\lim_{t\to\infty}\frac{\log(e^{(\frac{\alpha^2}{2}-r-\overline{\varepsilon})t}-e^{(\frac{\alpha^2}{2}-r-\overline{\varepsilon})T})}{t}=\frac{\alpha^2}{2}-r-\overline{\varepsilon}\ a.s.$$

Then we have

$$\liminf_{t\to\infty}\frac{\log\frac{1}{y(t)}}{t}\geq\frac{\alpha^2}{2}-r-\overline{\varepsilon}\ a.s.,$$

that is,

$$\limsup_{t\to\infty} \frac{\log y(t)}{t} \le r - \frac{\alpha^2}{2} + \bar{\varepsilon} \ a.s.$$

Let $\bar{\varepsilon} \to 0$, then

$$\limsup_{t\to\infty} \frac{\log y(t)}{t} \le r - \frac{\alpha^2}{2} \ a.s.$$

Therefore the solution $y(t)$ of system (6) exponentially tend to 0 almost surely under the condition $r - \frac{\alpha^2}{2} < 0$. □

4. Stationary distribution and ergodicity for system (6)

In this section, the main aim is to investigate the conditions for the existence of a unique stationary distribution of the system (6). It is useful to provide some information to prove the theorem in this part (see Lemma of [37]).

Let $X(t)$ be a homogeneous Markov process in $E_n \subset R^n$ described by the following stochastic differential equation :

$$dX(t) = b(X)dt + \sum_{m=1}^{n} g_m(X)dB_m(t),$$

the diffusion matrix is $A(x) = (a_{ij}(x))$, $a_{ij}(x) = \sum_{m=1}^{n} g_m^i(x)g_m^j(x)$.

Lemma 4.1. *(see [37]) We assume that there exists a bounded domain $D \subset E_n$ with regular boundary Γ, having the following properties:*

(B.1) In the domain D and some neighborhood thereof, the smallest eigenvalue of the diffusion matrix A(x) is bounded away from zero;

(B.2) if $x \in E_n \backslash D$ then the mean time τ at which a path issuing from x reaches the set D is finite, and $\sup_{x\in k} E_x\tau < \infty$ for every compact subset $k \subset E_n$.

We have the following result:

The Markov process X(t) has a stationary distribution $\mu(\cdot)$. Let $f(\cdot)$ be a function integrable with respect to the measure μ then

$$p_x\left\{\lim_{T\to\infty} \frac{1}{T}\int_0^T f(X(t))dt = \int_{E_n} f(x)\mu(dx)\right\} = 1 \ for\ all\ x \in E_n.$$

Remark 4.2. *The proof is given in [37]. The existence of stationary distribution with density is referred to Theorem 4.1, p.119 and Lemma 9.4, p.138. The weak convergence and the ergodicity is obtained in Theorem 5.1, p.121 and Theorem 7.1, p.130.*

To validate (B.1), it suffices to prove F is uniformly elliptical in any bounded domain D, where $F\mu = b(x)\mu_x + \frac{1}{2}tr(A(x)\mu_{xx})$, that is, there is a positive number M such that $\sum_{i,j=1}^{k} a_{ij}(x)\xi_i\xi_j \ge M\|\xi\|^2, x \in \overline{D}, \xi \in R^k$. (see chapter 3, p.103 of [38] and Rayleigh's principle in chapter 6, p.349 of [39]). To validate (B.2), it is enough to show that there exists some neighborhood D and a non-negative C^2-function V such that LV is negative for any $E_n \backslash D$ (details refer to p.1163 of [40]).

