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T-Stability of the Euler Method for Impulsive Stochastic Differential Equations Driven by Fractional Brownian Motion

Hui Yu[a]

[a]*College of Science, Heilongjiang Bayi Agricultural University, Daqing 163319, China*

Abstract. Due to the fact that a fractional Brownian motion (fBm) with the Hurst parameter $H \in (0, 1/2) \cup (1/2, 1)$ is neither a semimartingale nor a Markov process, relatively little is studied about the T-stability for impulsive stochastic differential equations (ISDEs) with fBm. Here, for such linear equations with $H \in (1/3, 1/2)$, by means of the average stability function, sufficient conditions of the T-stability are presented to their numerical solutions which are established from the Euler-Maruyama method with variable step-size. Moreover, some numerical examples are presented to support the theoretical results.

1. Introduction

The theory of stochastic differential equations (SDEs) driven by Brownian motion with the Hurst parameter $H = 1/2$ plays an important role on the stochastic analysis, and is widely used in many fields(see [23]). In fact, quite a lot of observed natural and social phenomena, such as the fluctuation of stock price and the rate of return in Financial markets, exhibit properties of "biased random walk" and "decile-shaped fat tail". It is suggested that fractional Brownian motion (fBm) with $H \in (0, 1/2) \cup (1/2, 1)$(see [2, 7, 8, 14]), which owns the self-similarity and the long-term memory, provides better modeling and description of such phenomena. In particular, such fractional equations are simple and effective, which were discussed early in [1, 17, 27, 31]. Moreover, It is worthwhile to note that the jump properties of impulsive systems have significant applications in physics, biology, control science and so on(see [20, 38]). However, the research of ISDEs with fBm with $H \in (0, 1/2) \cup (1/2, 1)$, which are neither semimartingale nor Markov processes, is just at the beginning (see [37]). And many significative properties of these equations, such as the T-stability of their numerical methods, are deserved to be discovered. The following sections summary the relevant work accomplished and outline what I would be doing.

The classical Black-Scholes model in financial markets, which was first studied in [3, 26], was extended by broadening the Hurst parameter from $H = 1/2$ to $H \in (0, 1)$ (see [10, 16]). The generalized Black-Scholes model can capture the characteristic of long-range dependence and heavy tailed distribution in miscellaneous financial data. By using the chaos decomposition approach, the existence of a unique continuous solution to linear equations with $H \in (0, 1/2)$ is given in [21]. To semilinear equations, the

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Email address: yuhui163@163.com (Hui Yu)

existence and uniqueness with $H \in (1/2, 1)$ and $H \in (0, 1)$ have been proved in [25] and [12, 22, 27] respectively. By means of the fractional integration and the classical Itô's stochastic calculus, the existence and uniqueness of multidimensional and time-dependent solutions for equations with multidimensional fractional Brownian motion and $H \in (1/2, 1)$ were given in [13]. In [11], an efficient method rooting in a stochastic operational matrix based on the block pulse functions with $H \in (1/2, 1)$ is proposed. Moreover, in [4], the existence of mild solutions for semilinear impulsive stochastic functional differential equations with $H \in (0, 1)$ was proved.

As far as I know, few results [9, 35] about almost sure stochastic stability and pth moment stochastic stability, which are important and widely used, were presented for stochastic processes with fBm. The behavior of the dissipativeness of semilinear SDEs with $H \in (0, 1)$ and their drift-implicit Euler method have been analysed in [12]. The exponential stability of semilinear SDEs with $H \in (1/2, 1)$ was analyzed in [22] via an auxiliary fuction. A result of stability is given for $H \in (1/2, 1)$ in [34]. In [41], by means of the Lyapunov exponents, necessary and sufficient conditions on the two stabilities were established for the generalized Black-Scholes model with $H \in (0, 1)$. Moreover, in [42], with the help of defining a new derivative operator and constructing Lyapunov functions, some sufficient conditions for stability in probability and moment exponential stability of a class of nonlinear equations were given.

Other authors also focused on the convergence of the numerical methods [12, 15, 18, 28, 29, 39] for some special fractional equations. The optimal rate of convergence in mean square of arbitrary approximation methods based on an equidistant discretization was derived for $H \in (1/2, 1)$ in [30]. By using wavelet approximation of multifractional Brownian motion, the approximating method was constructed in [36].

