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Hyers-Ulam Stability of Substitution Vector-Valued Integral Operator

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Abstract. For a substitution vector-valued integral operator T_u^{φ}, we determine necessary and sufficient conditions to have Hyers-Ulam stability using conditional expectation operators. Then, we present an example to illustrate our result.

1. Introduction and Preliminaries

It seems that S. M. Ulam [15] first raised the stability problem of functional equations. The problem can be stated as follows. Let G_1 be a group and (G_2, d) a metric group. Given $\epsilon > 0$, does there exists $\delta > 0$ such that if $f : G_1 \to G_2$ satisfies $d(f(xy), f(x)f(y)) < \delta$, for each $x, y \in G_1$, then a homomorphism $T : G_1 \to G_2$ exists with $d(f(x), T(x)) < \epsilon$, for each $x, y \in G_1$? The first (partial) answer to it was published in 1941 by Hyers[5]. It reads as follows. Let E and Y be Banach spaces and $\epsilon > 0$. Then, for each $g : E \to Y$ with

$$\sup_{x,y\in E} \|g(x + y) - g(x) - g(y)\| \le \epsilon,$$

there is a unique solution $f : E \to Y$ of the Cauchy equation $f(x + y) = f(x) + f(y)$ such that $\sup_{x\in E} \|g(x) - f(x)\| \le \epsilon$. This result is called the Hyers-Ulam stability of the additive Cauchy equation.
For the last 50 years, that issue has been a very popular subject of investigations and we refer the reader to [1, 2, 6–8, 10] for further information, some discussions, and examples of recent results.

T. Miura, S. Miyajima and S. -E. Takahasi [9] introduced the notion of the Hyers-Ulam stability of a mapping between two normed linear spaces as follows:

Definition 1.1 ([9]). *Let X, Y be normed linear spaces and T be a (not necessarily linear) mapping from X into Y. We say that T has the Hyers-Ulam stability if there exists a constant $M > 0$ with the following property:
For any $g \in T(X)$, $\epsilon > 0$ and $f \in T(X)$ satisfying $\|Tf - g\| \le \epsilon$, we can find $f_0 \in T(X)$ such that $Tf_0 = g$ and $\|f - f_0\| \le M\epsilon$.*

We call M a HUS constant for T, and denote the infimum of all HUS constants for T by M_T. We refer the reader for the Hyers-Ulam stability of substitution operators on function spaces to [4, 9, 13, 14] and the

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references cited therein.

From now on, by an operator we will a non-zero linear operator. Let $\mathcal{B}$ be a Banach space and let T be an operator from $\mathcal{B}$ into itself. The linearity of T implies that T has the H-U stability if and only if there exists a constant M with the following property:

For any $\epsilon > 0$ and $f \in \mathcal{B}$ with $\|Tf\| \leq \epsilon$ there exists $f_0 \in \mathcal{B}$ such that $Tf_0 = 0$ and $\|f - f_0\| \leq M\epsilon$.

For a bounded operator $T : \mathcal{B} \to \mathcal{B}$, we denote the null space of T by $\mathcal{N}(T)$, the range of T by $\mathcal{R}(T)$ and the induced one-to-one operator $\tilde{T}$ from the quotient space $\mathcal{B}/\mathcal{N}(T) \to \mathcal{B}$ defined by $\tilde{T}(f + \mathcal{N}(T)) = Tf$, for all $f \in \mathcal{B}$. Clearly $\mathcal{R}(T) = \mathcal{R}(\tilde{T})$.
Takagi, Miura and Takahashi [13] investigated the relation of the Hyers-Ulam stability of T and the inverse operator $\tilde{T}^{-1}$ from $\mathcal{R}(T)$ into $\mathcal{B}/\mathcal{N}(T)$ in the following sense.

Theorem A ([13],Theorem2). For a bounded linear operator T on a Banach space, the following statements are equivalent:

1. T has the Hyers-Ulam stability.
2. T has closed range.
3. $\tilde{T}^{-1}$ is bounded.

Moreover, in this case $M_T = \|\tilde{T}^{-1}\|$.

The aim of this paper is to carry some of the results obtained for the linear operators on function spaces in [4, 9, 13, 14] to a substitution vector-valued integral operator on $L^1(X)$ space.