Theorem 4.3. *If $r - \frac{c}{a} - \alpha^2 > 0$ and $s - \beta^2 > 0$ then for any given initial value $(x_0, y_0) \in R_+^2$, $(x(t), y(t))$ of system (6) has a unique stationary distribution $\mu(\cdot)$ and it is ergodic.*

Proof. By using Lemma 4.1, the system (6) firstly can be arranged in the form

$$d\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} rx - fx^2 - \dfrac{cxy^2}{x^2+bxy+ay^2} \\ sy - \dfrac{my^2}{x} \end{pmatrix} dt + \begin{pmatrix} \alpha x \\ 0 \end{pmatrix} dB_1(t) + \begin{pmatrix} 0 \\ \beta y \end{pmatrix} dB_2(t),$$

and the diffusion matrix is

$$\Lambda(x,y) = \begin{pmatrix} \alpha^2 x^2 & 0 \\ 0 & \beta^2 y^2 \end{pmatrix}.$$

There exists $M = \min\{\alpha^2x^2, \beta^2y^2, (x,y) \in \overline{D}\} > 0$ such that

$$\sum_{i,j=1}^{2} a_{ij}(x)\xi_i\xi_j = \alpha^2x^2\xi_1^2 + \beta^2y^2\xi_2^2 \geq M\|\xi\|^2, (x,y) \in \overline{D}, \xi \in R^2,$$

which implies that condition (B.1) is validated. Next we need to verify condition (B.2) in Lemma 4.1.

We denote $x(t)$ and $y(t)$ as x and y for convenience, and $(x(t), y(t)) \in R_+^2$ is also the solution of (6) for any initial value (x_0, y_0). Define a C^2-function V: $R_+^2 \to R_+$ by

$$V(x,y) = \frac{1}{x} + \frac{k}{y} + x + \log\frac{y}{k}, \tag{19}$$

where k satisfies $0 < k < \dfrac{r - \frac{c}{a} - \alpha^2}{m}$. We find $(1,k)$ is the minimum point of (19). Here $V(1,k) > 0$. and $V(x,y)$ is nonnegative. By Itô's formula one may write

$$\begin{aligned} LV = & -\frac{r}{x} + f + \left(\frac{cy^2}{x^2+bxy+ay^2}\right)\frac{1}{x} + \frac{\alpha^2}{x} - \frac{ks}{y} + \frac{km}{x} + \frac{k\beta^2}{y} + rx \\ & - fx^2 - \frac{cxy^2}{x^2+bxy+ay^2} + s - \frac{my}{x} - \frac{\beta^2}{2} \\ \leq & -\frac{r}{x} + f + \frac{c}{ax} + \frac{\alpha^2}{x} - \frac{ks}{y} + \frac{km}{x} + \frac{k\beta^2}{y} + rx - fx^2 + s - \frac{my}{x} - \frac{\beta^2}{2} \\ = & -(r - \frac{c}{a} - \alpha^2 - km)\frac{1}{x} - k(s-\beta^2)\frac{1}{y} + rx - fx^2 - \frac{my}{x} + f + s - \frac{\beta^2}{2}. \end{aligned}$$

Consider the bounded set

$$D = \{(x,y) \in R_+^2, \epsilon_1 \leq x \leq \frac{1}{\epsilon_1},\ \epsilon_2 \leq y \leq \frac{1}{\epsilon_2}\},$$

then

$$R_+^2 \setminus D = D_1 \cup D_2 \cup D_3 \cup D_4,$$

$$D_1 = \{(x,y) \in R_+^2, 0 < x < \epsilon_1\},\ \ D_2 = \{(x,y) \in R_+^2, x > \frac{1}{\epsilon_1}\},$$

$$D_3 = \{(x,y) \in R_+^2, 0 < y < \epsilon_2\},\ \ D_4 = \{(x,y) \in R_+^2, y > \frac{1}{\epsilon_2}, 0 < x < \frac{1}{\epsilon_1}\}.$$

We choose sufficiently small ϵ_1, ϵ_2 and $\epsilon_2 = \epsilon_1^2$ such that

$$-(r - \frac{c}{a} - \alpha^2 - km)\frac{1}{\epsilon_1} + r\epsilon_1 + f + s \leq -1, \tag{20}$$

$$K_1 - \frac{f}{2\epsilon_1^2} \leq -1, \tag{21}$$

$$K_2 - (s - \beta^2)\frac{1}{\epsilon_2} \leq -1, \tag{22}$$

$$K_2 - \frac{m}{\epsilon_1} \leq -1. \tag{23}$$

in which K_1 and K_2 are defined by the following (24) and (25).

