Filomat 32:18 (2018), 6375–6386
https://doi.org/10.2298/FIL1818375T

Published by Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Hyperconnectedness Modulo an Ideal

B.K. Tyagi[a], Manoj Bhardwaj[b], Sumit Singh[b]

[a]*Department of Mathematics, Atmaram Sanatan Dharma College, University of Delhi, New Delhi-110021, India*
[b]*Department of Mathematics, University of Delhi, New Delhi-110007, India*

Abstract. In this paper, hyperconnectedness with respect to an ideal, called hyperconnectedness modulo an ideal, of a topological space X is introduced. It is shown that hyperconnectedness and hyperconnectedness modulo an ideal coincide in case of trivial and codense ideal. Several characterisations of hyperconnectedness modulo an ideal I are obtained using semi-open, pre-open and semi-preopen sets. It is also shown that I-hyperconnectedness and hyperconnectedness modulo I, where I is an ideal in a space X, are equivalent. A new type of semi-open modulo ideal sets are defined and several characterisations of hyperconnectedness modulo an ideal using these sets are obtained.

1. Introduction

A topological space X is said to be hyperconnected [22] if every pair of nonempty open sets of X has nonempty intersection. Several notions which are equivalent to hyperconnectedness were defined and investigated in the literature. Levine [13] called a topological space X a D-space if every nonempty open set of X is dense in X and showed that X is a D-space if and only if it is hyperconnected. Pipitone and Russo [20] defined a topological space X to be semi-connected if X is not the union of two disjoint nonempty semi-open sets of X and showed that X is semi-connected if and only if it is a D-space. Maheshwari and Tapi [14] defined a topological space X to be s-connected if X is not the union of two nonempty semi-separated sets and showed the equivalence of s-connectedness and semi-connectedness. Hyperconnected spaces are also called irreducible in [23]. Recently, Ajmal and Kohli [2] have investigated further properties of hyperconnected spaces.

A nonempty collection I of subsets of X is called an ideal in X if it has the following properties: (i) If $A \in I$ and $B \subseteq A$, then $B \in I$ (hereditary) (ii) If $A \in I$ and $B \in I$, then $A \cup B \in I$ (finite additivity). According to Rose and Hamlet [21] (X, τ, I) denotes a set X with a topology τ and an ideal I on X. For a subset $A \subseteq X, A^{\star}(I) = \{x \in X : U \cap A \notin I$, for all open sets U containing $x\}$ is called the local function of A with respect to I and τ [6]. In 1999 J. Dontchev et a1.[6] called a subset A of a space (X, τ, I) to be I-dense if every point of X is in the local function of A with respect I and τ, that is, if $A^{\star}(I) = X$. An ideal I is codense if $I \cap \tau = \{\emptyset\}$

2010 *Mathematics Subject Classification*. Primary 54A05; Secondary 54A10
Keywords. Hyperconnectedness, hyperconnectedness modulo an ideal
Received: 24 May 2018; Revised: 10 October 2018; Accepted: 23 October 2018
Communicated by Ljubiša D.R. Kočinac
Corresponding author: Manoj Bhardwaj
The second and third author acknowledge the fellowship grant of University Grant Commission, India
Email addresses: brijkishore.tyagi@gmail.com (B.K. Tyagi), manojmnj27@gmail.com (Manoj Bhardwaj), sumitkumar405@gmail.com (Sumit Singh)

This paper is organized as follows. In Section 2, the notion of hyperconnected modulo an ideal is defined and the basic properties are developed. In Section 3, the characterizations of hyperconnectedness modulo an ideal are obtained and its relationship with other weaker and stronger forms of hyperconnectedness is investigated. Section 4 contains the properties of hyperconnected sets modulo an ideal. In Section 5, we introduced the notion of semi-open modulo an ideal set in a topological space and studied the behaviour of hyperconnected modulo an ideal spaces.

2. Preliminaries

Throughout the present paper, (X, τ) (or simply X) will denote a topological space on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a topological space X. The closure of A and the interior of A are denoted by $Cl(A)$ and $Int(A)$, respectively. A subset A is said to be semi-open [12] (resp. α-open [17], preopen [15], β-open [1]) if $A \subseteq Cl(Int(A))$ (resp. $A \subseteq Int(Cl(Int(A))), A \subseteq Int(Cl(A)), A \subseteq Cl(Int(Cl(A)))$). Andrijević [3] defined a subset A to be semi-preopen if there exists a preopen set V in X such that $V \subseteq A \subseteq Cl(V)$ and showed the equivalence of β-openness and semi-preopenness. The complement of a semi-open (resp. preopen, semi-preopen) set is said to be semi-closed (resp. preclosed, semipreclosed). The semi-closure [5] (resp. preclosure [7], semi-preclosure [3]) of A, denoted by $sCl(A)$ (resp. $pCl(A), spCl(A)$), is defined by the intersection of all semi-closed (resp. preclosed, semi-preclosed) sets of X containing A. The union of all semi-open sets contained in A is called the semi-interior of A and is denoted by $sInt(A)$. The family of all semi-open (resp. preopen, semi-preopen, regular open, regular closed) sets of X is denoted by $SO(X)$ (resp. $PO(X), SPO(X), RO(X), RC(X)$).

Definition 2.1. Let X be a space and I be an ideal in X. Then X is called hyperconnected modulo I if intersection of every two nonempty open sets is not in I.

Theorem 2.2. *Let X be a space and I be an ideal in X. Then the following statements are equivalent:*

1. *The intersection of two nonempty open sets is not in I.*
2. *There are no proper closed sets G and H such that $X \setminus (G \cup H) \in I$.*

Proof. (1) implies (2). Suppose that there are proper closed G and H such that $X \setminus (G \cup H) \in I$. If H empty, then $X \setminus G \in I$. Since $X \setminus G$ and X are nonempty open sets with $X \cap (X \setminus G) = (X \setminus G) \in I$, a contradiction. Hence, G and H both are nonempty proper closed sets. Then $X \setminus G$ and $X \setminus H$ are nonempty open sets. By (i), $(X \setminus G) \cap (X \setminus H) \notin I$ implies that $X \setminus (G \cup H) \notin I$, a contradiction.