Firs of all, we introduce notations, definitions and preliminary facts that are used throughout the paper.

Let (X, Σ, μ) be a σ-finite measure space and $\varphi : X \to X$ be a non-singular measurable transformation; i.e. $\mu \circ \varphi^{-1} \ll \mu$. Here the non-singularity of φ guarantees that the operator $f \to f \circ \varphi$ is well defined as a mapping on $L^0(\Sigma)$ where $L^0(\Sigma)$ denotes the linear space of all equivalence classes of Σ-measurable functions on X. Let $h_0 = d\mu \circ \varphi^{-1}/d\mu$ be the Radon-Nikodym derivative. We also assume that h_0 is almost everywhere finite-valued, or equivalently $\varphi^{-1}(\Sigma) \subseteq \Sigma$ is a sub-σ-finite algebra see[12]. As usual, Σ is said to be φ-invariant if $\varphi(\Sigma) \subseteq \Sigma$, where $\varphi(\Sigma) = \{\varphi(A), A \in \Sigma\}$. The measure μ is said to be normal if $\mu(A) = 0$ implies that $\varphi(A) \in \Sigma$ and $\mu(\varphi(A)) = 0$. The support of a measurable function f is defined by $\sigma(f) = \{x \in X : f(x) \neq 0\}$. All comparisons between two functions or two sets are to be interpreted as holding up to a μ-null set. For a sub-σ-finite algebra $\mathcal{A} \subseteq \Sigma$, the conditional expectation operator associated with $\mathcal{A}$ is the mapping $f \to E^{\mathcal{A}}f$, defined for all non-negative f as well as for all $f \in L^p(\Sigma), 1 \leq p \leq \infty$, where $E^{\mathcal{A}}f$, by Radon-Nikodym Theorem, is the unique $\mathcal{A}$-measurable function satisfying

$$\int_A f d\mu = \int_A E^{\mathcal{A}} f d\mu, \quad \forall A \in \mathcal{A}.$$

We recall that $E^{\mathcal{A}} : L^2(\Sigma) \to L^2(\mathcal{A})$ is an orthogonal projection. For more details on the properties of $E^{\mathcal{A}}$ see [11]. Throughout this paper, we assume that $\mathcal{A} = \varphi^{-1}(\Sigma)$ and $E^{\varphi^{-1}(\Sigma)} = E$.

For a given complex Hilbert space $\mathcal{H}$, let $u : X \to \mathcal{H}$ be a mapping. We say that u is weakly measurable if for each $h \in \mathcal{H}$ the mapping $x \mapsto \langle u(x), h\rangle$ of X to C is measurable. We will denote this map by $\langle u, h\rangle$. Let $L^p(\Sigma)$ be the class of all measurable mappings $f : X \to \mathrm{C}$ such that $\|f\|_p^p = \int_X |f(x)|^p d\mu < \infty$ for $p \geq 1$.

Let $\varphi : X \to X$ be a non-singular measurable transformation and let $u : X \to \mathcal{H}$ be a weakly measurable function. Then the pair (u, φ) induces a substitution vector-valued integral operator $T_u^{\varphi} : L^p(\Sigma) \to \mathcal{H}$ defined by

$$\langle T_u^{\varphi} f, h\rangle = \int_X \langle u, h\rangle f \circ \varphi d\mu, \quad h \in \mathcal{H}, \ f \in L^p(\Sigma).$$

It is easy to see that T_u^{φ} is well defined and linear. Moreover for each $f \in L^p(\Sigma)$,

$$\sup_{h\in\mathcal{H}_1} |\langle T_u^{\varphi} f, h\rangle| \leq \sup_{h\in\mathcal{H}_1} \|T_u^{\varphi} f\|\|h\| = \|T_u^{\varphi} f\| = |\langle T_u^{\varphi} f, \frac{T_u^{\varphi} f}{\|T_u^{\varphi} f\|}\rangle| \leq \sup_{h\in\mathcal{H}_1} |\langle T_u^{\varphi} f, h\rangle|,$$

where $\mathcal{H}_1$ is the closed unit ball of $\mathcal{H}$. Hence $\|T_u^{\varphi} f\| = \sup_{h\in\mathcal{H}_1} |\langle T_u^{\varphi} f, h\rangle|$, for each $f \in L^p(\Sigma)$. Some fundemental properties of this operator on $L^2(\Sigma)$ space are studied by the author et al in [3].

