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Hölder's Means and Fixed Points for Multivalued Nonexpansive Mappings

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Abstract. In this paper, we show some geometric conditions on Banach spaces by considering Hölder's means and many well known parameters namely the James constant, the von Neumann-Jordan constant, the weakly convergent sequence coefficient, the normal structure coefficient, the coefficient of weak orthogonality, which imply the existence of fixed points for multivalued nonexpansive mappings and normal structure of Banach spaces. Some of our main results improve and generalize several known results in the recent literature on this topic. We also show that some of our results are sharp.

1. Introduction

Fixed point theory for multivalued mappings has many useful applications in various fields, in particular game theory, control theory, convex optimization, differential equations and mathematical economics. Thus, it is natural to extend the known fixed point results for singlevalued mappings to the setting of multivalued mappings.

In 1969, Nadler [35] extended the Banach Contraction Principle to multivalued contractive mappings in complete metric spaces. Since then, the metric fixed point theory of multivalued mappings has been rapidly developed. Some classical fixed point theorems for singlevalued nonexpansive mappings have been extended to multivalued nonexpansive mappings. Nevertheless, the fixed point theory of multivalued nonexpansive mappings is much more complicated and difficult than its singlevalued counterpart and a lot of problems remain open, for instance, the possibility of extending the well known Kirk's theorem [30], that is, do Banach spaces with weak normal structure have the fixed point property (FPP, in short) for multivalued nonexpansive mappings?

The notions of normal structure and uniform normal structure play an important role in metric fixed point theory for nonexpansive mappings. Since under various geometric properties of a Banach space often measured by different geometric constants, normal structure or uniform normal structure of the space is guaranteed, it is natural to study if those properties imply the FPP for multivalued mappings. Dhompongsa et al. [9, 11] introduced the Domínguez-Lorenzo condition ((DL)-condition, in short) and property (D) which imply the FPP for multivalued nonexpansive mappings and normal structure in reflexive Banach spaces. A possible approach to the above problem is to look for geometric conditions in a Banach space X which imply either the (DL)-condition or property (D). In 2007, Domínguez-Benavides and Gavira [17] had established

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the FPP for multivalued nonexpansive mappings in terms of the modulus of squareness, universal infinite-dimensional modulus, and Opial modulus. The James and von Neumann-Jordan constants are two most widely studied constants by many authors. In [9, 11–15, 19, 20, 25–27, 34] one can find some recent works and results which imply the FPP for multivalued nonexpansive mappings and normal structure of Banach spaces in terms of the James and von Neumann-Jordan constants.

The main aim of the present paper is to give some sufficient conditions for fixed points of multivalued nonexpansive mappings and normal structure of Banach spaces by considering Hölder's means and many geometrical constants. Some of our results improve and generalize a number of recent well known results on this subject. Furthermore, we give different examples which show that some of our results are sharp.

2. Preliminaries

Before going to the results, let us recall some concepts and results which will be used in the following sections.

Throughout the paper, let X be a real Banach space with $\dim(X) \geq 2$ and X^* denotes the dual space of X. We will use $B_X = \{x \in X : \|x\| \leq 1\}$ and $S_X = \{x \in X : \|x\| = 1\}$ to denote the closed unit ball and the unit sphere of X, respectively.

We recall that a Banach space X is said to have normal structure (weak normal structure, respectively) [4] if for every bounded closed (weakly compact, respectively) convex subset K of X that contains more than one point, there exists a point $x_0 \in E$ such that

$$\sup\Big\{\|x_0 - y\| : y \in K\Big\} < \sup\Big\{\|x - y\| : x, y \in K\Big\}.$$

In reflexive Banach spaces, normal structure and weak normal structure are the same. A Banach space X is said to have uniform normal structure if there exists $0 < c < 1$ such that for any closed bounded convex subset K of X that contains more than one point, there exists $x_0 \in E$ such that

$$\sup\Big\{\|x_0 - y\| : y \in K\Big\} < c \sup\Big\{\|x - y\| : x, y \in K\Big\}.$$

It was proved by Kirk [30] that if a weakly compact convex subset K of X has normal structure then any nonexpansive mapping on K has a fixed point. Whether or not a Banach space has normal structure depends on the geometry of the unit sphere.

Recall that a Banach space X is called uniformly non-square provided that there exists $\delta > 0$ such that either $\|x + y\| \leq 2 - \delta$ or $\|x - y\| \leq 2 - \delta$ for all $x, y \in B_X$. In [23] it was proved that uniformly non-square Banach spaces are reflexive, indeed super-reflexive.

Let E be a nonempty subset of X. We shall denote by $CB(X)$ the family of all nonempty bounded closed subsets of X and by $KC(X)$ the family of all nonempty compact convex subsets of X. A multivalued mapping $T : E \to CB(X)$ is said to be nonexpansive if

$$H(Tx, Ty) \leq \|x - y\|, \qquad x, y \in E,$$

where $H(\cdot,\cdot)$ denotes the Hausdorff metric on $CB(X)$ defined by

$$H(A, B) = \max\Big\{\sup_{x\in A}\inf_{y\in B}\|x - y\|, \sup_{y\in B}\inf_{x\in A}\|x - y\|\Big\}, \qquad A, B \in CB(X).$$

The following method and results deal with the concept of asymptotic centers.

Let $\{x_n\}$ be a bounded sequence in X. The asymptotic radius $r\big(E, \{x_n\}\big)$ and the asymptotic center $A\big(E, \{x_n\}\big)$ of $\{x_n\}$ in E are defined by

$$r\big(E, \{x_n\}\big) = \inf\Big\{\limsup_{n\to\infty}\|x_n - x\| : x \in E\Big\},$$

and

$$A\Big(E,\{x_n\}\Big)=\Big\{x\in E\ :\ \limsup_{n\to\infty}\|x_n-x\|=r\Big(E,\{x_n\}\Big)\Big\},$$

respectively. It is known that $A\Big(E,\{x_n\}\Big)$ is a nonempty weakly compact convex set whenever E is [22].

The sequence $\{x_n\}$ is called regular with respect to E if $r\Big(E,\{x_n\}\Big)=r\Big(E,\{x_{n_i}\}\Big)$ for all subsequences $\{x_{n_i}\}$ of $\{x_n\}$, and $\{x_n\}$ is called asymptotically uniform with respect to E if $A\Big(E,\{x_n\}\Big)=A\Big(E,\{x_{n_i}\}\Big)$ for all subsequences $\{x_{n_i}\}$ of $\{x_n\}$. Furthermore, $\{x_n\}$ is called regular asymptotically uniform with respect to E if $\{x_n\}$ is regular and asymptotically uniform with respect to E.