(2) implies (1). Let A and B be any nonempty open sets in X. Then $X \setminus A$ and $X \setminus B$ are proper closed sets in X, $X \setminus [(X \setminus A) \cup (X \setminus B)] \notin I$ implies that $X \setminus [X \setminus (A \cap B)] \notin I$. Thus, $(A \cap B) \notin I$. □

Theorem 2.3. *Let X be a space and I be an ideal in X. Then the following statements are equivalent:*

1. *The interior of every proper closed set is in I.*
2. *For every nonempty open set V in X, $X \setminus Cl(V) \in I$.*

Proof. (1) implies (2). Let V be any nonempty proper open set in X. Since $X \setminus V$ is a proper closed set, $Int(X \setminus V) \in I$ implies that $(X \setminus Cl(V)) \in I$.

(2) implies (1). Let A be any nonempty proper closed set in X. Since $X \setminus A$ is a nonempty proper open set in X, by (ii), $[X \setminus Cl(X \setminus A)] \in I$. Thus, $Int(A) \in I$. □

Theorem 2.4. *Let X be a space and I be an ideal in X and there are no proper closed sets G and H such that $X \setminus (G \cup H) \in I$. Then the interior of every proper closed set is in I.*

Proof. Let A be a nonempty proper closed set in X. Suppose that $Int(A) \notin I$. Then $X \setminus Int(A)$ is nonempty proper closed set in X. Then $[X \setminus (A \cup (X \setminus Int(A)))] = \emptyset \in I$, a contradiction. □

Theorem 2.5. *Let X be a space and I be an ideal in X. If X is hyperconnected modulo I, then*

1. *The intersection of two nonempty open sets is not in I.*
2. *There are no proper closed sets G and H such that $X \setminus (G \cup H) \in I$.*
3. *The interior of every proper closed set is in I.*
4. *For every nonempty open set V in X, $X \setminus Cl(V) \in I$.*

Theorem 2.6. *Let X be a space and I be an ideal in X and for $Int(A) \in I$ and each $J \in I$, $Int(A \cup J) \in I$. Then the following statements are equivalent:*

1. *The topological space X is hyperconnected modulo I.*
2. *There are no proper closed sets G and H such that $X \setminus (G \cup H) \in I$.*
3. *The interior of every proper closed set is in I.*
4. *For every nonempty open set V in X, $X \setminus Cl(V) \in I$.*

3. Characterizations of Hyperconnectedess Modulo *I*

In case of trivial or codense ideal, the two notions, hyperconnectedness modulo an ideal and hyperconnectedness concide.

Theorem 3.1. *Let X be a space and I be a codense ideal in X. Then X is hyperconnected modulo I if and only if X is hyperconnected.*

Proof. Let (X, τ) be hyperconnected modulo I. Then by the definition (X, τ) is hyperconnected.
Conversely, let (X, τ) be hyperconnected and $A, B \in \tau \setminus \{\emptyset\}$. Then $A \cap B \in \tau \setminus \{\emptyset\}$. Since I is codense, $A \cap B \notin I$. Thus, (X, τ) is hyperconnected modulo I. □

Recall that a space X is called I-hyperconnected [6] if every nonempty open set is I-dense in X.

Theorem 3.2. ([6]) *Let X be a space and I be an ideal in X. Then X is I-hyperconnected if and only if X is hyperconnected and I is codense.*

Theorem 3.3. *Let X be a space and I be an ideal in X. Then X is I-hyperconnected if and only if X is hyperconnected modulo I.*

Proof. The proof follows from Theorem 3.1 and Theorem 3.2. □

Theorem 3.4. *Let X be a space and I be an ideal in X. If X is hyperconnected modulo I, then I is codense.*

Proof. The proof follows from Theorem 3.2 and Theorem 3.3. □

Theorem 3.5. ([9]) *Let (X, τ, I) be a space, where I is codense. Then a set D is I-dense if and only if $(U - A) \cap D \neq \emptyset$, for all $\emptyset \neq U \in \tau$ and $A \in I$.*

Theorem 3.6. *Let (X, τ) be a space and I be codense in X. Then X is hyperconnected modulo I if and only if $(U - A) \cap D \neq \emptyset$, for all $\emptyset \neq U \in \tau, A \in I$ and $D \in \tau$.*

Proof. The proof follows from Theorem 3.2 and Theorem 3.5. □

Theorem 3.7. *Let X be a space and I be an ideal in X. If X is hyperconnected modulo I, then X is pseudocompact.*

Proof. The proof follows from the fact that hyperconnectedness of X implies pseudocompactness of X. □

Theorem 3.8. *Let X be a space and I be an ideal in X. If X is hyperconnected modulo I, then X is locally connected.*

Proof. The proof follows from the fact that hyperconnectedness of X implies local connectedness of X. □

Theorem 3.9. *Let (X, τ) be a space and I be an ideal in X. Then (X, τ) is hyperconnected modulo I if and only if for each subsets E and F of X, $X \setminus Cl(E)$ and $X \setminus Cl(F) \notin I$ implies $X \setminus Cl(E \cup F) \notin I$ and I is codense.*

Proof. Let (X, τ) be hyperconnected modulo I and E and F be subsets of X such that $X \setminus Cl(E), X \setminus Cl(F) \notin I$. Then $X \setminus Cl(E), X \setminus Cl(F) \in \tau \setminus \{\emptyset\}$. Since X is hyperconnected modulo I, $X \setminus Cl(E \cup F) = X \setminus Cl(E) \cup Cl(F) = (X \setminus Cl(E)) \cap (X \setminus Cl(F)) \notin I$.