But now, the results of above-mentioned stabilities for stochastic systems are far more than T-stability (see [6, 32]), although the simulation of T-stability requires a small number of samples and can be easily implemented by computer programming. My major aim here is to fill this gap in the T-stability of the Euler-Maruyama method for the model of linear ISDEs with fBm with $H \in (1/3, 1/2)$, which are further developed from the generalized Black-Scholes model in [10, 16] with the help of introducing impulse. More precisely, by means of the average stability function, sufficient conditions of the T-stability of such equations are presented with variable step-size based on the distances between impulsive times (section 3). To show these conditions, the strong convergence of the Euler-Maruyama method is proved in section 2, and the stochastical and asymptotical stability in the large are firstly given to such equations in section 3.

2. Preliminary

Throughout this paper, let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$ be a complete filtered probability space satisfying standard conditions. $|\cdot|$ is the Euclidean norm in $\mathbf{R}^d$, $d \in \mathbf{N}$ and $L^2_{\mathcal{F}_0}(\Omega; \mathbf{R}^d)$ denote the family of $\mathbf{R}^d$-valued $\mathcal{F}_0$-measurable random variables ξ with $\mathbf{E}|\xi|^2 < \infty$.

The following d-dimensional ISDE driven by fBm is concerned in my paper

$$\begin{aligned} &dx(t) = \lambda x(t)dt + \mu x(t)dB^H(t), \quad t \geq 0, \quad t \neq \tau_k, \\ &x(\tau_k) = \beta_k x(\tau_k^-), \quad k = 1, 2, \cdots \\ &x(t) = x_0 \end{aligned} \tag{1}$$

where $x(t-) = \lim_{s\to t-} x(s)$, initial value $x_0 \in L^2_{\mathcal{F}_0}(\Omega; \mathbf{R}^d)$ and $\lambda, \mu \in \mathbf{R}$. The fractional Brownian motion $B^H(t) = (B^{H_1}(t), \cdots, B^{H_M}(t))^T$ with independent scalar components is defined on the filtered probability space and the Hurst parameter $H := \min\{H_j \in (1/3, 1/2) \mid j = 1, \cdots, M\}$.

Assumption 1. *The impulse of equation (1) satisfies the undermentioned conditions.*
(i) *The impulsive functions $\{\beta_k\}_{k=1,2,\cdots}$ are independent real-valued random variables. And there exists a constant L*

such that $\beta_k \leq L$.
(*ii*) *There are infinitely many impulsive times*

$$0 < \tau_1 < \tau_2 < \cdots < \infty,$$

which distances d_k *are bounded by two constants* θ_1 *and* θ_2*, that is to say,*

$$0 < \theta_1 \leq d_k = \tau_{k+1} - \tau_k \leq \theta_2 < \infty.$$

(*iii*) *Let* $\tau_0 = 0$ *and* $\beta_0 = 1$.

Based on the fact that the distances d_k $(k = 0, 1, 2, \cdots)$ might be unequal, the Euler-Maruyama method with variable step-size $h_k = d_k/m$ $(m \in \mathbf{N}_+)$ applied to the equation (1) has the form

$$\begin{aligned} X_{km,p} &= X_{km,p-1} + \lambda h_k X_{km,p-1} + \mu X_{km,p-1} \triangle B^H_{km,p-1}, \quad p = 1, 2, \cdots, m, \\ X_{km,0} &= \beta_k X_{(k-1)m,m} \\ X(0) &= x_0 \end{aligned} \tag{2}$$

where $X_{km,p} \approx x(t_{km,p})$, $t_{km,p} = (km + p)h_k$, $p = 1, 2, \cdots, m$, $X_{km,0} \approx x(\tau_k^-)$.

In [33], the increments of the fractional Brownian motion have the property that $B^H_t - B^H_s \sim N(0, (t - s)^{2H})$, $0 \leq s < t$. Therefore, the independent increments

$$\triangle B^H_{km,p-1} := B^H(t_{km,p}) - B^H(t_{km,p-1})$$

obey the normal distribution $N(0, h_k^{2H})$.

The existence and uniqueness of solutions of the equation (1) can be confirmed on the basis of [21, 37]. Before discussing the stability of the Euler-Maruyama method (2), we prove its convergence.