Case 1. When $(x, y) \in D_1$ then we have

$$\begin{aligned} LV &\leq -(r - \frac{c}{a} - \alpha^2 - km)\frac{1}{x} + rx + f + s \\ &\leq -(r - \frac{c}{a} - \alpha^2 - km)\frac{1}{\epsilon_1} + r\epsilon_1 + f + s, \end{aligned}$$

We have from (20) that

$$LV \leq -1.$$

Case 2. For any $(x, y) \in D_2$ we show

$$LV \leq rx - fx^2 + f + s \leq K_1 - \frac{fx^2}{2} \leq K_1 - \frac{f}{2\epsilon_1^2}$$

where

$$K_1 = \sup_{x\in(0,\infty)} \{rx - \frac{fx^2}{2} + f + s\} < \infty. \tag{24}$$

In view of (21) we get

$$LV \leq -1.$$

Case 3. For $(x, y) \in D_3$ we have

$$\begin{aligned} LV &\leq -k(s - \beta^2)\frac{1}{y} + rx - fx^2 + f + s \\ &\leq K_2 - k(s - \beta^2)\frac{1}{y} \\ &\leq K_2 - k(s - \beta^2)\frac{1}{\epsilon_2}, \end{aligned}$$

with

$$K_2 = \sup_{x\in(0,\infty)} \{rx - fx^2 + f + s\} < \infty. \tag{25}$$

In view of (22) it is obvious that

$$LV \leq -1.$$

Case 4. Letting $(x, y) \in D_4$ and $\epsilon_2 = \epsilon_1^2$, we have

$$\begin{aligned} LV &\le rx - fx^2 + f + s - \frac{my}{x} \\ &\le K_2 - \frac{my}{x} \\ &\le K_2 - \frac{m}{\epsilon_1}, \end{aligned}$$

By condition (23), we also obtain

$$LV \le -1.$$

We can take D to be a neighborhood of the rectangular and a non-negative C^2-function such that LV is negative for any $R_+^2 \backslash D$. It implies condition (B.2) in Lemma 4.1 is satisfied. Therefore the stochastic system (6) has a unique stationary distribution $\mu(\cdot)$ and it is ergodic. □

Now based on the ergodic property, we give the property of the solution of system (6). From (18) and the conclusion of Theorem 4.3, we derive:

Theorem 4.4. *Assume the same conditions as in Theorem 4.3. Then for any initial value $(x_0, y_0) \in R_+^2$, the solution $(x(t), y(t))$ of system (6) has the property*

$$p\left\{\lim_{t\to\infty} \frac{1}{t}\int_0^t \frac{y(s)}{x(s)}ds = \int_{R_+^2} \frac{z_2}{z_1}\mu(dz_1, dz_2) = \frac{s - \frac{\beta^2}{2}}{m}\right\} = 1.$$

5. Numerical simulation and discussion

Finally our outcomes are verified by the method mentioned [41] using matlab software. Consider the discretization equation:

$$\begin{cases} x_{k+1} = x_k + (rx_k - fx_k^2 - \dfrac{cx_k y_k^2}{ay_k^2 + bx_k y_k + x_k^2})\Delta t + \alpha x_k \sqrt{\Delta t}\xi_k + \dfrac{\alpha^2}{2} x_k(\Delta t\xi_k^2 - \Delta t), \\ y_{k+1} = y_k + (sy_k - \dfrac{my_k^2}{x_k})\Delta t + \beta y_k \sqrt{\Delta t}\eta_k + \dfrac{\beta^2}{2} y_k(\Delta t\eta_k^2 - \Delta t), \end{cases}$$

where ξ_k and η_k $(k = 1, 2, \cdots n)$ are the Gaussian random variables $N(0, 1)$. We get the following numerical simulation through appropriate choice of the parameters.