Conversely, let $A, B \in \tau \setminus \{\emptyset\}$. Since I is codense, $A = X \setminus Cl(X \setminus A) \notin I$ and $B = X \setminus Cl(X \setminus B) \notin I$. Then $X \setminus Cl((X \setminus A) \cup (X \setminus B)) \notin I \Rightarrow A \cap B \notin I$. Thus, (X, τ) is hyperconnected modulo I. □

Theorem 3.10. ([3]) *Let X be a space. Then*

1. *$sCl(W) = W \cup Int(Cl(W))$, for every nonempty set $W \in SPO(X)$.*
2. *$pCl(W) = W \cup Cl(Int(W))$, for every nonempty set $W \in SO(X)$.*
3. *$spCl(W) = W \cup Int(Cl(Int(W)))$, for every nonempty set $W \in SO(X)$.*

Theorem 3.11. ([19]) *The following are equivalent for a topological space X:*

1. *X is hyperconnected.*
2. *$X = Cl(W)$, for every nonempty set $W \in SPO(X)$.*
3. *$X = sCl(W)$, for every nonempty set $W \in SPO(X)$.*
4. *$X = pCl(W)$, for every nonempty set $W \in SO(X)$.*
5. *$X = spCl(W)$, for every nonempty set $W \in SO(X)$.*

Theorem 3.12. ([18]) *A space X is hyperconnected if and only if $sCl(U) = X$, for every nonempty set $U \in SO(X)$.*

Theorem 3.13. *Let X be a space and I be a codense ideal in X. Then:*

1. *$X \setminus Cl(W) \in I$ if and only if $X = Cl(W)$, for every nonempty set W.*
2. *$X \setminus sCl(W) \in I$ if and only if $X = sCl(W)$, for every nonempty set $W \in SPO(X)$.*
3. *$X \setminus pCl(W) \in I$ if and only if $X = pCl(W)$, for every nonempty set $W \in SO(X)$.*
4. *$X \setminus spCl(W) \in I$ if and only if $X = spCl(W)$, for every nonempty set $W \in SO(X)$.*
5. *$X \setminus sCl(W) \in I$ if and only if $X = sCl(W)$, for every nonempty set $W \subseteq X$.*

Proof. 1. Since I is codense, $X \setminus Cl(W) \in I \Leftrightarrow X \setminus Cl(W) = \emptyset \Leftrightarrow X = Cl(W)$.

2. Let $X \setminus sCl(W) \in I$ and $W \in SPO(X)$. By Theorem 3.10, $sCl(W) = W \cup Int(Cl(W))$. Then $W \subseteq Cl(Int(Cl(W))) \Rightarrow W \cup Int(Cl(W)) \subseteq Cl(Int(Cl(W))) \subseteq Cl(W) \Rightarrow X \setminus Cl(W) \subseteq X \setminus (W \cup Int(Cl(W))) = X \setminus sCl(W) \in I$. Since I is codense, $X \setminus Cl(W) = \emptyset \Rightarrow X = Cl(W)$. Thus, $sCl(W) = W \cup Int(Cl(W)) = X$. The converse is obvious.

3. Let $X \setminus pCl(W) \in I$ and $W \in SO(X)$. By Theorem 3.10, $pCl(W) = W \cup Cl(Int(W))$. Then $W \subseteq Cl(Int(W)) \Rightarrow W \cup Cl(Int(W)) = Cl(Int(W)) \Rightarrow X \setminus Cl(Int(W)) = X \setminus pCl(W) \in I$. Since I is codense, $X \setminus Cl(Int(W)) = \emptyset \Rightarrow X = Cl(Int(W))$. Thus, $pCl(W) = W \cup Cl(Int(W)) = X$. The converse is obvious.

4. Let $X \setminus spCl(W) \in I$ and $W \in SO(X)$. By Theorem 3.10, $spCl(W) = W \cup Int(Cl(Int(W)))$. Then $W \subseteq Cl(Int(W)) \Rightarrow W \cup Int(Cl(Int(W))) \subseteq Cl(Int(W)) \Rightarrow X \setminus Cl(Int(W)) \subseteq X \setminus (W \cup Int(Cl(Int(W))) = X \setminus spCl(W) \in I$. Since I is codense, $X \setminus Cl(Int(W)) = \emptyset \Rightarrow X = Cl(Int(W)) = Int(Cl(Int(W)))$. Thus, $spCl(W) = W \cup Int(Cl(Int(W))) = X$. The converse is obvious.

5. Let $X \setminus sCl(W) \in I$ and $W \in SO(X)$. Then $sCl(W) \subseteq Cl(W) \Rightarrow X \setminus Cl(W) \subseteq X \setminus sCl(W) \in I$. Since I is codense, $X \setminus Cl(W) = \emptyset \Rightarrow X = Cl(W) = Int(Cl(W))$. Thus, $sCl(W) = W \cup Int(Cl(W)) = X$. The converse is obvious. □

Theorem 3.14. *Let X be a space and I be an ideal in X. Then X is hyperconnected modulo I if and only if*

1. *$X \setminus Cl(W) \in I$, for every $W \in SPO(X)$ and I is codense.*
2. *$X \setminus sCl(W) \in I$, for every $W \in SPO(X)$ and I is codense.*
3. *$X \setminus pCl(W) \in I$, for every $W \in SO(X)$ and I is codense.*
4. *$X \setminus spCl(W) \in I$, for every $W \in SO(X)$ and I is codense.*

5. *$X \setminus sCl(W) \in I$, for every $W \in SO(X)$ and I is codense.*
6. *$U \cap V \notin I$, for every nonempty $U, V \in SO(X)$.*
7. *$U \cap W \notin I$, for every nonempty $U \in SO(X)$ and $W \in \alpha O(X)$.*

Proof. 1. Let X be hyperconnected modulo I. By Theorem 3.4, X is hyperconnected and I is codense. By Theorem 3.11, $X = Cl(W)$ for every nonempty $W \in SPO(X)$. Then $X \setminus Cl(W) \in I$.

Conversely, let $X \setminus Cl(W) \in I$ for every nonempty $W \in SPO(X)$ and I be a codense. Then $X = Cl(W)$. By Theorem 3.11, X is hyperconnected. Since I is codense, by Theorem 3.1, X is hyperconnected modulo I.

2. Let X be hyperconnected modulo I. By Theorem 3.4, X is hyperconnected and I is codense. By Theorem 3.11, $X = sCl(W)$ for every nonempty $W \in SPO(X)$. Then $X \setminus sCl(W) \in I$.

Conversely, let $X \setminus sCl(W) \in I$ for every nonempty $W \in SPO(X)$ and I be codense ideal in X. By Theorem 3.13, $X = sCl(W)$. By Theorem 3.11, X is hyperconnected. Since I is codense, by Theorem 3.1, X is hyperconnected modulo I.