Theorem 2.1. *Under Assumption 1, the Euler-Maruyama method (2) is strong convergent to the equation (1).*

Proof. In the interval $[\tau_k, \tau_{k+1})$, $k = 0, 1, 2, \cdots$, the equation (1) is a SDE with fBm $H \in (1/3, 1/2)$. According to [39], the Euler-Maruyama method (2) is strong convergent with order $2H$ in the interval $[\tau_k, \tau_{k+1})$, $k = 0, 1, 2, \cdots$. Therefore, in the same way as `Lemma 10.2.2` in [19], the Euler-Maruyama method (2) is strong convergent to the equation (1) with order $2H$ in the interval $[0, \infty)$. □

To demonstrate T-stability of the Euler-Maruyama method (2), stochastical and asymptotical stability in the large of the equation (1) is firstly considered. Some definitions on stability (see [5, 23]) are consequently introduced.

Definition 2.2. *The equation (1) is said to be stochastically stable (otherwise known as stable in probability), if for any* $\epsilon \in (0, 1)$ *, there exists a positive constant* $\delta = \delta(\epsilon) > 0$ *such that for* $|x_0| < \delta$ *a.s.*

$$\mathbf{p}\{\lim_{t\to\infty} x(t; x_0) = 0\} \geq 1 - \epsilon.$$

Definition 2.3. *The equation (1) is said to be stochastical and asymptotical stability in the large, if the equation (1) is stochastically stable and for all* $x_0 \in L^2_{\mathcal{F}_0}(\Omega; \mathbf{R}^d)$

$$\mathbf{p}\{\lim_{t\to\infty} x(t; x_0) = 0\} = 1.$$

Definition 2.4. *The Euler-Maruyama method (2) of the equation (1) is said to be T-stable, if the equation (1) is stochastical and asymptotical stability in the large and the solutions* $X_k = (X_{km,0}, X_{km,1}, \cdots, X_{km,m})^T$ *of the equation (2) satisfy*

$$\lim_{k\to\infty} |X_k| = 0.$$

3. T-stability

In this section, the T-Stability of the Euler-Maruyama method (2) for the equation (1) is focused on. For this purpose, the stochastical and asymptotical stability in the large of the equation (1) is firstly given by means of the Lyapunov exponent.

Theorem 3.1. *Under Assumption 1, if there is a positive constant q such that*

$$|\beta_k|\exp[\lambda(\tau_{k+1}-\tau_k)-\frac{\mu^2}{2}(\tau_{k+1}^{2H}-\tau_k^{2H})]<q<1,\;\; k=0,1,2,\cdots, \tag{3}$$

where $1<\tau_k<\tau_{k+1}$, the equation (1) is stochastical and asymptotical stability in the large.

Proof. For any $t\geq 0$, there exists an appropriate $k\in\mathbf{N}$ such that $t\in[\tau_k,\tau_{k+1})$. Applying the conditions of reducibility in [40] and the Liouville formula in [24] to the equation (1), we have

$$\begin{aligned} x(t) &= x_0\prod_{i=0}^{k}\beta_i\exp[\lambda t-\frac{\mu^2}{2}t^{2H}+\mu B^H(t)] \\ &= \beta_k\exp[\lambda(t-\tau_k)-\frac{\mu^2}{2}(t^{2H}-\tau_k^{2H})+\mu B^H(t)]\prod_{i=0}^{k-1}\beta_i\exp[\lambda(\tau_{i+1}-\tau_i)-\frac{\mu^2}{2}(\tau_{i+1}^{2H}-\tau_i^{2H})]. \end{aligned} \tag{4}$$

Taking norms and logarithms on both sides yields

$$\begin{aligned} \log|x(t)| &= \log|\beta_k x_0|+[(t-\tau_k)-\frac{\mu^2}{2}(t^{2H}-\tau_k^{2H})+\mu B^H(t)] \\ &\quad+\sum_{i=0}^{k-1}\log\{|\beta_i|\exp[\lambda(\tau_{i+1}-\tau_i)-\frac{\mu^2}{2}(\tau_{i+1}^{2H}-\tau_i^{2H})]\}. \end{aligned} \tag{5}$$

I construct the function

$$f(x)=b^x-a^x,\;\; 1<a<b,\;\; x\in(0,1], \tag{6}$$

which is obviously monotonically increasing. I thus have

$$\log|x(t)| \;\leq\; \log|Lx_0|+|\lambda-\frac{\mu^2}{2}|\theta_2+\mu B^H(t)+\sum_{i=0}^{k-1}\log\{|\beta_i|\exp[\lambda(\tau_{i+1}-\tau_i)-\frac{\mu^2}{2}(\tau_{i+1}^{2H}-\tau_i^{2H})]\}, \tag{7}$$

where $1<\tau_k<\tau_{k+1}$.