Lemma 2.1. *Let $\{x_n\}$ and E be as above. Then the following assertions hold.*

(i) (Goebel [21], Lim [32]) *There always exists a subsequence of $\{x_n\}$ which is regular with respect to E.*

(ii) (Kirk [31]) *If E is separable, then $\{x_n\}$ contains a subsequence which is asymptotically uniform with respect to E.*

Let C be a nonempty bounded subset of X. The Chebyshev radius of C relative to E is defined by

$$r_E(C)=\inf\Big\{r_x(C)\ :\ x\in E\Big\},$$

where $r_x(C)=\sup\{\|x-y\|\ :\ y\in C\}$. We denote $r_C(C)=r(C)$.

The following two properties of Banach spaces were introduced and used to guarantee the existence of fixed points for multivalued nonexpansive mappings (see [9, 11]).

Definition 2.2. ([9])A Banach space X is said to satisfy property (D) if there exists $\lambda\in[0,1)$ such that for any nonempty weakly compact convex subset E of X, any sequence $\{x_n\}\subset E$ which is regular asymptotically uniform with respect to E, and any sequence $\{y_n\}\subset A\Big(E,\{x_n\}\Big)$ which is regular asymptotically uniform with respect to E we have

$$r\Big(E,\{y_n\}\Big)\le\lambda\, r\Big(E,\{x_n\}\Big).$$

Definition 2.3. ([11])A Banach space X is said to satisfy the (DL)-condition if there exists $\lambda\in[0,1)$ such that for every weakly compact convex subset E of X and for every bounded sequence $\{x_n\}$ in E which is regular with respect to E,

$$r_E\Big(A\Big(E,\{x_n\}\Big)\Big)\le\lambda\, r\Big(E,\{x_n\}\Big).$$

It is clear from the definition that property (D) is weaker than the (DL)-condition. The next results show that property (D) and the (DL)-condition are stronger than weak normal structure and also imply the existence of fixed points for multivalued nonexpansive mappings.

Theorem 2.4. ([9, 11]) *Let E be a nonempty weakly compact convex subset of a Banach space X which satisfies (the (DL)-condition) property (D). Let $T:E\to KC(E)$ be a nonexpansive mapping. Then T has a fixed point.*

Theorem 2.5. ([9, 11]) *Let X be a Banach space satisfying (the (DL)-condition) property (D). Then X has weak normal structure.*

There are various geometric constants for a Banach space in the literature. Among them the James and von Neumann-Jordan constants are two most widely studied constants, due to their connections with various geometric structure of Banach spaces. The following two constants of a Banach space X,

$$C_{NJ}(X)=\sup\left\{\frac{\|x+y\|^2+\|x-y\|^2}{2(\|x\|^2+\|y\|^2)}\ :\ x,y\in X,\ \|x\|+\|y\|>0\right\},$$

$$J(X) = \sup\Big\{ \min\{\|x+y\|, \|x-y\|\} \ : \ x, y \in B_X \Big\},$$

are called the von Neumann-Jordan [7] and James constants [19], respectively.

The following result regarding the relationship between the James and von Neumann-Jordan constants was proved in [20].

Theorem 2.6. ([20]) *If X is a Banach space such that $C_{NJ}(X) < 1 + \frac{1}{(J(X))^2}$, then X satisfies the (DL)-condition.*

The following theorem can be found in [10].

Theorem 2.7. ([10]) *If a Banach space X verifies $C_{NJ}(X) < 1 + \frac{1}{(J(X))^2}$, then X has uniform normal structure.*

We recall that Hölder's means (also called power means) between two positive numbers a and b are defined by

$$M_p(a,b) = \left(\frac{a^p + b^p}{2}\right)^{\frac{1}{p}} \quad \text{for } p \neq 0,$$

$$M_0(a,b) = \lim_{p \to 0} M_p(a,b) = \sqrt{ab}.$$

In particular, the arithmetic mean $A := M_1$ and the geometric mean $G := M_0$ are well known. For two real numbers $p \leq q$,

$$\min(a,b) \leq M_p(a,b) \leq M_q(a,b) \leq \max(a,b),$$

where " = " holds only for the case $a = b$.

Recently, Cui and Lu [8] introduced the constant

$$H_p(X) = \sup\Big\{ M_p\big(\|x+y\|, \|x-y\|\big) \ : \ x, y \in B_X \Big\},$$

for a real number p, by considering Hölder's means. It is obvious that $A_2(X) = H_1(X)$ (see [3]) and $T(X) = H_0(X)$ (see [1]), that is,

$$A_2(X) = \sup\left\{ \frac{\|x+y\| + \|x-y\|}{2} \ : \ x, y \in B_X \right\},$$

and

$$T(X) = \sup\Big\{ \sqrt{\|x+y\| \, \|x-y\|} \ : \ x, y \in B_X \Big\}.$$

The constant $A_2(X)$ was defined by Baronti, Casini and Papini in [3] by considering the arithmetic mean of $\|x+y\|$ and $\|x-y\|$ and the constant $T(X)$ was introduced by Alonso and Llorens-Fuster in [1] by considering the geometric mean between $\|x+y\|$ and $\|x-y\|$. It is noteworthy that X is uniformly non-square if and only if $H_p(X) < 2$ (see [8]).

Now, let us collect some useful properties concerning these constants (see [1, 3, 28]):

(a) $C_{NJ}(X) = C_{NJ}(X^*)$;

(b) $A_2(X) = A_2(X^*)$;

(c) $J(X) \leq T(X) \leq A_2(X)$.

The weakly convergent sequence coefficient $WCS(X) \in [1,2]$ [2] of X is equivalently defined by

$$WCS(X) = \inf\left\{\frac{\lim_{n\neq m}\|x_n - x_m\|}{\limsup_n \|x_n\|}\right\},$$

where the infimum is taken over all weakly (not strongly) null sequences $\{x_n\}$ with $\lim_{n\neq m}\|x_n - x_m\|$ existing.