Definition 1.2. *Let $u : X \to \mathcal{H}$ be a weakly measurable function. We say that $(u, \varphi, \mathcal{H})$ has absolute property, if for each $f \in L^p(X)$, there exists $h_f \in \mathcal{H}_1$ such that $\sup_{h\in\mathcal{H}_1} \int_X |\langle u, h\rangle||f \circ \varphi| d\mu = \int_X |\langle u, h_f\rangle||f \circ \varphi| d\mu$, and $\langle u, h_f\rangle = e^{i(-\arg f\circ\varphi+\theta_f)}|\langle u, h_f\rangle|$, for a constant θ_f.*

Proposition 1.3 ([3]). *Assume that $(u, \varphi, \mathcal{H})$ has the absolute property. Then*

$$\sup_{h\in\mathcal{H}_1} |\int_X \langle u, h\rangle f \circ \varphi d\mu| = \sup_{h\in\mathcal{H}_1} \int_X |\langle u, h\rangle||f \circ \varphi| d\mu.$$

Throughout of this paper we assume that $(u, \varphi, \mathcal{H})$ has the absolute property.

2. The main results

In this section, we determine the Hyers-Ulam stability the substitution vector-valued integral operator $T_u^{\varphi} : L^1(\Sigma) \to \mathcal{H}$, with the norm of the inverse of the one-to-one operator induced by this operator.

First, we present an auxiliary lemma which plays a key role in the sequel.

Lemma 2.1. *Let Σ be φ-invariant. If $T_u^{\varphi} : L^1(\Sigma) \to \mathcal{H}$ is a bounded substitution vector-valued integral operator. Then we have*

$$\|f + \mathcal{N}(T_u^{\varphi})\| = \int_{\varphi(\cup_{h\in\mathcal{H}_1} \sigma|\langle u,h\rangle|)} |f|.$$

Proof. Put $\mathcal{D} = \varphi(\cup_{h\in\mathcal{H}_1} \sigma|\langle u, h\rangle|)$ and $\mathcal{D}^c = X \setminus \mathcal{D}$. we can write

$$L^1(X, \Sigma, \mu) = L^1(\mathcal{D}, \Sigma_1, \mu) \oplus L^1(\mathcal{D}^c, \Sigma_2, \mu),$$

where $\Sigma_1 = \Sigma \cap \mathcal{D}$ and $\Sigma_2 = \Sigma \cap \mathcal{D}^c$. Moreover we have

$$\mathcal{N}(T_u^{\varphi}) = \{f \in L^1(\Sigma) : |f| = 0 \quad \text{on} \quad \mathcal{D}\} = L^1(\Sigma_2).$$

If T_u^{φ} is one-to-one, then $\mu(\mathcal{D}^c) = 0$ and hence there is nothing to prove. Choose $g \in \mathcal{N}(T_u^{\varphi})$ arbitrary. Thus for each $f \in L^1(\Sigma)$ we obtain

$$\int_{\mathcal{D}} |f| d\mu = \int_{\mathcal{D}} |f + g| d\mu \leq \int_X |f + g| d\mu = \|f + g\|.$$

Therefore, we deduce that $\int_{\mathcal{D}} |f| d\mu \leq \|f + \mathcal{N}(T_u^{\varphi})\|$. Now, put $p = -\chi_{\mathcal{D}^c} f$. It is easy to see that $p \in \mathcal{N}(T_u^{\varphi})$. Hence, we get that for all $f \in L^1(\Sigma)$,

$$\|f + \mathcal{N}(T_u^{\varphi})\| \leq \|f + p\| = \|f(1 - \chi_{\mathcal{D}^c})\| = \|f\chi_{\mathcal{D}}\| = \int_{\mathcal{D}} |f| d\mu.$$

Therefore the lemma is proved. □

In the following theorem we give necessary and sufficient conditions for $T_u^{\varphi} : L^1(\Sigma) \to \mathcal{H}$ to have the Hyers-Ulam stability.