3. Let X be hyperconnected modulo I. By Theorem 3.4, X is hyperconnected and I is codense. By Theorem 3.11, $X = pCl(W)$ for every nonempty $W \in SO(X)$. Then $X \setminus pCl(W) \in I$.

Conversely, let $X \setminus pCl(W) \in I$ for every nonempty $W \in SO(X)$ and I be a codense ideal in X. By Theorem 3.13, $X = pCl(W)$. By Theorem 3.11, X is hyperconnected. Since I is codense, by Theorem 3.1, X is hyperconnected modulo I.

4. Let X be hyperconnected modulo I. By Theorem 3.4, X is hyperconnected and I is codense. By Theorem 3.11, $X = spCl(W)$ for every nonempty $W \in SO(X)$. Then $X \setminus spCl(W) \in I$.

Conversely, let $X \setminus spCl(W) \in I$ for every nonempty $W \in SO(X)$ and I be codense ideal in X. By Theorem 3.13, $X = spCl(W)$. By Theorem 3.11, X is hyperconnected. Since I is codense, by Theorem 3.1, X is hyperconnected modulo I.

5. Let X be hyperconnected modulo I. By Theorem 3.4, X is hyperconnected and I is codense. By Theorem 3.11, $X = sCl(W)$ for every nonempty $W \in SO(X)$. Then $X \setminus sCl(W) \in I$.

Conversely, let $X \setminus sCl(W) \in I$ for every nonempty $W \in SO(X)$ and I be a codense ideal in X. By Theorem 3.13, $X = sCl(W)$. By Theorem 3.12, X is hyperconnected. Since I is codense, by Theorem 3.1, X is hyperconnected modulo I.

6. Let X be hyperconnected modulo I and $U, V \in SO(X)$. There are $A, B \in \tau \setminus \{\emptyset\}$ such that $A \subseteq U$ and $B \subseteq V$. Since X is hyperconnected modulo I, $A \cap B \notin I \Rightarrow U \cap V \notin I$.

Conversely, let $A, B \in \tau \setminus \{\emptyset\}$. Then $A, B \in SO(X) \Rightarrow A \cap B \notin I$. Thus, X is hyperconnected modulo I.

7. Let X be hyperconnected modulo I, $U \in SO(X)$ and $V \in \alpha O(X)$ be non-empty sets. Then $Int(U) \neq \emptyset \neq Int(V)$. Since X is hyperconnected modulo I, $Int(U) \cap Int(V) \notin I$ so that $U \cap V \notin I$.

Conversely, let $A, B \in \tau \setminus \{\emptyset\}$. Then $A \in SO(X)$ and $B \in \alpha O(X)$, so that $A \cap B \notin I$. Thus, X is hyperconnected modulo I. □

Theorem 3.15. ([8]) *A space is hyperconnected if and only if the collection of not dense sets and the nowhere dense sets are equal.*

Theorem 3.16. *Let X be a space and I be a codense ideal in X. Then X is hyperconnected modulo I if and only if the collection of not dense sets and the nowhere dense sets are equal.*

Proof. The proof follows from Theorem 3.15. □

4. Properties of Hyperconnectedess Modulo *I* Spaces

It is known that any continuous image of a hyperconnected space is hyperconnected.

Theorem 4.1. *Let X be a space and I be an ideal in X. If X is hyperconnected modulo I, then every continuous mapping from X to a Hausdorff space is constant.*

Proof. The proof follows from the fact that hyperconnectedness modulo an ideal implies hyperconnectedness. □

Let X and Y be sets and let $f : X \to Y$ be a mapping. For an ideal I in Y we denote $f^{-1}(I) = \{A \subseteq X : f(A) \in I\}$ which is an ideal in X.

Theorem 4.2. *Let X and Y be spaces and let $f : X \to Y$ be a continuous surjection. Let I be an ideal in Y. Then, if X is hyperconnected modulo $f^{-1}(I)$, then Y is hyperconnected modulo I.*

Proof. Suppose that Y is not hyperconnected modulo I. Then there are nonempty open sets A and B such that $(A \cap B) \in I$. Then $f^{-1}(A)$ and $f^{-1}(B)$ are nonempty open sets in X with $f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B) \in f^{-1}(I)$, a contradiction. □

Recall that for a space X and an ideal I in X, X is called connected modulo I [11] if there is no continuous mapping $f : X \to [0, 1]$ such that

- $f^{-1}(0)$ and $f^{-1}(1)$ are neither in I
- $X \setminus (f^{-1}(0) \cup f^{-1}(1))$ is in I.

Let X be a space and I be an ideal in X. A mapping $f : X \to [0, 1]$ is called 2-valued modulo I [11] if

- $f^{-1}(0)$ and $f^{-1}(1)$ are neither in I
- $X \setminus (f^{-1}(0) \cup f^{-1}(1))$ is in I.

Theorem 4.3. *If X is hyperconnected modulo I, then X is connected modulo I.*

Proof. Suppose X is not connected modulo I. Then there is a continuous mapping $f : X \to [0, 1]$ which is 2-valued modulo I. The sets $f^{-1}(0)$ and $f^{-1}(1)$ are neither in I hence are nonempty. Since f is continuous, $f^{-1}(0)$ and $f^{-1}(1)$ are closed sets in X. As f is 2-valued modulo I, $(X \setminus (f^{-1}(0) \cup f^{-1}(1))) \in I$, a contradiction. □

Next theorem deals with Theorem 2.7 [11] in case of hyperconnected modulo I.

Theorem 4.4. *Let X be a space and let I be an ideal in X. Suppose that X is hyperconnected modulo I. Then, there is a maximal (with respect to set-theoretic inclusion $\subseteq$) ideal M in X which contains I and X is hyperconnected modulo M.*

Proof. The proof is similar to the proof of Theorem 2.7 [11]. □

It is well known that a continuous mapping from a hyperconnected space to a Hausdorff space is constant. Here it is generlized for hyperconnected modulo an ideal.