The condition (3) means that

$$\log\{|\beta_k|\exp[\lambda(\tau_{k+1}-\tau_k)-\frac{\mu^2}{2}(\tau_{k+1}^{2H}-\tau_k^{2H})]\}<0,\;\; k=0,1,2,\cdots. \tag{8}$$

From (7), (8) and the result (`Lemma 2.2` in [41])

$$\lim_{t\to\infty}\frac{B^H(t)}{t}=0,\;\; a.s.,$$

I can obtain the Lyapunov exponent

$$\limsup_{t\to\infty}\frac{1}{t}\log|x(t)|\leq\limsup_{t\to\infty}\frac{1}{t}\sum_{i=0}^{k-1}\log\{|\beta_i|\exp[\lambda(\tau_{i+1}-\tau_i)-\frac{\mu^2}{2}(\tau_{i+1}^{2H}-\tau_i^{2H})]\}<0, \tag{9}$$

which means that the equation (1) is stochastical and asymptotical stability in the large.

I therefore observe that the equation (1) is stochastical and asymptotical stability in the large if its Hurst parameter, impulsive functions, impulsive times and coefficients satisfy (3). □

To analyze the T-Stability of the Euler-Maruyama method (2), I begin by deducing the recurrence relation

$$\begin{aligned}
X_{km,p} &= (1+\lambda h_k+\mu\triangle B^H_{km,p-1})X_{km,p-1}\\
&= (1+\lambda h_k+\mu\triangle B^H_{km,p-1})(1+\lambda h_k+\mu\triangle B^H_{km,p-2})X_{km,p-2}\\
&= \cdots\\
&= \prod_{i=1}^{p-1}(1+\lambda h_k+\mu\triangle B^H_{km,i})X_{(k-1)m,1}\\
&= \beta_k\prod_{i=1}^{p-1}(1+\lambda h_k+\mu\triangle B^H_{km,i})X_{(k-1)m,m}.
\end{aligned}$$

This implies

$$X_{km,p} = X_0\beta_k\prod_{i=0}^{p-1}(1+\lambda h_k+\mu\triangle B^H_{km,i})\prod_{j=0}^{k-1}[\beta_j\prod_{q=0}^{m-1}(1+\lambda h_j+\mu\triangle B^H_{jm,q})]. \tag{10}$$

In this paper, the normal distribution $\triangle B^H_{jm,q}$ is taken as $U_{jm,q}h_j^H$ whose probability distribution is given by $\mathbf{P}\{U_{jm,q}=\pm1\}=1/2$. The average stability function of the Euler-Maruyama method (2) is thus obtained by

$$\begin{aligned}
R(h_k;\lambda,\mu,\beta_k) &= \beta_k\prod_{q=0}^{m-1}(1+\lambda h_k+\mu\triangle B^H_{km,q})\\
&= \beta_k\prod_{q=0}^{m-1}(1+\lambda h_k+\mu U_{km,q}h_k^H).
\end{aligned}\tag{11}$$

Making use of the results (3) and (6), I can compute

$$\begin{aligned}
R^2(h_k;\lambda,\mu,\beta_k) &= \beta_k^2\prod_{q=0}^{m-1}(1+\lambda h_k+\mu U_{km,q}h_k^H)^2\\
&= \beta_k^2\prod_{q=0}^{m-1}(1+\lambda h_k+\mu h_k^H)(1+\lambda h_k-\mu h_k^H)\\
&= \beta_k^2\prod_{q=0}^{m-1}[(1+\lambda h_k)^2-\mu^2h_k^{2H}]\\
&= \beta_k^2[(1+\lambda h_k)^2-\mu^2h_k^{2H}]^m\\
&\le \{[(1+\lambda h_k)^2-\mu^2h_k^{2H}]\exp[-2\lambda h_k+\frac{\mu^2}{m}(\tau_{k+1}^{2H}-\tau_k^{2H})]\}^m\\
&\le \{[(1+\lambda h_k)^2-\mu^2h_k^{2H}]\exp[(\mu^2-2\lambda)h_k]\}^m,
\end{aligned}\tag{12}$$

where $1<\tau_k<\tau_{k+1}$.