The normal structure coefficient $N(X) \in [1,2]$ of X defined by Bynum [5] is the number

$$N(X) = \inf\left\{\frac{\operatorname{diam}(E)}{r(E)} \ : \ E \subset X \text{ bounded and convex and } \operatorname{diam}(E) > 0\right\}.$$

The modulus of convexity of X [6] is the function $\delta_X : [0,2] \to [0,1]$ defined by

$$\delta_X(\varepsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} \ : \ x, y \in B_X, \ \|x - y\| \geq \varepsilon\right\}.$$

The characteristic of convexity of X [6] is the number

$$\varepsilon_0(X) = \sup\left\{\varepsilon \ : \ \delta_X(\varepsilon) = 0\right\}.$$

The function δ_X can be discontinuous at 2, but in spite of this

$$\varepsilon_0(X) = 2\Big(1 - \lim_{\varepsilon\to 2^-}\delta(\varepsilon)\Big),$$

(see [33]). As a consequence of Lindenstrauss' formulae [33],

$$\varepsilon_0(X) = 2\rho'_{X^*}(0) \qquad \text{and} \qquad \varepsilon_0(X^*) = 2\rho'_X(0).$$

We recall that

$$\rho'_X(0) = \lim_{t\to 0^+}\frac{\rho_X(t)}{t},$$

where $\rho_X : [0,\infty) \to [0,\infty)$ is the modulus of smoothness of X [22] defined by

$$\rho_X(t) = \sup\left\{\frac{\|x+ty\| + \|x-ty\|}{2} - 1 \ : \ x, y \in B_X\right\}.$$

In section 4 we prove a result that generalizes, in an strict sense, the following theorems.

Theorem 2.8. ([17]) *If X is a Banach space such that $\rho'_X(0) < \frac{1}{2}$, then X satisfies the (DL)-condition.*

Theorem 2.9. ([36]) *If a Banach space X verifies $\rho'_X(0) < \frac{1}{2}$, then X has uniform normal structure.*

The coefficient $M(X) \in [1,2]$ of X, introduced by Domínguez Benavides [16], is defined by

$$M(X) = \sup\left\{\frac{1+a}{R(a,X)} \ : \ a \geq 0\right\},$$

with

$$R(a,X) = \sup\left\{\liminf_{n\to\infty}\|x_n + x\|\right\},$$

where the supremum is taken over all $x \in X$ with $\|x\| \leq a$ and all weakly null sequences $\{x_n\}$ in B_X such that

$$D(\{x_n\}) = \limsup_{n\to\infty}\Big(\limsup_{m\to\infty}\|x_n - x_m\|\Big) \leq 1.$$

The following theorems can be found in [20] and [34].

Theorem 2.10. ([20]) *If X is a Banach space such that* $\rho'_X(0) < \frac{M(X)}{2}$*, then X satisfies the (DL)-condition.*

Theorem 2.11. ([34]) *If a Banach space X verifies* $\rho'_X(0) < \frac{M(X)}{2}$*, then X has normal structure.*

Theorem 2.12. ([20]) *If X is a Banach space such that* $J(X) < 1 + \frac{1}{R(1,X)}$*, then X satisfies the (DL)-condition.*

Theorem 2.13. ([34]) *If a Banach space X verifies* $J(X) < 1 + \frac{1}{R(1,X)}$*, then X has normal structure.*

In [24], Jiménez-Melado and Llorens-Fuster defined the coefficient of weak orthogonality $\mu(X) \in [1,3]$ of X by

$$\mu(X) = \inf\Big\{\lambda \;:\; \limsup_{n\to\infty} \|x_n + x\| \le \lambda \limsup_{n\to\infty} \|x_n - x\|\Big\},$$

where the infimum is taken over all $x \in X$ and all weakly null sequences $\{x_n\}$ in X. It is known that $\mu(X) = \mu(X^*)$ if X is reflexive (see [25]).

The following result regarding the relationship between the James constant and the coefficient of weak orthogonality was proved in [27].

Theorem 2.14. ([27]) *If X is a Banach space such that* $J(X) < 1 + \frac{1}{\mu(X)}$*, then X satisfies the (DL)-condition.*

The following theorem can be found in [25].

Theorem 2.15. ([25]) *If a Banach space X verifies* $J(X) < 1 + \frac{1}{\mu(X)}$*, then X has normal structure.*

3. Hölder's means and property (D)

We first give a condition which implies property (D) and so the FPP for multivalued nonexpansive mappings and normal structure in a reflexive Banach space.

Theorem 3.1. *Let X be a Banach space such that*

$$A_2(X) < 1 + \frac{WCS(X)}{2}.$$

Then X has property (D).

Proof. Let E be a nonempty weakly compact convex subset of X. Denote $r = r\big(E, \{x_n\}\big)$ and $A = A\big(E, \{x_n\}\big)$. We can assume that $r > 0$. Let $\{x_n\} \subset E$ and $\{y_n\} \subset A$ be regular asymptotically uniform sequences with respect to E. Passing through a subsequence of $\{y_n\}$ if necessary, we can also assume that $\{y_n\}$ is weakly convergent to a point $y \in E$ and $d := \lim_{n,m\to\infty, n\neq m} \|y_n - y_m\|$ exists. By using the convexity of A and again, passing through a subsequence of $\{x_n\}$ if necessary, we assume in addition that

$$\|x_n - y_{2n}\| \le r + \frac{1}{n}, \qquad \|x_n - y_{2n+1}\| \le r + \frac{1}{n},$$

and

$$\Big\|x_n - \frac{1}{2}\big(y_{2n} + y_{2n+1}\big)\Big\| \ge r - \frac{1}{n},$$

for all $n \in \mathbb{N}$. Consider

$$u_n = \frac{1}{r + \frac{1}{n}}\big(x_n - y_{2n}\big), \qquad v_n = \frac{1}{r + \frac{1}{n}}\big(x_n - y_{2n+1}\big).$$

It is easy to see that $\lim_{n\to\infty}\|u_n+v_n\|=2$ and $\lim_{n\to\infty}\|u_n-v_n\|=\frac{d}{r}$. Thus, we have

$$A_2(X)\geq\frac{1}{2}\Big(2+\frac{d}{r}\Big)=1+\frac{d}{2r}.$$

Now, we estimate d as follows:

$$\begin{aligned}d&=\lim_{n\neq m}\|y_n-y_m\|=\lim_{n\neq m}\Big\|(y_n-y)-(y_m-y)\Big\|\\&\geq WCS(X)\limsup_{n\to\infty}\|y_n-y\|\\&\geq WCS(X)r\Big(E,\{y_n\}\Big).\end{aligned}$$

Therefore, we conclude

$$r\Big(E,\{y_n\}\Big)\leq\frac{2\Big(A_2(X)-1\Big)}{WCS(X)}\,r.$$

Since $A_2(X)<1+\frac{WCS(X)}{2}$, it follows that X satisfies property (D). □

Since $WCS(X)\leq 2$, if $A_2(X)<1+\frac{WCS(X)}{2}$, then $A_2(X)<2$, which implies that X is uniformly non-square, and consequently X is reflexive. Thus, by applying Theorems 2.4, 2.5 and 3.1, we obtain the following sufficient conditions so that a Banach space X has the fixed point theory for multivalued nonexpansive mappings and normal structure.