If $r - \frac{c}{a} - \alpha^2 > 0$ and $s - \beta^2 > 0$, then the results of Theorem 3.1-3.3 are obvious and system (6) has a stationary distribution. Fig.1 and Fig.2 examine the results clearly. The difference between Fig.1 and Fig.2 is the intensity of white noise. Specially we choose $\alpha = 0.1, \beta = 0.1$ in Fig.1 and $\alpha = 0.02, \beta = 0.04$ in Fig.2. We clearly observed that with the decreasing values of α, β, the dynamics of stochastic system is getting more similar to the deterministic system's.

In Fig.3, $\alpha = 0.01$, $\beta = 0.21$ such that $r - \frac{\alpha^2}{2} > 0$ and $s - \frac{\beta^2}{2} < 0$. We note that the prey is persistent and the predator is going to die out as said in Theorem 3.5 (i); while in Fig.4, $\alpha = 0.5$, $\beta = 0.01$ such that $r - \frac{\alpha^2}{2} < 0$ and $s - \frac{\beta^2}{2} \ge 0$, the both species will die out as said in Theorem 3.5 (ii). Theorem 3.5 also shows that both species of the system (6) are extinct if $r - \frac{\alpha^2}{2} < 0$ and $s - \frac{\beta^2}{2} < 0$. Numerical simulation support this conclusion. For example in Fig.5 we choose the parameters to satisfy the case of Theorem 3.5. Hence, the prey and the predator are all extinct represented by the red line in Fig.5. In Figs.3-5, we simulate the corresponding deterministic system represented by the blue line at the same values of parameters except for the intensity of white noise. We find that the corresponding deterministic system is persistent and the strong noise may take the species to be extinct which only occurs in the stochastic model.

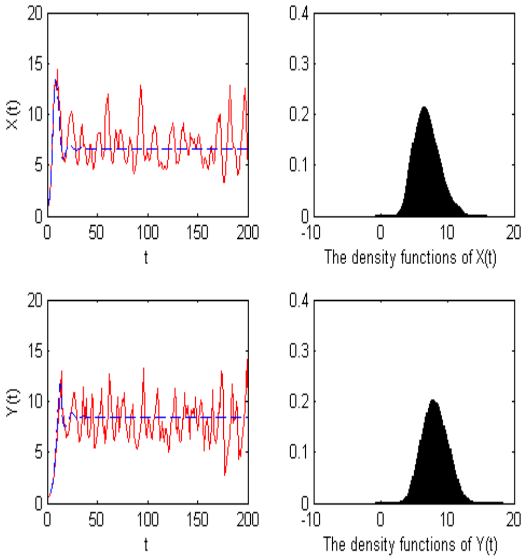

Figure 1: In the left two pictures, the red line represents the solution $(x(t), y(t))$ of system (6); the blue lines represent the solution $(x(t), y(t))$ of model (5) for the same initial value $(x_0, y_0) = (0.5, 0.5)$. Here, $r = 1, f = 0.07, c = 0.8, a = 0.8, b = 0.1, s = 0.51, m = 0.4, \alpha = 0.1$ and $\beta = 0.1$. The right two figures show system (6) has a unique stationary distribution.

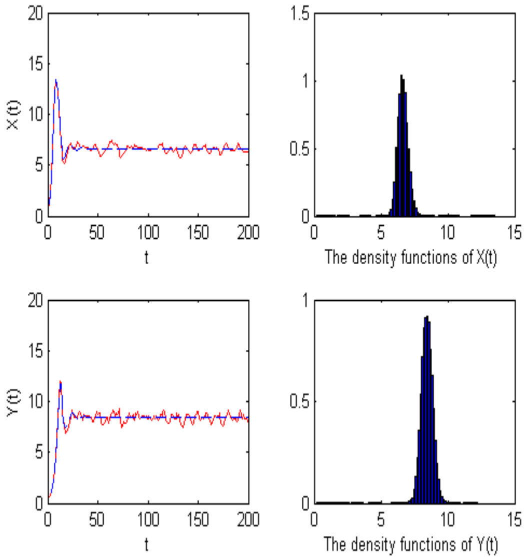

Figure 2: The parameters $\alpha = 0.02$ and $\beta = 0.04$, the others are the same as Figure 1. In the left two pictures, the red line represents the solution $(x(t), y(t))$ of system (6); the blue lines represent the solution $(x(t), y(t))$ of model (5) for the same initial value $(x_0, y_0) = (0.5, 0.5)$. The right ones show system (6) has a unique stationary distribution.