Theorem 2.2. *Let T_u^{φ} be a bounded operator from $L^1(\Sigma)$ into $\mathcal{H}$. Also let Σ be φ-invariant. If μ is normal, then the following assertions are equivalent:*

(i) *T_u^{φ} has the Hyers-Ulam stability.*

(ii) *T_u^{φ} has closed range.*

(iii) *There exists $r > 0$ such that $\sup_{h\in\mathcal{H}_1} h_0E(|\langle u,h\rangle|)\circ\varphi^{-1} \geq r$ for μ-almost all $x \in \cup_{h\in\mathcal{H}_1}\sigma(J_h)$, where $J_h := h_0E(|\langle u,h\rangle|)\circ\varphi^{-1}$.*

(iv) *$\varphi(\cup_{h\in\mathcal{H}_1}\sigma|\langle u,h\rangle|) \subseteq \{x \in X; \sup_{h\in\mathcal{H}_1} h_0E(|\langle u,h\rangle|)\circ\varphi^{-1}(x) \geq r\}$, for some $r > 0$.*

(v) *There exists $M > 0$ such that $\|f + \mathcal{N}(T_u^{\varphi})\| \leq M\|T_u^{\varphi} f\|$, for each $f \in L^1(\Sigma)$.*

Proof. The implication (i) ⇒ (ii) is direct consequent of Theorem A.

(ii) ⇒ (iii) Assume T_u^{φ} has closed range. Then $T_u^{\varphi}|_{\sigma(J_h)}(L^1|_{J_h}(\Sigma))$ is closed in $\mathcal{H}$ for each $h \in \mathcal{H}_1$. Since $T_u^{\varphi}|_{\sigma(J_h)}$ is injective for each $h \in \mathcal{H}_1$ see([3],Theorem 2.12). Hence we can deduce that there exists a constant $d > 0$ such that $\|T_u^{\varphi}|_{\sigma(J_h)} f\| \geq d\|f\|$ for any $f \in L^1|_{\sigma(J_h)}(\Sigma)$ and for each $h \in \mathcal{H}_1$. Now, by the contrary assume that (iii) is not hold. Then for each $r > 0$ and for each $h \in \mathcal{H}_1$ we have $h_0E(|\langle u,h\rangle|)\circ\varphi^{-1} < r$ on $\cup_{h\in\mathcal{H}_1}\sigma(J_h)$. Put $f = \chi_B$ with $\mu(B) < \infty$ and $B \subseteq \cup_{h\in\mathcal{H}_1}\sigma(J_h)$. Therefore we have

$$d\|\chi_B\| \leq \|T_u^{\varphi}|_{\sigma(J_h)}\chi_B\| = \sup_{h\in\mathcal{H}_1}\int_X h_0E(|\langle u,h\rangle|)\circ\varphi^{-1}\chi_B d\mu < r\mu(B).$$

It is sufficient put $r = d$, but this is a contradiction. Hence we conclude that (iii) is hold.

(iii) ⇒ (iv) We have $\sup_{h\in\mathcal{H}_1} h_0E(|\langle u,h\rangle|)\circ\varphi^{-1} \geq r$ on $\cup_{h\in\mathcal{H}_1}\sigma(J_h)$ for some $r > 0$. It is enough to prove that $\varphi(\cup_{h\in\mathcal{H}_1}\sigma|\langle u,h\rangle|) \subseteq \cup_{h\in\mathcal{H}_1}\sigma(J_h)$. If T_u^{φ} is one-to-one, then $\cup_{h\in\mathcal{H}_1}\sigma(J_h) = X$ and hence there is nothing to prove. If $\varphi(\cup_{h\in\mathcal{H}_1}\sigma|\langle u,h\rangle|) \nsubseteq \cup_{h\in\mathcal{H}_1}\sigma(J_h)$, then we can choose $C \subseteq \cup_{h\in\mathcal{H}_1}\sigma|\langle u,h\rangle|$ such that $0 < \mu(\varphi(C)) < \infty$ with $\varphi(C) \cap (\cup_{h\in\mathcal{H}_1}\sigma(J_h)) = \emptyset$. For any $h \in \mathcal{H}_1$ we have

$$0 = \int_X \chi_{\varphi(C)} h_0E(|\langle u,h\rangle|)\circ\varphi^{-1} d\mu = \int_X \chi_{\varphi^{-1}(\varphi(C))}|\langle u,h\rangle| d\mu.$$