Corollary 3.2. *Let E be a nonempty bounded closed convex subset of a Banach space X such that*

$$A_2(X)<1+\frac{WCS(X)}{2},$$

and $T:E\to KC(E)$ be a nonexpansive mapping. Then T has a fixed point.

Corollary 3.3. *Let X be a Banach space such that*

$$A_2(X)<1+\frac{WCS(X)}{2}.$$

Then X has normal structure.

4. Hölder's means and the (DL)-condition

In this section, we present some properties concerning geometrical constants of Banach spaces which also imply the (DL)-condition and so the FPP for multivalued nonexpansive mappings and normal structure in a reflexive Banach space.

Theorem 4.1. *Let E be a nonempty weakly compact convex subset of a Banach space X and let $\{x_n\}$ be a bounded sequence in E regular with respect to E. Then*

$$r_E\Big(A\Big(E,\{x_n\}\Big)\Big)\leq\frac{2\Big(A_2(X)-1\Big)}{N(X)}\,r\Big(E,\{x_n\}\Big).$$

Proof. Denote $r=r\Big(E,\{x_n\}\Big)$ and $A=A\Big(E,\{x_n\}\Big)$. We can assume that $r>0$. We note that since $\{x_n\}$ is regular with respect to E, passing through a subsequence does not have any effect to the asymptotic radius of the whole sequence $\{x_n\}$. If $\mathrm{diam}(A)=0$, then $r_E(A)=0$ and hence we are done. So we can assume that

$\operatorname{diam}(A) > 0$. Let $\varepsilon > 0$ and $u, v \in A$ be such that $\|u - v\| \geq \operatorname{diam}(A) - \varepsilon > 0$. Convexity of A implies $\frac{u+v}{2} \in A$. By the definition of A, we have

$$\limsup_{n\to\infty} \|x_n - u\| = \limsup_{n\to\infty} \|x_n - v\| = \limsup_{n\to\infty} \left\| x_n - \left(\frac{u+v}{2}\right) \right\| = r.$$

Since $\|u - v\| > 0$, there exists a subsequence $\{x_{n'}\}$ of $\{x_n\}$ such that $x_{n'} - u$ and $x_{n'} - v$ are not both zero. Thus, we have

$$\begin{aligned}\Big(\operatorname{diam}(A) - \varepsilon\Big) + \left\| 2\Big(x_{n'} - \Big(\frac{u+v}{2}\Big)\Big) \right\| &\leq \|u - v\| + \left\| 2\Big(x_{n'} - \Big(\frac{u+v}{2}\Big)\Big) \right\| \\ &\leq A_2(X)\Big(\|x_{n'} - u\| + \|x_{n'} - v\|\Big).\end{aligned}$$

By taking the upper limit as $n' \to \infty$ throughout, we have

$$\Big(\operatorname{diam}(A) - \varepsilon\Big) + 2r \leq A_2(X)(r + r),$$

from which it follows that

$$\operatorname{diam}(A) - \varepsilon \leq 2\Big(A_2(X) - 1\Big)r.$$

Because ε is arbitrarily small, we conclude

$$\operatorname{diam}(A) \leq 2\Big(A_2(X) - 1\Big)r. \tag{1}$$

Since A is a bounded convex subset of X with $\operatorname{diam}(A) > 0$, it follows that

$$r_E(A) \leq r(A) \leq \frac{\operatorname{diam}(A)}{N(X)}. \tag{2}$$

Combining (1) and (2), we obtain

$$r_E(A) \leq \frac{2\Big(A_2(X) - 1\Big)}{N(X)}\, r.$$

□

Since $N(X) \leq 2$, if $A_2(X) < 1 + \frac{N(X)}{2}$, then $A_2(X) < 2$, which implies that X is uniformly non-square, and consequently X is reflexive. Thus, by applying Theorems 2.4, 2.5 and 4.1, we obtain the following sufficient conditions so that a Banach space X has the fixed point theory for multivalued nonexpansive mappings and normal structure.

Corollary 4.2. *Let E be a nonempty bounded closed convex subset of a Banach space X such that*

$$A_2(X) < 1 + \frac{N(X)}{2},$$

and $T : E \to KC(E)$ be a nonexpansive mapping. Then T has a fixed point.

Corollary 4.3. *Let X be a Banach space such that*

$$A_2(X) < 1 + \frac{N(X)}{2}.$$

Then X has normal structure.

By using the relation (c), it is easy to prove the following result.

Proposition 4.4. *Let X be a Banach space and consider the following statements:*

(i) $C_{NJ}(X) < 1 + \frac{1}{(A_2(X))^2}$,

(ii) $C_{NJ}(X) < 1 + \frac{1}{(T(X))^2}$,

(iii) $C_{NJ}(X) < 1 + \frac{1}{(J(X))^2}$.

Each of these conditions implies the next.

Since $A_2(X) \geq \sqrt{2}$, if $C_{NJ}(X) < 1 + \frac{1}{(A_2(X))^2}$, then $C_{NJ}(X) < 2$, which implies that X is uniformly non-square, and consequently X is reflexive. Hence, we obtain the following results.

Corollary 4.5. *Let E and E^* be a nonempty bounded closed convex subset of a Banach space X and its dual space X^*, respectively. If*

$$C_{NJ}(X) < 1 + \frac{1}{(A_2(X))^2},$$

and $T : E \to KC(E)$ and $T^* : E^* \to KC(E^*)$ are nonexpansive mappings, then T and T^* have a fixed point.