Theorem 4.5. *Let X be a space and I be an ideal in X. If X is hyperconnected modulo I, then there is no continuous mapping f from X to a T_1-space Y with at least two points $y_1, y_2 \in Y$ such that*

1. *$f^{-1}(y_1)$ and $f^{-1}(y_2)$ are neither in I.*
2. *$(X \setminus (f^{-1}(y_1) \cup f^{-1}(y_2)) \in I$.*

Proof. Suppose that there is a continuous mapping f from X to a T_1-space Y with at least two points such that (i) and (ii) hold. Since Y is T_1-space, $\{y_1\}$ and $\{y_2\}$ are closed sets in Y. Then $f^{-1}(y_1)$ and $f^{-1}(y_2)$ are nonempty proper closed sets in X and $(X \setminus (f^{-1}(y_1) \cup f^{-1}(y_2)) \in I$, a contradiction. □

Corollary 4.6. *Let X be a space and I be an ideal in X. If X is hyperconnected modulo I, then there is no continuous mapping f from X to a Hausdorff space Y with at least two points $y_1, y_2 \in Y$ such that*

1. *$f^{-1}(y_1)$ and $f^{-1}(y_2)$ are neither in I.*
2. *$(X \setminus (f^{-1}(y_1) \cup f^{-1}(y_2)) \in I$.*

Theorem 4.7. *Let X be a space and I be an ideal in X. If X is hyperconnected modulo I, then X cannot be Hausdorff unless it contains only one point.*

Proof. Let X be a Hausdorff space with more than one point. Let x_1 and x_2 are two distinct points in X. There are disjoint open sets U and V in X containing points x_1 and x_2, respectively. Then $U \cap V = \emptyset \in I$, a contradiction. □

Theorem 4.8. *Let Y be a subspace of a topological space X and I be an ideal in X. Then $I_Y = \{A \cap Y : A \in I\}$ is an ideal in Y.*

Proof. Let G and H be in I_Y. Then $G = A \cap Y$ and $H = B \cap Y$, for some $A, B \in I$. Since $(A \cup B) \in I$, $(G \cup H) = (A \cap Y) \cup (B \cap Y) = (A \cup B) \cap Y \in I_Y$. Now let $L \in I_Y$ and $M \subseteq L$. Then $L = N \cap Y$ for some $N \in I$. Since $M \subseteq L \subseteq N \in I$ and $(M \cap N) \in I$, $M = (M \cap N) \cap Y$ implies that $M \in I_Y$. □

It is known that every open subspace of a hyperconnected space is hyperconnected. This fact is generalized for hyperconnectedness modulo an ideal.

Theorem 4.9. *Let Y be a subspace of a topological space X with nonempty interior and I be an ideal in X. If X is hyperconnected modulo I, then Y is hyperconnected modulo I_Y.*

Proof. Suppose that Y is not hyperconnected modulo I_Y. Then there are nonempty open sets A and B in Y such that $(A \cap B) \in I_Y$. Then $A = C \cap Y$ and $B = D \cap Y$, for some open sets C and D in X. Now $(C \cap D) \cap Y = (A \cap B) \in I_Y$ implies that $(C \cap D) \cap Y = (M \cap Y)$ for some $M \in I$. Therefore, $(C \cap D) \cap Int_X(Y) \subseteq (C \cap D) \cap Y \subseteq M \in I$, a contradiction. □

Corollary 4.10. *Let Y be a subspace of a topological space X with Y α-open (semi-open) in X and I be an ideal in X. If X is hyperconnected modulo I,then Y is hyperconnected modulo I_Y.*

Theorem 4.11. *Let Y be a subspace of a topological space X and I be an ideal in X. Then*

1. *$I_{Int_X(Y)} = \{A \cap Int_X(Y) : A \in I\}$ is an ideal in Y.*
2. *$I_{Cl_X(Y)} = \{A \cap Cl_X(Y) : A \in I\}$ is an ideal in X.*

Proof. 1. Let G and H be in $I_{Int_X(Y)}$. Then $G = A \cap Int_X(Y)$ and $H = B \cap Int_X(Y)$, for some $A, B \in I$. Since $(A \cup B) \in I$, $(G \cup H) = (A \cap Int_X(Y)) \cup (B \cap Int_X(Y)) = (A \cup B) \cap Int_X(Y) \in I$. Now let $L \in I_{Int_X(Y)}$ and $M \subseteq L$. Then $L = N \cap Int_X(Y)$ for some $N \in I$. Since $M \subseteq L \subseteq N \in I$ and $(M \cap N) \in I$, $M = (M \cap N) \cap Int_X(Y)$. Thus, $M \in I_{Int_X(Y)}$.

2. Let G and H be in $I_{Cl_X(Y)}$. Then $G = A \cap Cl_X(Y)$ and $H = B \cap Cl_X(Y)$, for some $A, B \in I$. Since $(A \cup B) \in I$, $(G \cup H) = (A \cap Cl_X(Y)) \cup (B \cap Cl_X(Y)) = (A \cup B) \cap Cl_X(Y) \in I$. Now let $L \in I_{Cl_X(Y)}$ and $M \subseteq L$. Then $L = N \cap Cl_X(Y)$ for some $N \in I$. Since $M \subseteq L \subseteq N \in I$ and $(M \cap N) \in I$, $M = (M \cap N) \cap Cl_X(Y)$. Thus, $M \in I_{Cl_X(Y)}$. □

Theorem 4.12. *Let Y be a subspace of a topological space X with nonempty interior and I be an ideal in X. If X is hyperconnected modulo I, then Y is hyperconnected modulo $I_{Int_X(Y)}$.*

Proof. Suppose that Y is not hyperconnected modulo $I_{Int_X(Y)}$. Then there are nonempty open sets A and B in Y such that $(A \cap B) \in I_{Int_X(Y)}$. Then $A = C \cap Y$ and $B = D \cap Y$ for some open sets C and D in X. Now $(C \cap D) \cap Int_X(Y) \subseteq (C \cap D) \cap Y = (A \cap B) \in I_{Int_X(Y)}$ implies that $(C \cap D) \cap Int_X(Y) = M \cap Int_X(Y)$ for some $M \in I$. Therefore, $(C \cap D) \cap Int_X(Y) \subseteq M \in I$, a contradiction. □