Recalling the definition 2.4, I can infer that the Euler-Maruyama method (2) of the equation (1) is T-stability if and only if the average stability function (11) satisfies $|R(h_k;\lambda,\mu,\beta_k)|<1$, where $1<\tau_k<\tau_{k+1}$. The region of T-stability is thus structured by

$$H(h_k)=\{h_k\,\big|\,|(1+\lambda h_k)^2-\mu^2h_k^{2H}|<\exp[(2\lambda-\mu^2)h_k]\}, \tag{13}$$

where $1<\tau_k<\tau_{k+1}$. In the following theorem, I seek appropriate h_k to fulfil

$$(1+\lambda h_k)^2-\mu^2h_k^{2H}<\exp[(2\lambda-\mu^2)h_k]. \tag{14}$$

Modestly using the Taylor's formula to the right side of (14) and choosing fitting h_k , we can have

$$(1+\lambda h_k)^2 - \mu^2 h_k^{2H} < 1 + (2\lambda - \mu^2)h_k + \frac{(2\lambda-\mu^2)^2}{2} h_k^2. \tag{15}$$

Theorem 3.2. *Under Assumption 1, the Euler-Maruyama method (2) for the equation (1) is T-stable when one of the following cases is met*
(*i*)

$$H_1(h_k) = \{h_k \mid \lambda > \frac{\mu^2}{2},\ \lambda^2 > \frac{(2\lambda-\mu^2)^2}{2},\ \ \Delta_{1,1} < h_k < \Delta_{1,2}\}, \tag{16}$$

where $\Delta_{1,1} = \max\{0,\ \sqrt[1-2H]{\frac{\mu^2}{2\lambda}}\ \}$, $\ \Delta_{1,2} = \min\{1,\ \tau_{k+1} - \tau_k,\ \ \sqrt[1-2H]{\frac{\mu^2}{\mu^2+\lambda^2-\frac{(2\lambda-\mu^2)^2}{2}}}\ \}$.
(*ii*)

$$H_2(h_k) = \{h_k \mid \lambda > \frac{\mu^2}{2},\ \lambda^2 = \frac{(2\lambda-\mu^2)^2}{2},\ \ \Delta_{2,1} < h_k < \Delta_{2,2}\}, \tag{17}$$

where $\Delta_{2,1} = \max\{0,\ \sqrt[1-2H]{\frac{\mu^2}{2\lambda}}\ \}$, $\ \Delta_{2,2} = \min\{1,\ \tau_{k+1} - \tau_k\ \}$.
(*iii*)

$$H_3(h_k) = \{h_k \mid \lambda > \frac{\mu^2}{2},\ \lambda^2 < \frac{(2\lambda-\mu^2)^2}{2},\ \ \Delta_{3,1} < h_k < \Delta_{3,2}\}, \tag{18}$$

where $\Delta_{3,1} = \max\{0,\ \sqrt[1-2H]{\frac{\mu^2}{2\lambda}},\ \frac{\mu^2}{\frac{(2\lambda-\mu^2)^2}{2}-\lambda^2+\mu^2}\}$, $\ \Delta_{3,2} = \min\{1,\ \tau_{k+1} - \tau_k\ \}$.
(*iv*)

$$H_4(h_k) = \{h_k \mid \lambda > \frac{\mu^2}{2},\ \lambda^2 < \frac{(2\lambda-\mu^2)^2}{2},\ \ \Delta_{4,1} < h_k = 1 < \Delta_{4,2}\}, \tag{19}$$

where $\Delta_{4,1} = \max\{0,\ \sqrt[1-2H]{\frac{\mu^2}{2\lambda}}\ \}$, $\ \Delta_{4,2} = \tau_{k+1} - \tau_k$.
(*v*)

$$H_5(h_k) = \{h_k \mid \lambda > \frac{\mu^2}{2},\ \lambda^2 < \frac{(2\lambda-\mu^2)^2}{2},\ \ \Delta_{5,1} < h_k < \Delta_{5,2}\}, \tag{20}$$

where $\Delta_{5,1} = \max\{1,\ \sqrt[1-2H]{\frac{\mu^2}{2\lambda}}\ \}$, $\ \Delta_{5,2} = \min\{\tau_{k+1} - \tau_k,\ \ \sqrt[1-2H]{\frac{\frac{(2\lambda-\mu^2)^2}{2}-\lambda^2+\mu^2}{\mu^2}}\ \}$.