Proof. Since $C_{NJ}(X) < 1+\frac{1}{(A_2(X))^2}$, it implies that X satisfies the (DL)-condition by Theorem 2.6 and Proposition 4.4. Thus, T has a fixed point by Theorem 2.4. Now, Since $C_{NJ}(X) = C_{NJ}(X^*)$ and $A_2(X) = A_2(X^*)$, we get $C_{NJ}(X^*) < 1 + \frac{1}{(A_2(X^*))^2}$. Hence, by Theorem 2.6 and Proposition 4.4 again we conclude that X^* satisfies the (DL)-condition. Therefore, T^* has a fixed point by Theorem 2.4. □

Corollary 4.6. *Let X be a Banach space such that*

$$C_{NJ}(X) < 1 + \frac{1}{(A_2(X))^2}.$$

Then X and X^* have uniform normal structure.

Proof. Since $C_{NJ}(X) < 1 + \frac{1}{(A_2(X))^2}$, it follows that X has uniform normal structure by Theorem 2.7 and Proposition 4.4. Now, Since $C_{NJ}(X) = C_{NJ}(X^*)$ and $A_2(X) = A_2(X^*)$, we get $C_{NJ}(X^*) < 1 + \frac{1}{(A_2(X^*))^2}$. Thus, by Theorem 2.7 and Proposition 4.4 again we conclude that X^* has uniform normal structure. □

Since $T(X) \geq \sqrt{2}$, if $C_{NJ}(X) < 1 + \frac{1}{(T(X))^2}$, then $C_{NJ}(X) < 2$, which implies that X is uniformly non-square, and consequently X is reflexive. Hence, by applying Theorems 2.6 and 2.7 and Proposition 4.4, we obtain the following sufficient conditions so that a Banach space X has the fixed point theory for multivalued nonexpansive mappings and uniform normal structure.

Corollary 4.7. *Let E be a nonempty bounded closed convex subset of a Banach space X such that*

$$C_{NJ}(X) < 1 + \frac{1}{(T(X))^2},$$

and $T : E \to KC(E)$ be a nonexpansive mapping. Then T has a fixed point.

Corollary 4.8. *Let X be a Banach space such that*

$$C_{NJ}(X) < 1 + \frac{1}{(T(X))^2}.$$

Then X has uniform normal structure.

Proposition 4.9. *Let X be a Banach space. If* $\rho'_X(0) < \frac{1}{A_2(X)}$, *then* $\rho'_X(0) < \frac{M(X)}{2}$.

Proof. In [34] it is proved that

$$M(X) \geq \sup_{t>0} \frac{1+t}{1+\rho_X(t)} = \sup_{t>0} \left(\frac{1}{1+\rho_X(t)} + \frac{1}{1+\rho_X(1/t)} \right).$$

By this inequality, we conclude that $M(X) \geq \frac{2}{A_2(X)}$ since $A_2(X) = 1 + \rho_X(1)$ (see [3]). Therefore, we deduce the desired inequality. □

Corollary 4.10. *Let E be a nonempty bounded closed convex subset of a Banach space X such that*

$$\rho'_X(0) < \frac{1}{A_2(X)},$$

and $T : E \to KC(E)$ *be a nonexpansive mapping. Then T has a fixed point.*

Proof. Since $A_2(X) \geq \sqrt{2}$, it follows that $\rho'_X(0) < 1$, and therefore X is reflexive. As $\rho'_X(0) < \frac{1}{A_2(X)}$, it implies that X satisfies the (DL)-condition by Theorem 2.10 and Proposition 4.9. Thus, T has a fixed point by Theorem 2.4. □

Remark 4.11. Let us see that Corollary 4.10 is a strict generalization of Theorem 2.8. In Theorem 2.8 it was proved that $\rho'_X(0) < \frac{1}{2}$ implies the (DL)-condition. Since $A_2(X) \leq 2$, the condition $\rho'_X(0) < \frac{1}{2}$ implies $\rho'_X(0) < \frac{1}{A_2(X)}$. So that, Corollary 4.10 generalizes Theorem 2.8. Moreover, the following example shows that the generalization is strict.

Consider the Bynum space $\ell_{2,1}$ defined as $\ell_{2,1} := (\ell_2, \|\cdot\|_{2,1})$ where $\|x\|_{2,1} := \max\{\|x^+\|_2, \|x^-\|_2\}$ with $x^+(i) = \max\{x(i), 0\}$ for each $i \geq 1$ and $x^- = x^+ - x$. It is known that $A_2(\ell_{2,1}) = A_2(\ell_{2,\infty}) = 1 + \frac{1}{\sqrt{2}}$ (see [25]) and $\rho'_{\ell_{2,1}}(0) = \frac{1}{2}$ (see [22]). Thus, we have

$$\rho'_{\ell_{2,1}}(0) = \frac{1}{2} < \frac{\sqrt{2}}{\sqrt{2}+1} = \frac{1}{A_2(\ell_{2,1})}.$$

Therefore, $\ell_{2,1}$ verifies the hypothesis in Corollary 4.10, but lies out of the scope of Theorem 2.8.

Corollary 4.12. *Let X be a Banach space such that*

$$\rho'_X(0) < \frac{1}{A_2(X)}.$$

Then X has uniform normal structure.

Proof. Since $A_2(X) \geq \sqrt{2}$, it follows that $\rho'_X(0) < 1$. So X is reflexive, indeed superreflexive. If $\widetilde{X}$ be a ultrapower of X, then $\rho'_X(0) = \rho'_{\widetilde{X}}(0)$ (see [22]) and $A_2(X) = A_2(\widetilde{X})$. Thus, we get $\rho'_{\widetilde{X}}(0) < \frac{1}{A_2(\widetilde{X})}$. Hence, $\widetilde{X}$ has normal structure by Theorem 2.11 and Proposition 4.9. It is known that if X is superreflexive, that is $\widetilde{X^*} = (\widetilde{X})^*$, then X has uniform normal structure if and only if $\widetilde{X}$ has normal structure (see [29]). Consequently, X has uniform normal structure. □

Remark 4.13. As in Remark 4.11, Corollary 4.12 is a strict generalization of Theorem 2.9.

Proposition 4.14. *In any Banach space X,*

(i) $A_2(X) \geq 1 + \frac{\varepsilon_0(X)}{2}$.

(ii) $A_2(X) \geq 1 + \rho'_X(0)$.