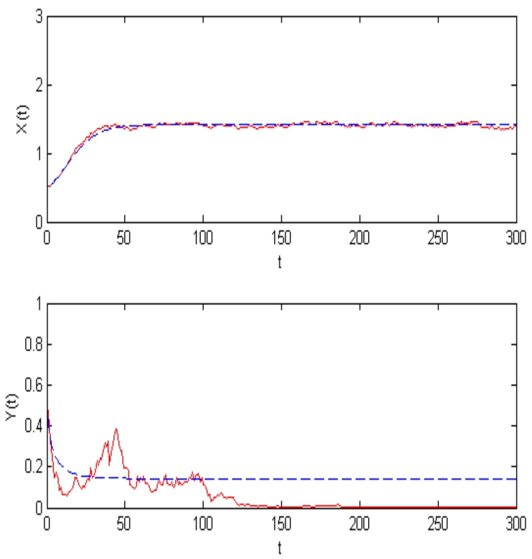

Figure 3: These two figures express the solutions of systems (6) and (5) respectively for $(x_0, y_0) = (0.5, 0.5)$, here $r = 0.1, f = 0.07, c = 0.09, a = 1, b = 0.1, s = 0.02, m = 0.2, \alpha = 0.01$ and $\beta = 0.21$.

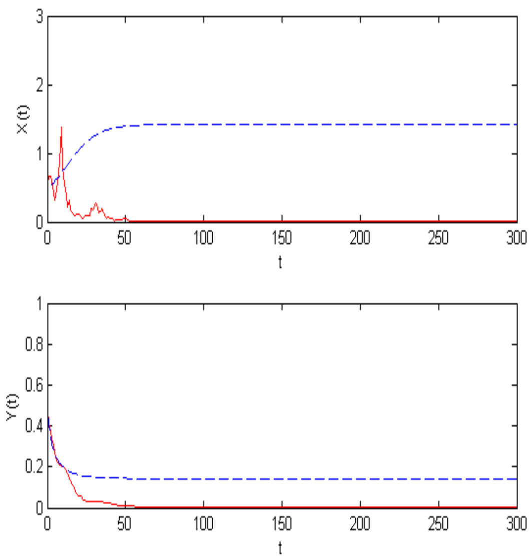

Figure 4: These two figures express the solutions of systems (6) and (5) respectively for $(x_0, y_0) = (0.5, 0.5)$, here $r = 0.1, f = 0.07, c = 0.09, a = 1, b = 0.1, s = 0.02, m = 0.2, \alpha = 0.5$ and $\beta = 0.01$.

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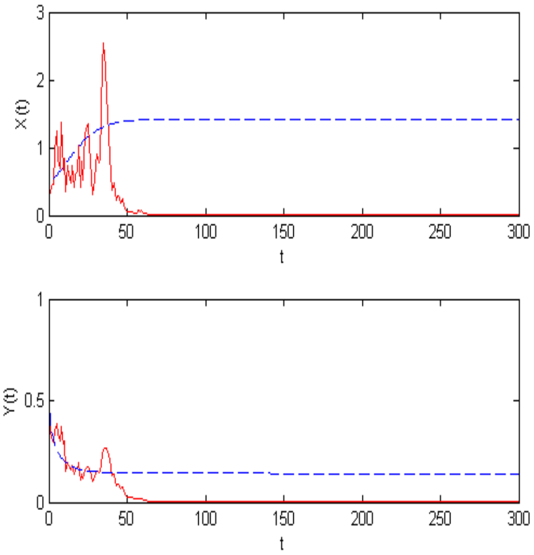

Figure 5: These two figures express the solutions of systems (6) and (5) respectively for $(x_0, y_0) = (0.5, 0.5)$, here $r = 0.1, f = 0.07, c = 0.09, a = 1, b = 0.1, s = 0.02, m = 0.2$, $\alpha = 0.5$ and $\beta = 0.21$.

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