Proof. The conditions $\lambda > \frac{\mu^2}{2}$ and $h_k > \sqrt[1-2H]{\frac{\mu^2}{2\lambda}}$ can imply $(1+\lambda h_k)^2 - \mu^2 h_k^{2H} > 0$ and $2\lambda - \mu^2 > 0$ in (15). The proof will be continued in five cases as follows.
(*i*) If $h_k \in H_1(h_k)$, it is easy to see

$$0 < h_k < 1, \tag{21}$$

$$h_k^{1-2H} < \frac{\mu^2}{\mu^2 + \lambda^2 - \frac{(2\lambda-\mu^2)^2}{2}}, \tag{22}$$

and

$$\lambda^2 - \frac{(2\lambda - \mu^2)^2}{2} > 0. \tag{23}$$

The inequality (22) can be reduced to

$$[\mu^2 + \lambda^2 - \frac{(2\lambda - \mu^2)^2}{2}]h_k < \mu^2 h_k^{2H}. \tag{24}$$

The inequality (21) means

$$[\lambda^2 - \frac{(2\lambda - \mu^2)^2}{2}]h_k^2 + \mu^2 h_k < [\lambda^2 - \frac{(2\lambda - \mu^2)^2}{2}]h_k + \mu^2 h_k. \tag{25}$$

From the inequalities (24) and (25), I have

$$[\lambda^2 - \frac{(2\lambda - \mu^2)^2}{2}]h_k^2 + \mu^2 h_k < \mu^2 h_k^{2H}, \tag{26}$$

which can give

$$1 + \lambda^2 h_k^2 + 2\lambda h_k - \mu^2 h_k^{2H} < 1 + 2\lambda h_k - \mu^2 h_k + \frac{(2\lambda - \mu^2)^2}{2} h_k^2. \tag{27}$$

From (26), such h_k can fulfil (15) and (14). Therefore the Euler-Maruyama method (2) for the equation (1) is T-stable.

(*ii*) The inequality (14) is satisfied, if the inequality

$$[\lambda^2 - \frac{(2\lambda - \mu^2)^2}{2}]h_k^2 < \mu^2 (h_k^{2H} - h_k). \tag{28}$$

holds. In addition, if $h_k \in H_2(h_k)$, the both sides of (27) have the distinguishing feature that

$$\lambda^2 - \frac{(2\lambda - \mu^2)^2}{2}]h_k^2 = 0 \tag{29}$$

and

$$\mu^2 (h_k^{2H} - h_k) > 0. \tag{30}$$

From (29) and (30), I thus can obtain that (28) always holds for $h_k \in H_2(h_k)$.

(*iii*) For any $h_k \in H_3(h_k)$, we have

$$h_k^2 < h_k < h_k^{2H}, \tag{31}$$

and

$$h_k > \frac{\mu^2}{\frac{(2\lambda-\mu^2)^2}{2} - \lambda^2 + \mu^2}, \tag{32}$$

which lead to

$$[\frac{(2\lambda - \mu^2)^2}{2} - \lambda^2]h_k^2 + \mu^2 h_k^{2H} > [\frac{(2\lambda - \mu^2)^2}{2} - \lambda^2 + \mu^2]h_k^2 > \mu^2 h_k. \tag{33}$$

Hence (28) holds and then the Euler-Maruyama method (2) is T-stable.

(*iv*) In $H_4(h_k)$, the condition

$$\lambda^2 < \frac{(2\lambda - \mu^2)^2}{2} \tag{34}$$

can be reduced to

$$[\lambda^2 - \frac{(2\lambda-\mu^2)^2}{2}]\cdot 1^2 + \mu^2\cdot 1 < \mu^2\cdot 1^{2H}, \tag{35}$$

which can give the inequality (28).
(v) If $h_k \in H_5(h_k)$, we can obtain

$$h_k^2 > h_k > h_k^{2H} \tag{36}$$

and

$$h_k^{1-2H} < \frac{\frac{(2\lambda-\mu^2)^2}{2} - \lambda^2 + \mu^2}{\mu^2}, \tag{37}$$

which mean

$$[\frac{(2\lambda-\mu^2)^2}{2} - \lambda^2]h_k^2 + \mu^2 h_k^{2H} > [\frac{(2\lambda-\mu^2)^2}{2} - \lambda^2 + \mu^2]h_k^{2H} > \mu^2 h_k. \tag{38}$$