Proof. (i) Let $\varepsilon \in [0,2]$. Consider $x, y \in B_X$ such that $\|x - y\| \geq \varepsilon$. Thus, we have

$$A_2(X) \geq \frac{\|x+y\| + \|x-y\|}{2} \geq \frac{\|x+y\| + \varepsilon}{2},$$

and therefore,

$$1 - \frac{\|x+y\|}{2} \geq 1 - A_2(X) + \frac{\varepsilon}{2}.$$

By the definition of δ_X, we obtain that

$$\delta_X(\varepsilon) \geq 1 - A_2(X) + \frac{\varepsilon}{2},$$

or which is the same

$$A_2(X) \geq \frac{\varepsilon}{2} + \big(1 - \delta_X(\varepsilon)\big).$$

Hence, we conclude

$$A_2(X) \geq \sup\left\{\frac{\varepsilon}{2} + \big(1 - \delta_X(\varepsilon)\big) \; : \; \varepsilon \in [0,2]\right\},$$

and in particular

$$A_2(X) \geq \lim_{\varepsilon \to 2^-} \left(\frac{\varepsilon}{2} + \big(1 - \delta_X(\varepsilon)\big)\right) = 1 + \frac{\varepsilon_0(X)}{2}.$$

(ii) $A_2(X) = A_2(X^*) \geq 1 + \frac{\varepsilon_0(X^*)}{2} = 1 + \rho'_X(0)$. □

Proposition 4.15. *Let X be a Banach space and consider the following statements:*

(i) $A_2(X) < \frac{1+\sqrt{5}}{2}$,

(ii) $A_2(X) < 1 + \frac{1}{T(X)}$,

(iii) $A_2(X) < 1 + \frac{1}{J(X)}$,

(iv) $A_2(X) < 1 + \frac{M(X)}{2}$,

(v) $\rho'_X(0) < \frac{M(X)}{2}$.

Each of these conditions implies the next.

Proof. (i)⇒ (ii). Since the inequality $x(x-1) < 1$ holds if and only if $x \in (-\infty, \frac{1-\sqrt{5}}{2}] \cup [0, \frac{1+\sqrt{5}}{2}]$, we have $A_2(X)(A_2(X) - 1) < 1$. On the other hand, $T(X) \leq A_2(X)$. Hence,

$$T(X)(A_2(X) - 1) \leq A_2(X)(A_2(X) - 1) < 1.$$

Therefore, $A_2(X) < 1 + \frac{1}{T(X)}$.

(ii)⇒ (iii). The result follows from the inequality $J(X) \leq T(X)$.

(iii)⇒ (iv). Since $J(X) \geq R(1, X)$ (see [34]), we get $J(X) \geq \frac{2}{M(X)}$, which gives the desired inequality.

(iv)⇒ (v). This implication derives from from the inequality $A_2(X) \geq 1 + \rho'_X(0)$ in Proposition 4.14. □

Remark 4.16. In view of Proposition 4.15, we can see that there are spaces satisfying condition (iv) which do not satisfy (i), (ii) and (iii). To do this, we show that $A_2(X) < 1 + \frac{M(X)}{2}$ does not imply $A_2(X) < 1 + \frac{1}{J(X)}$. For example, if we consider $x = (\frac{1}{2}, -\frac{1}{2}, 0, \cdots)$ and $y = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, \cdots)$ in the Bynum space $\ell_{2,1}$, then we obtain that $J(\ell_{2,1}) \geq \frac{3}{2}$. It is also known that $A_2(\ell_{2,1}) = A_2(\ell_{2,\infty}) = 1 + \frac{1}{\sqrt{2}}$ (see [25]) and $M(\ell_{2,1}) = \sqrt{\frac{5}{2}}$ (see [18]). Hence,

$$A_2(\ell_{2,1}) = 1 + \frac{1}{\sqrt{2}} > \frac{5}{3} \geq 1 + \frac{1}{J(\ell_{2,1})},$$

while

$$A_2(\ell_{2,1}) = 1 + \frac{1}{\sqrt{2}} < 1 + \frac{\sqrt{5}}{2\sqrt{2}} = 1 + \frac{M(\ell_{2,1})}{2}.$$

Consequently, the scope of Conditions (i), (ii) and (iii) is strictly more limited than the scope of Condition (iv) (and then of Condition (v)).

Corollary 4.17. *Let E and E^* be a nonempty bounded closed convex subset of a Banach space X and its dual space X^*, respectively. If*

$$A_2(X) < \frac{1 + \sqrt{5}}{2},$$

and $T : E \to KC(E)$ and $T^* : E^* \to KC(E^*)$ are nonexpansive mappings, then T and T^* have a fixed point.

Proof. Since $A_2(X) < \frac{1+\sqrt{5}}{2}$, it follows that $A_2(X) < 2$ and hence X is uniformly non-square, and consequently X is reflexive. As $A_2(X) = A_2(X^*)$, it follows from Theorem 2.10 and Proposition 4.15 that X and X^* satisfy the (DL)-condition. Thus, T and T^* have a fixed point by Theorem 2.4. □

Remark 4.18. In [37, Corollary 2.4] it was proved that if X is a Banach space with $A_2(X) < \frac{1+\sqrt{5}}{2}$, then X and X^* have normal structure, and it is proved in [11, Theorem 3.2] that the (DL)-condition implies the weak normal structure. Thus, our Corollary 4.17 is stronger than [37, Corollary 2.4].

The following result improves Corollary 2.4 of [37].

Corollary 4.19. *Let X be a Banach space such that*

$$A_2(X) < \frac{1 + \sqrt{5}}{2}.$$

Then X and X^* have uniform normal structure.

Proof. As $A_2(X) < \frac{1+\sqrt{5}}{2}$, it implies that $A_2(X) < 2$ and hence X is uniformly non-square, and consequently X is reflexive, indeed superreflexive. If $\widetilde{X}$ be a ultrapower of X, then $A_2(X) = A_2(\widetilde{X})$. Thus, we get $A_2(\widetilde{X}) < \frac{1+\sqrt{5}}{2}$. Hence, $\widetilde{X}$ has normal structure by Theorem 2.11 and Proposition 4.15. It is known that if X is superreflexive, that is $\widetilde{X^*} = (\widetilde{X})^*$, then X has uniform normal structure if and only if $\widetilde{X}$ has normal structure (see [29]). Consequently, X and X^* have uniform normal structure since $A_2(X) = A_2(X^*)$. □

Corollary 4.20. *Let E be a nonempty bounded closed convex subset of a Banach space X such that one of the following two conditions is satisfied:*

(i) $A_2(X) < 1 + \frac{1}{T(X)}$,

(ii) $A_2(X) < 1 + \frac{1}{J(X)}$,

and let $T : E \to KC(E)$ be a nonexpansive mapping. Then T has a fixed point.