Corollary 4.13. *Let Y be a subspace of a topological space X with Y pre-open (β-open) in X and I be an ideal in X. If X is hyperconnected modulo I, then Y is hyperconnected modulo $I_{Cl_X(Y)}$.*

Theorem 4.14. *Let Y and Z be subspaces of a topological space X and I be an ideal in X. Then $I_Y \cup I_Z \subseteq I_{Y \cup Z}$.*

Proof. Let $A \in I_Y \cup I_Z$. Then $A = (J \cap Y)$ or $A = (J \cap Z)$. So $A \subseteq J \cap (Y \cup Z) \in I_{Y \cup Z}$, for some $J \in I$. □

Theorem 4.15. *Let Y and Z be subspaces of a topological space X with at least one having nonempty interior and I be an ideal in X. If X is hyperconnected modulo I, then the subspace $Y \cup Z$ is hyperconnected modulo $I_{Y \cup Z}$.*

Proof. Suppose that $Y \cup Z$ is not hyperconnected modulo $I_{Y\cup Z}$. Then there are nonempty open sets A and B in $Y \cup Z$ such that $(A \cap B) \in I_{Y\cup Z}$. Then $A = C \cap (Y \cup Z)$ and $B = D \cap (Y \cup Z)$ for some open sets C and D in X. Thus, $A \cap B = (C \cap D) \cap (Y \cup Z) \in I_{Y\cup Z}$ implies that $(C \cap D) \cap (Y \cup Z) = J \cap (Y \cup Z)$ for some $J \in I$. Now $(C \cap D) \cap (Y \cup Z) \subseteq J \in I$ implies that $(C \cap D) \cap (Int(Y) \cup Int(Z)) \in I$, a contradiction. □

Theorem 4.16. *Let Y and Z be subspaces of a topological space X with intersecting interiors and I be an ideal in X. If X is hyperconnected modulo I, then the subspace $Y \cap Z$ is hyperconnected modulo $I_{Y\cap Z}$.*

Proof. Suppose that $Y \cap Z$ is not hyperconnected modulo $I_{Y\cap Z}$. Then there are nonempty open sets A and B in $Y \cap Z$ such that $(A \cap B) \in I_{Y\cap Z}$. Then $A = C \cap (Y \cap Z)$ and $B = D \cap (Y \cap Z)$ for some open sets C and D in X. Thus, $A \cap B = (C \cap D) \cap (Y \cap Z) \in I_{Y\cap Z}$ implies that $(C \cap D) \cap (Y \cap Z) = J \cap (Y \cap Z)$ for some $J \in I$. Now $(C \cap D) \cap (Y \cap Z) \subseteq J \in I$ implies that $(C \cap D) \cap (Int_X(Y) \cap Int_X(Z)) \in I$, a contradiction. □

Theorem 4.17. *Let Y be a dense subspace of a topological space X and I be an ideal in X. If Y is hyperconnected modulo I_Y, then X is hyperconnected modulo I.*

Proof. Suppose that X is not hyperconnected modulo I. Then there are nonempty open sets U and V such that $U \cap V \in I$. Since Y is dense in X, $U \cap Y$ and $V \cap Y$ are nonempty open sets in Y. Then $(U \cap Y) \cap (V \cap Y) = (U \cap V) \cap Y \in I_Y$, a contradiction. □

Definition 4.18. Let X be a space and I be an ideal in X. Then a subset $C \subseteq X$ is called irreducible component modulo I of X, if C is hyperconnected modulo I_C and if $C \subseteq M$ and M is hyperconnected modulo I_M, then $M = C$.

Theorem 4.19. *Let X be a space and I be an ideal in X. If X is hyperconnected modulo I, then the irreducible component C modulo I_C of X is X.*

Theorem 4.20. *Let X be a space and I be an ideal in X. Then the irreducible component C modulo I_C lie in a connected component X.*

Proof. The proof follows from the fact that hyperconnectedness modulo an ideal implies hyperconnectedness and hyperconnectedness implies connectedness. □

Theorem 4.21. *Let X be a Hausdorff space and I be an ideal in X. If C is irreducible component modulo I_C of X, then C is a singleton in X.*

Proof. Suppose that C is not a singleton. Then for distinct x and y in C, there are disjoint open sets U and V in X containing x and y, respectively. Then $U \cap C$ and $V \cap C$ are disjoint nonempty open sets in C. Therefore $(U \cap C) \cap (V \cap C) = \emptyset \in I_C$, a contradiction. □

Theorem 4.22. *Let X be a space and I be an ideal in X. If I is codense, then $I \cap SO(X) = \{\emptyset\}$.*

Proof. Suppose that V is nonempty semi-open set and $V \in I$. Then there exists a nonempty open set U such that $U \subseteq V \subseteq Cl(U)$. Then $U \in I$, a contradiction. □

Theorem 4.23. ([16]) *Let X be a space. Then $SO(X, \tau) \setminus \{\emptyset\}$ forms a filter on X if and only if X is hyperconnected.*

Theorem 4.24. *Let X be a space and I be a codense ideal in X. Then $SO(X, \tau) \setminus \{\emptyset\}$ forms a filter on X if and only if X is hyperconnected modulo I.*

Proof. Let $A, B \in \tau \setminus \{\emptyset\}$. Then $A \cap B \in SO(X, \tau) \setminus \{\emptyset\}$. Since I is codense, $A \cap B \notin I$. The converse follows from Theorem 4.23. □

Definition 4.25. ([4]) Let X be a space and I be an ideal in X. Then X is said to be submaximal if for every dense set is open.

Theorem 4.26. *Let X be a submaximal space and I be codense ideal in X. Then X is hyperconnected modulo I if and only if $\tau \setminus \{\emptyset\}$ is filter on X.*

Proof. Let X be hyperconnected modulo I. Then every two nonempty open sets intersect. Now if a nonempty open set A is contained in B, then $X \setminus Cl(B) \subseteq X \setminus Cl(A)$. Since X is hyperconnected modulo I, $X \setminus Cl(B) \subseteq X \setminus Cl(A) \in I$. By submaximality of X, B is a nonempty open subset in X. Thus, $\tau \setminus \{\emptyset\}$ is filter on X.