From (37),it is easy to see $h_k \in H_4(h_k)$ satisfies (28). Therefore, the Euler-Maruyama method (2) is T-stable. □

4. Numerical example

The influence of the parameters λ, μ, β_k and the stepsize h_k on T-stability of the Euler-Maruyama method (2) to the equation (1) is shown in this section. Here, three groups of the parameters and their regions of T-stability from Theorem 3.2 are given as

$$\begin{aligned}
&\texttt{I}: \ \lambda = 3,\ \mu^2 = 2,\ H = \frac{3}{8},\ \tau_k = k,\ \beta_k = \exp(-5),\ k = 1, 2, \cdots,\\
&\quad H_1(h_k) = \{h_k \mid 0.0123 < h_k < 0.1795\}.
\end{aligned}$$

$$\begin{aligned}
&\texttt{II}: \ \lambda = \frac{1}{2},\ \mu^2 = 1 - \frac{\sqrt{2}}{2},\ H = \frac{2}{5},\ \tau_k = 2k,\ \beta_k = \exp(-2),\ k = 1, 2, \cdots,\\
&\quad H_2(h_k) = \{h_k \mid 0.002155 < h_k < 1\}.
\end{aligned}$$

$$\begin{aligned}
&\texttt{III}: \ \lambda = 2,\ \mu^2 = 1,\ H = \frac{5}{12},\ \tau_k = 3k,\ \beta_k = \exp(-7),\ k = 1, 2, \cdots,\\
&\quad H_3(h_k) = \{h_k \mid 0.0002441 < h_k < 1\},\\
&\quad H_4(h_k) = \{h_k \mid 0.6667 < h_k = 1 < 3\},\\
&\quad H_5(h_k) = \{h_k \mid 1 < h_k < 3\}.
\end{aligned}$$

I fix the parameters with I-III and choose the stepsizes inside and outside the regions of T-stability H_1-H_5 in Figs.1-3, which describe the T-stability of the Euler-Maruyama method (2) to the equation (1) by Matlab. The observations show that these numerical examples are consistent with the results of Theorem 3.2 in my paper.

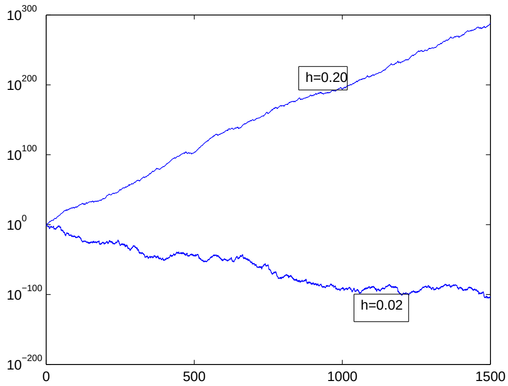

Figure 1: T-stability of the Euler-Maruyama method for I

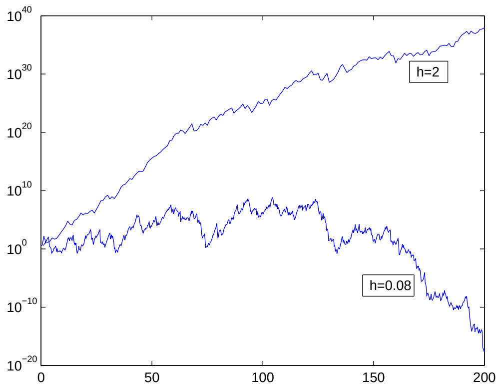

Figure 2: T-stability of the Euler-Maruyama method for II

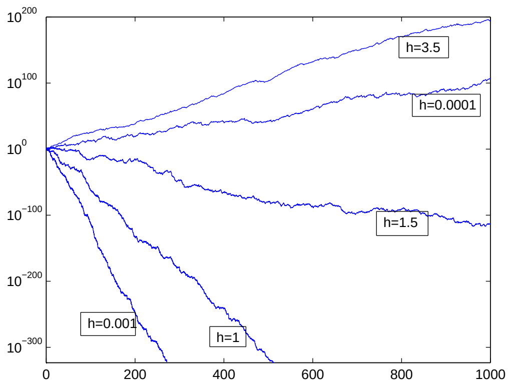

Figure 3: T-stability of the Euler-Maruyama method for III

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