On the other hand we have

$$\begin{aligned}\mu(C) &= \mu(C \cap (\cup_{h\in\mathcal{H}_1}\sigma|\langle u,h\rangle|)) = \mu(\cup_{h\in\mathcal{H}_1}(C \cap \sigma|\langle u,h\rangle|)) \\ &\leq \mu(\cup_{h\in\mathcal{H}_1}(\varphi^{-1}(\varphi(C)) \cap \sigma|\langle u,h\rangle|)) \leq \sum_{h\in\mathcal{H}_1}\mu(\varphi^{-1}(\varphi(C)) \cap \sigma|\langle u,h\rangle|)\end{aligned}$$

Since μ is normal, we get that $\mu(\varphi(C)) = 0$. But this is a contradiction.

(iv) ⇒ (v) Put $A := \{x \in X : \sup_{h\in\mathcal{H}_1} h_0E(|\langle u,h\rangle|)\circ\varphi^{-1} \geq r\}$. Take ϵ arbitrary, then there exists $h_1 \in \mathcal{H}_1$ such that $B := \{x \in X : h_0E(|\langle u, h_1\rangle|) \circ \varphi^{-1} \geq r - \epsilon\}$ with $A \subseteq B$. By Lemma 2.1, we obtain that

$$\begin{aligned}\|f + \mathcal{N}(T_u^\varphi)\| &= \int_{\varphi(\cup_{h\in\mathcal{H}_1}\sigma|\langle u,h\rangle|)} |f| d\mu \leq \int_A |f| d\mu = \frac{1}{r-\epsilon}\int_A (r-\epsilon)|f| d\mu \\ &\leq \frac{1}{r-\epsilon}\int_B h_0E(|\langle u, h_1\rangle|) \circ \varphi^{-1}|f| d\mu \\ &\leq \frac{1}{r-\epsilon}\sup_{h\in\mathcal{H}_1}\int_X h_0E(|\langle u, h\rangle|) \circ \varphi^{-1}|f| d\mu \\ &= \frac{1}{r-\epsilon}\|T_u^\varphi f\|,\end{aligned}$$

for each $f \in L^1(X)$. Since ϵ was arbitrary, consequently there is a constant $M = \frac{1}{r}$.

(v) $\Rightarrow$ (i) It is trivial by using Theorem A and definition of the Hyers-Ulam stability. □

Theorem 2.3. *Under the same assumptions as in Theorem 2.2, if* $R = \sup\{r > 0 : \varphi(\cup_{h\in\mathcal{H}_1}\sigma|\langle u, h\rangle|) \subseteq \{x \in X; \sup_{h\in\mathcal{H}_1} h_0E(|\langle u, h\rangle|) \circ \varphi^{-1} \geq r\}$. *Then* $M_{T_u^\varphi} = 1/R$.

Proof. By theorem 2.2, if r is taken over all numbers satisfying

$$\varphi(\cup_{h\in\mathcal{H}_1}\sigma|\langle u, h\rangle|) \subseteq \{\sup_{h\in\mathcal{H}_1} h_0E(|\langle u, h\rangle|) \circ \varphi^{-1} \geq r\},$$

we obtain $M_{T_u^\varphi} = \|\widetilde{T_u^\varphi}^{-1}\| \leq 1/R$. For the opposite inequality, assume that $\|\widetilde{T_u^\varphi}^{-1}\| \leq 1/r$ and for each $h \in \mathcal{H}_1$, $\varphi(\cup_{h\in\mathcal{H}_1}\sigma|\langle u, h\rangle|) \nsubseteq \{J_h \geq r\}$ for some $r > 0$. Hence we can choose $A \subseteq \varphi(\cup_{h\in\mathcal{H}_1}\sigma|\langle u, h\rangle|)$, with $0 < \mu(A) < \infty$ such that $\sup_{h\in\mathcal{H}_1} h_0E(|\langle u, h\rangle|) \circ \varphi^{-1}|_A < r$. Put $f_0 = \frac{\chi_A}{\mu(A)}$. Then we get that

$$\|T_u^\varphi f_0\| = \sup_{h\in\mathcal{H}_1}\int_X h_0E(|\langle u, h\rangle|) \circ \varphi^{-1}\frac{\chi_A}{\mu(A)} d\mu = \sup_{h\in\mathcal{H}_1}\int_A h_0E(|\langle u, h\rangle|) \circ \varphi^{-1}\frac{1}{\mu(A)} d\mu < r.$$