Proof. From (i) or (ii) we know that X is reflexive. Now, by applying Theorem 2.10 and Proposition 4.15, we conclude that X satisfies the (DL)-condition. Therefore, T has a fixed point by Theorem 2.4 □

Corollary 4.21. *Let X be a Banach space such that one of the following two conditions is satisfied:*

(i) $A_2(X) < 1 + \frac{1}{T(X)}$,

(ii) $A_2(X) < 1 + \frac{1}{J(X)}$.

Then X has uniform normal structure.

Proof. From (i) or (ii) we know that X is reflexive, indeed superreflexive. If $\widetilde{X}$ be a ultrapower of X, then $J(X) = J(\widetilde{X})$, $T(X) = T(\widetilde{X})$ and $A_2(X) = A_2(\widetilde{X})$. Thus, we get $A_2(\widetilde{X}) < 1 + \frac{1}{T(\widetilde{X})}$ and $A_2(\widetilde{X}) < 1 + \frac{1}{J(\widetilde{X})}$. Hence, $\widetilde{X}$ has normal structure by Theorem 2.11 and Proposition 4.15. It is known that if X is superreflexive, that is $\widetilde{X^*} = (\widetilde{X})^*$, then X has uniform normal structure if and only if $\widetilde{X}$ has normal structure (see [29]). Consequently, X has uniform normal structure. □

Since $M(X) \leq 2$, if $A_2(X) < 1 + \frac{M(X)}{2}$, then $A_2(X) < 2$, which implies that X is uniformly non-square, and consequently X is reflexive. Thus, by applying Theorems 2.10 and 2.11 and Proposition 4.15, we obtain the following sufficient conditions so that a Banach space X has the fixed point theory for multivalued nonexpansive mappings and normal structure.

Corollary 4.22. *Let E be a nonempty bounded closed convex subset of a Banach space X such that*

$$A_2(X) < 1 + \frac{M(X)}{2},$$

and $T : E \to KC(E)$ be a nonexpansive mapping. Then T has a fixed point.

Corollary 4.23. *Let X be a Banach space such that*

$$A_2(X) < 1 + \frac{M(X)}{2}.$$

Then X has normal structure.

Remark 4.24. Corollaries 4.22 and 4.23 are sharp in the sense that there is a Banach space X such that $A_2(X) = 1 + \frac{M(X)}{2}$ and X does not satisfy the (DL)-condition. Consider the Bynum space $\ell_{2,\infty}$ defined as $\ell_{2,\infty} := (\ell_2, \|\cdot\|_{2,\infty})$ where $\|x\|_{2,\infty} := \max\{\|x^+\|_2, \|x^-\|_2\}$ with $x^+(i) = \max\{x(i), 0\}$ for each $i \geq 1$ and $x^- = x^+ - x$. It is known that $A_2(\ell_{2,\infty}) = 1 + \frac{1}{\sqrt{2}}$ (see [25]) and $M(\ell_{2,\infty}) = \sqrt{2}$ (see [16]). Thus, we have

$$A_2(\ell_{2,\infty}) = 1 + \frac{1}{\sqrt{2}} = 1 + \frac{M(\ell_{2,\infty})}{2},$$

and $\ell_{2,\infty}$ fails to have weak normal structure. Therefore, $\ell_{2,\infty}$ does not satisfy the (DL)-condition.

Proposition 4.25. *Let X be a Banach space and consider the following statements:*

(i) $T(X) < \frac{1+\sqrt{5}}{2}$,

(ii) $T(X) < 1 + \frac{1}{J(X)}$,

(iii) $T(X) < 1 + \frac{1}{R(1,X)}$,

(iv) $J(X) < 1 + \frac{1}{R(1,X)}$.

Each of these conditions implies the next.

Proof. (i) ⇒ (ii). Since the inequality $x(x-1) < 1$ holds if and only if $x \in (-\infty, \frac{1-\sqrt{5}}{2}] \cup [0, \frac{1+\sqrt{5}}{2}]$, we have $T(X)(T(X)-1) < 1$. On the other hand, $J(X) \le T(X)$. So

$$J(X)(T(X)-1) \le T(X)(T(X)-1) < 1.$$

Therefore, $T(X) < 1 + \frac{1}{J(X)}$.

(ii) ⇒ (iii). Since $J(X) \ge R(1, X)$ (see [34]), it then follows that $T(X) < 1 + \frac{1}{R(1,X)}$.

(iii) ⇒ (iv). By using the inequality $J(X) \le T(X)$, the result follows. □

Remark 4.26. Note that we can see that there are spaces satisfying condition (iii) which do not satisfy (i) nor (ii), that is, $T(X) < 1 + \frac{1}{R(1,X)}$ does not imply $T(X) < \frac{1+\sqrt{5}}{2}$. For example, if we consider for $\beta \ge 1$, the space $X_\beta := (\ell_2, |\cdot|_\beta)$ endowed with the norm

$$|x|_\beta := \max\{\|x\|_2, \beta\|x\|_\infty\}.$$

The space X_β verifies $T(X_\beta) = \min(2, \beta\sqrt{2})$ (see [1]) and $R(1, X_\beta) = \max(\frac{\beta}{\sqrt{2}}, \frac{\sqrt{3}}{\sqrt{2}})$ (see [18]). Then, for any $\beta \in [\frac{1+\sqrt{5}}{2\sqrt{2}}, \frac{1+\sqrt{2/3}}{\sqrt{2}})$, we have

$$\frac{1+\sqrt{5}}{2} \le \beta\sqrt{2} = T(X_\beta) < 1 + \sqrt{\frac{2}{3}} = 1 + \frac{1}{R(1, X_\beta)}.$$

Corollary 4.27. *Let E be a nonempty bounded closed convex subset of a Banach space X such that one of the following two conditions is satisfied:*

(i) $T(X) < \frac{1+\sqrt{5}}{2}$,

(ii) $T(X) < 1 + \frac{1}{J(X)}$,

and let $T : E \to KC(E)$ be a nonexpansive mapping. Then T has a fixed point.

Proof. From (i) or (ii) we know that X is reflexive. Now, by applying Theorem 2.12 and Proposition 4.25, we conclude that X satisfies the (DL)-condition. Therefore, T has a fixed point by Theorem 2.4. □

Corollary 4.28. *Let X be a Banach space such that one of the following two conditions is satisfied:*

(i) $T(X) < \frac{1+\sqrt{5}}{2}$,

(ii) $T(X) < 1 + \frac{1}{J(X)}$.