Conversely, let $\tau \setminus \{\emptyset\}$ be a filter on X. Suppose that X is not hyperconnected modulo I. Then there are $A, B \in \tau \setminus \{\emptyset\}$ such that $A \cap B \in I$. Since I is codense, $A \cap B = \emptyset$, a contradiction. □

5. Semi-Open Modulo I Sets and Hyperconnected Modulo I Spaces

Definition 5.1. Let (X, τ) be a space and I be an ideal in X. A set V in X is said to be semi-open modulo I if either A is empty or there is a nonempty open set U such that $U \setminus V \in I$, $V \setminus Cl(U) \in I$ and $U \cap V \neq \emptyset$. The class of semi-open modulo I sets is denoted by $SO_I(X)$ or $SO_I(X, \tau)$.

Theorem 5.2. *Let (X, τ) be a space and I be an ideal in X. Then any finite union of semi-open modulo I sets is semi-open modulo I.*

Proof. Let $A, B \in SO_I(X, \tau)$. Then there are nonempty open sets U and V such that $U \cap A \neq \emptyset, V \cap B \neq \emptyset, U \setminus A, V \setminus B \in I$ and $A \setminus Cl(U), B \setminus Cl(V) \in I$. Then $(U \cup V) \cap (A \cup B) = (U \cap (A \cup B)) \cup (V \cap (A \cup B)) \neq \emptyset$. Now $(U \cup V) \setminus (A \cup B) = (U \cup V) \cap (X \setminus (A \cup B)) = (U \cup V) \cap (X \setminus A) \cap (X \setminus B) = (U \cap (X \setminus A) \cap (X \setminus B)) \cup (V \cap (X \setminus A) \cap (X \setminus B)) \subseteq (U \cap (X \setminus A)) \cup (V \cap (X \setminus B)) = (U \setminus A) \cup (V \setminus B) \in I$. Now $(A \cup B) \setminus Cl(U \cup V) = (A \cup B) \cap (X \setminus (Cl(U) \cup Cl(V))) = (A \cup B) \cap (X \setminus Cl(U)) \cap (X \setminus Cl(V)) \subseteq (A \cap (X \setminus Cl(U))) \cup (B \cap (X \setminus Cl(V))) = (A \setminus Cl(U)) \cup (B \setminus Cl(V)) \in I$. □

Theorem 5.3. *Let X be a space and I be an ideal in X. Then every semi-open set is semi-open modulo I.*

Theorem 5.4. *Let X be a space and I be a codense ideal in X. Then X is hyperconnected modulo I if and only if $SO_I(X) \setminus \{\emptyset\}$ is a filter on X.*

Proof. Let X be hyperconnected modulo I and $A, B \in SO_I(X) \setminus \{\emptyset\}$. There are nonempty open sets U and V such that $U \setminus A, V \setminus B, A \setminus Cl(U), B \setminus Cl(V) \in I$. To show that $A \cap B \in SO_I(X) \setminus \{\emptyset\}$, since X is hyperconnected modulo I, $U \cap V$ is nonempty. Now $(U \cap V) \setminus (A \cap B) = (U \cap V) \cap (X \setminus (A \cap B)) = (U \cap V) \cap ((X \setminus A) \cup (X \setminus B)) = ((U \cap V) \cap (X \setminus A)) \cup ((U \cap V) \cap (X \setminus B)) \subseteq (U \cap (X \setminus A)) \cup (V \cap (X \setminus B)) = (U \setminus A) \cup (V \setminus B) \in I$. Now if $A \cap B = \emptyset$, then $(U \cap V) \setminus (A \cap B) \in I$ implies that $U \cap V \in I$, a contradiction. So $A \cap B \neq \emptyset$. By Theorem 2.5, $X \setminus Cl(U \cap V) \in I$. Then $(A \cap B) \setminus Cl(U \cap V) \in I$. Now to show that $(U \cap V) \cap (A \cap B) \neq \emptyset$, suppose that $(U \cap V) \cap (A \cap B) = \emptyset$. Then $(U \cap V) \subseteq (X \setminus (A \cap B))$ which implies that $U \cap V = (U \cap V) \cap (X \setminus (A \cap B)) = (U \cap V) \setminus (A \cap B) \in I$, a contradiction. Now let $C \in SO_I(X) \setminus \{\emptyset\}$ and $C \subseteq D$. Now we show that D is semi-open modulo I, there is nonempty open set W such that $W \setminus C, C \setminus Cl(W) \in I$. Then $W \setminus D \subseteq W \setminus C \in I$ and $D \setminus Cl(W) = D \cap (X \setminus Cl(W) \subseteq X \setminus Cl(W) \in I$ as X is hyperconnected modulo I and W is a nonempty open set in X. Thus, $SO_I(X) \setminus \{\emptyset\}$ is a filter on X.

Conversely, let $SO_I(X) \setminus \{\emptyset\}$ be filter on X. Let A and B be nonempty open sets in X. Then $A \cap B$ is a nonempty open set in X. Thus, X is hyperconnected modulo I. □

Theorem 5.5. *Let X be a space and I be a codense ideal in X. Then X is hyperconnected if and only if $SO_I(X) \setminus \{\emptyset\}$ is a filter on X.*

Proof. The proof follows from Theorem 5.4 and Theorem 3.1. □

Definition 5.6. ([16]) Let X be a space. If (X, τ) is hyperconnected and (X, τ^1) is not hyperconnected for any $\tau \subset \tau^1$, then (X, τ) is called maximal hyperconnected.

Definition 5.7. Let X be a space and I be an ideal in X. If (X, τ) is hyperconnected modulo I and (X, τ^1) is not hyperconnected modulo I for any $\tau \subset \tau^1$, then (X, τ) is maximal hyperconnected modulo I.

The equivalence class of all topologies on a set X, which have the same semi-open modulo ideal I sets as the semi-open modulo ideal I sets by τ, is denoted by $[\tau]$.