Therefore

$$1 = \|f_0\| = \int_{\varphi(\cup_{h\in\mathcal{H}_1}\sigma|\langle u,h\rangle|)} |f_0| d\mu = \|f_0 + \mathcal{N}(T_u^\varphi)\| = \|\widetilde{T_u^\varphi}^{-1}(T_u^\varphi f_0)\| \leq \|\widetilde{T_u^\varphi}^{-1}\|\|T_u^\varphi f_0\| < 1.$$

Which is a contradiction. Hence we deduce that if $\|\widetilde{T_u^\varphi}^{-1}\| < 1/r$ then $\varphi(\cup_{h\in\mathcal{H}_1}\sigma|\langle u, h\rangle|) \subseteq \{\sup_{h\in\mathcal{H}_1} h_0E(|\langle u, h\rangle|) \circ \varphi^{-1} \geq r\}$. This follows that $1/R \leq \|\widetilde{T_u^\varphi}^{-1}\|$. □

Example 2.4. *Let* $X = (0, 1)$, Σ *be the Lebesgue subsets of* X *and let* μ *be the Lebesgue measure on* X. *Also let* $\varphi : X \to X$ *be defined by*

$$\varphi(x) = \begin{cases} 2x & 0 < x < \frac{1}{2}, \\ 2 - 2x & \frac{1}{2} \leq x < 1. \end{cases}$$

Direct computation shows that $h_0(x) = 1$. *Define* $u : X \to \mathcal{R}$ *by* $u(x) = x + 1$. *Then for each* $h \in \mathcal{H}_1$, *we have*

$$\begin{aligned}\int_X h_0 E|\langle u(x), h\rangle| \circ \varphi^{-1} d\mu &= \int_{(0,\frac{1}{2})} E|\langle u(x), h\rangle| \circ \varphi^{-1} d\mu \circ \varphi^{-1} \\ &+ \int_{(\frac{1}{2},1)} E|\langle u(x), h\rangle| \circ \varphi^{-1} d\mu \circ \varphi^{-1} \\ &= \int_{\varphi^{-1}(0,\frac{1}{2})} E|\langle u(x), h\rangle| d\mu + \int_{\varphi^{-1}(\frac{1}{2},1)} E|\langle u(x), h\rangle| d\mu \\ &= \int_{\varphi^{-1}(0,\frac{1}{2})} |\langle u(x), h\rangle| d\mu + \int_{\varphi^{-1}(\frac{1}{2},1)} |\langle u(x), h\rangle| d\mu \\ &= \frac{1}{2}\int_{(0,1)} |\langle u(\frac{x}{2}), h\rangle| dx + \frac{1}{2}\int_{(0,1)} |\langle u(1-\frac{x}{2}), h\rangle| dx\end{aligned}$$

Hence for each $h \in \mathcal{H}_1$ *we get that*

$$h_0 E|\langle u(x), h\rangle| \circ \varphi^{-1} = \frac{1}{2}\left(|\langle u(\frac{x}{2}), h\rangle| + |\langle u(1-\frac{x}{2}), h\rangle|\right).$$

Therefore

$$h_0 E|\langle u(x), h\rangle| \circ \varphi^{-1} = \frac{1}{2}\left(|\frac{xh}{2} + h| + |2h - \frac{xh}{2}|\right) \geq \frac{1}{2}|3h|.$$

This implies that $\sup_{h\in\mathcal{R}_1} h_0 E|\langle u(x), h\rangle| \circ \varphi^{-1} \geq \frac{3}{2}$, *where* $\mathcal{R}_1$ *is the closed unit ball of* $\mathcal{R}$. *It is sufficient put* $r = \frac{3}{2} - \epsilon$, *for* ϵ *arbitrary. Then, by Theorem 2.2 we deduce that* T_u^{φ} *on* $L^1(\Sigma)$ *has the Hyers-Ulam stability.*

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