Then X has uniform normal structure.

Proof. From (i) or (ii) we know that X is reflexive, indeed superreflexive. If $\widetilde{X}$ be a ultrapower of X, then $J(X) = J(\widetilde{X})$ and $T(X) = T(\widetilde{X})$. Thus, we get $T(\widetilde{X}) < \frac{1+\sqrt{5}}{2}$ and $T(\widetilde{X}) < 1 + \frac{1}{J(\widetilde{X})}$. Hence, $\widetilde{X}$ has normal structure by Theorem 2.13 and Proposition 4.25. It is known that if X is superreflexive, that is $\widetilde{X^*} = (\widetilde{X})^*$, then X has uniform normal structure if and only if $\widetilde{X}$ has normal structure (see [29]). Consequently, X has uniform normal structure. □

Since $R(1, X) \leq 2$, if $T(X) < 1 + \frac{1}{R(1,X)}$, then $T(X) < 2$, which implies that X is uniformly non-square, and consequently X is reflexive. Thus, by using Theorems 2.12 and 2.13 and Proposition 4.25, we obtain the following sufficient conditions so that a Banach space X has the fixed point theory for multivalued nonexpansive mappings and normal structure.

Corollary 4.29. *Let E be a nonempty bounded closed convex subset of a Banach space X such that*

$$T(X) < 1 + \frac{1}{R(1, X)},$$

and $T : E \to KC(E)$ be a nonexpansive mapping. Then T has a fixed point.

Corollary 4.30. *Let X be a Banach space such that*

$$T(X) < 1 + \frac{1}{R(1, X)}.$$

Then X has normal structure.

Remark 4.31. Corollaries 4.29 and 4.30 are sharp in the sense that there is a Banach space X such that $T(X) = 1 + \frac{1}{R(1,X)}$ and X does not satisfy the (DL)-condition. If we consider the Bynum space $\ell_{2,\infty}$ which fails to have weak normal structure. It is known that $T(\ell_{2,\infty}) = 1 + \frac{1}{\sqrt{2}}$ (see [25]) and $R(1, \ell_{2,\infty}) = \sqrt{2}$ (see [16]). Then, we have

$$T(\ell_{2,\infty}) = 1 + \frac{1}{\sqrt{2}} = 1 + \frac{1}{R(1, \ell_{2,\infty})}.$$

By using the relation (c), it is easy to prove the following result.

Proposition 4.32. *Let X be a Banach space and consider the following statements:*

(i) $A_2(X) < 1 + \frac{1}{\mu(X)}$,

(ii) $T(X) < 1 + \frac{1}{\mu(X)}$,

(iii) $J(X) < 1 + \frac{1}{\mu(X)}$.

Each of these conditions implies the next.

Since $\mu(X) \geq 1$, if $A_2(X) < 1 + \frac{1}{\mu(X)}$, then $A_2(X) < 2$, which implies that X is uniformly non-square, and consequently X is reflexive. Hence, we obtain the following results.

Corollary 4.33. *Let E and E^* be a nonempty bounded closed convex subset of a Banach space X and its dual space X^*, respectively. If*

$$A_2(X) < 1 + \frac{1}{\mu(X)},$$

and $T : E \to KC(E)$ and $T^* : E^* \to KC(E^*)$ are nonexpansive mappings, then T and T^* have a fixed point.

Proof. Since $A_2(X) < 1 + \frac{1}{\mu(X)}$, it implies that X satisfies the (DL)-condition by Theorem 2.14 and Proposition 4.32. Thus, T has a fixed point by Theorem 2.4. Now, Since $A_2(X) = A_2(X^*)$ and $\mu(X) = \mu(X^*)$, we get $A_2(X^*) < 1 + \frac{1}{\mu(X^*)}$. Hence, by Theorem 2.14 and Proposition 4.32 again we conclude that X^* satisfies the (DL)-condition. Therefore, T^* has a fixed point by Theorem 2.4. □

Corollary 4.34. *Let X be a Banach space such that*

$$A_2(X) < 1 + \frac{1}{\mu(X)}.$$

Then X and X^* have normal structure.

Proof. Since $A_2(X) < 1 + \frac{1}{\mu(X)}$, it follows that X has normal structure by Theorem 2.15 and Proposition 4.32. Now, Since $A_2(X) = A_2(X^*)$ and $\mu(X) = \mu(X^*)$, we get $A_2(X^*) < 1 + \frac{1}{\mu(X^*)}$. Thus, by Theorem 2.15 and Proposition 4.32 again we conclude that X^* has normal structure. □

Remark 4.35. Corollaries 4.33 and 4.34 are sharp in the sense that there is a Banach space X such that $A_2(X) = 1 + \frac{1}{\mu(X)}$ and X does not satisfy the (DL)-condition. Consider the Bynum space $\ell_{2,\infty}$. It is known that $A_2(\ell_{2,\infty}) = 1 + \frac{1}{\sqrt{2}}$ (see [25]) and $\mu(\ell_{2,\infty}) = \sqrt{2}$ (see [25]). Hence, we have

$$A_2(\ell_{2,\infty}) = 1 + \frac{1}{\sqrt{2}} = 1 + \frac{1}{\mu(\ell_{2,\infty})},$$

and $\ell_{2,\infty}$ fails to have weak normal structure. Therefore, $\ell_{2,\infty}$ does not satisfy the (DL)-condition.

Since $\mu(X) \geq 1$, if $T(X) < 1 + \frac{1}{\mu(X)}$, then $T(X) < 2$, which implies that X is uniformly non-square, and consequently X is reflexive. Thus, by applying Theorems 2.14 and 2.15 and Proposition 4.32, we obtain the following sufficient conditions so that a Banach space X has the fixed point theory for multivalued nonexpansive mappings and normal structure.

Corollary 4.36. *Let E be a nonempty bounded closed convex subset of a Banach space X such that*

$$T(X) < 1 + \frac{1}{\mu(X)},$$

and $T : E \to KC(E)$ be a nonexpansive mapping. Then T has a fixed point.

Corollary 4.37. *Let X be a Banach space such that*

$$T(X) < 1 + \frac{1}{\mu(X)}.$$

Then X has normal structure.

Remark 4.38. As in Remark 4.35, Corollaries 4.36 and 4.37 are sharp.

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