Theorem 5.8. *Let X be a space and I be an ideal in X. If (X,τ) is hyperconnected modulo I, then (X,S), where $S \in [\tau]$ is also hyperconnected modulo I.*

Proof. Let (X,τ) be hyperconnected modulo I. Then $SO_I(X,\tau) \setminus \{\emptyset\}$ is a filter on X and $SO_I(X,S) = SO_I(X,\tau)$. Thus, (X,S) is hyperconnected modulo I. □

Theorem 5.9. *Let X be a space and I be an ideal in X. If (X,τ) is hyperconnected modulo I, then $(X, SO_I(X,\tau))$ is hyperconnected modulo I.*

Proof. Let X be hyperconnected modulo I. Then $SO_I(X) \setminus \{\emptyset\}$ is filter on X. It is sufficient to show that $SO_I(X,\tau) = SO_I(X, SO_I(X,\tau))$. Obviously, $SO_I(X,\tau) \subseteq SO_I(X, SO_I(X,\tau))$. Now suppose that $A \in SO_I(X, SO_I(X,\tau)) \setminus \{\emptyset\}$. There is a $U \in SO_I(X,\tau) \setminus \{\emptyset\}$ such that $U \cap A \neq \emptyset, U \setminus A \in I$ and $A \setminus S_ICl(U) \in I$. Since $U \in SO_I(X,\tau) \setminus \{\emptyset\}$, there is a $W \in \tau \setminus \{\emptyset\}$ such that $W \cap U \neq \emptyset, W \setminus U \in I$ and $U \setminus Cl(W) \in I$. Now we show that $A \cap W \neq \emptyset, W \setminus A \in I$ and $A \setminus Cl(W) \in I$. As X is hyperconnected modulo I, I is codense. Then $A \cap W \neq \emptyset, A \setminus Cl(W) \subseteq X \setminus Cl(W) \in I$ and $W \setminus A \subseteq (W \setminus U) \cup (U \setminus A) \in I$. Thus, $SO_I(X,\tau) = SO_I(X, SO_I(X,\tau))$. Hence, $(X, SO_I(X,\tau))$ is hyperconnected modulo I. □

In particular, if (X,τ) is hyperconnected modulo an ideal I, $SO_I(X,\tau)$ is the largest topology in $[\tau]$ such that $(X, SO_I(X,\tau))$ is hyperconnected modulo I. If we consider inclusion as "$\subseteq$" in $[\tau]$, then $SO_I(X,\tau)$ is the greatest element in $[\tau]$ and it is denoted by $F(\tau)$.

Definition 5.10. ([16]) A topological property R is called contractive (expensive) if (X,τ) has the property R and $\tau' \subseteq \tau (\tau \subseteq \tau')$, then (X,τ') has the property R.

Theorem 5.11. *Hyperconnectedness modulo an ideal is contractive property.*

Proof. Let I be an ideal in a space (X,τ) and (X,τ) be hyperconnected modulo I. Consider a topology τ' such that $\tau' \subseteq \tau$. To show that (X,τ') is hyperconnected modulo I, let $A, B \in \tau' \setminus \{\emptyset\}$. Then $A, B \in \tau \setminus \{\emptyset\}$ and $A \cap B \notin I$. Thus, (X,τ') is hyperconnected modulo I. □

Theorem 5.12. *Let (X,τ) be a space and I be an ideal in X. If (X,τ) is maximal hyperconnected modulo I, then $SO_I(X,\tau) \setminus \{\emptyset\}$ is an ultrafilter on X and, $\tau = SO_I(X,\tau)$.*

Proof. Let (X,τ) be hyperconnected modulo I. Then, by Theorem 5.4, $SO_I(X,\tau) \setminus \{\emptyset\}$ is a filter on X. Assume that $A \subseteq X$ such that $A \notin SO_I(X,\tau) \setminus \{\emptyset\}$. Then $A \notin \tau$. Now $\tau(A)$, is the simple expansion of τ by A. Since $\tau \subseteq \tau(A), \tau(A)$ is not hyperconnected modulo I. Then there exist two nonempty open sets C_1 and C_2 in $(X, \tau(A))$ such that $C_1 \cap C_2 \in I$. Then $C_1 = U_1 \cup (V_1 \cap A)$ and $C_2 = U_2 \cup (V_2 \cap A)$, where $U_1, U_2, V_1, V_2 \in \tau$. Now $C_1 \cap C_2 \in I$ implies that $U_1 \cap U_2 \in I$. Since (X,τ) is hyperconnected modulo I, $U_1 \cap U_2 = \emptyset$ implies that $U_1 = \emptyset$ or $U_2 = \emptyset$. Without loss of generality, assume that $U_1 = \emptyset$. Two cases may arise.

Case 1: $U_2 = \emptyset$.

Then $V_1 \cap V_2 \cap A = (V_1 \cap A) \cap (V_2 \cap A) = C_1 \cap C_2 \in I$ and $V_1 \cap V_2 \notin I$ as V_1 and V_2 are nonempty open sets. Now we show that $X \setminus A \in SO_I(X,\tau) \setminus \{\emptyset\}$.

Since $(V_1 \cap V_2) \cap (X \setminus A) \notin I$, $(V_1 \cap V_2) \cap (X \setminus A) \neq \emptyset$ and $(V_1 \cap V_2) \setminus (X \setminus A) = (V_1 \cap V_2) \cap A \in I$. Since X is hyperconnected modulo I, $(X \setminus A) \setminus Cl(V_1 \cap V_2) \subseteq X \setminus Cl(V_1 \cap V_2) \in I \Rightarrow (X \setminus A) \in SO_I(X,\tau) \setminus \{\emptyset\}$.

Case 2: $U_2 \neq \emptyset$.

Then $U_2 \cap V_1 \notin I$. Since $C_1 \cap C_2 \in I$, $V_1 \cap U_2 \cap A = (V_1 \cap U_2 \cap A) \cap (U_2 \cup V_2) = (V_1 \cap A) \cap (U_2 \cup V_2) \cap (U_2 \cap A) = C_1 \cap C_2 \in I$. Then $(U_2 \cap V_1) \setminus (X \setminus A) = (U_2 \cap V_1) \cap A \in I$ and $(X \setminus A) \cap (U_2 \cap V_1) \notin I$ as $U_2 \cap V_1 \notin I$. Now since X is hyperconnected modulo I and $U_2 \cap V_1 \neq \emptyset$, $(X \setminus A) \setminus Cl(U_2 \cap V_1) \subseteq X \setminus Cl(U_2 \cap V_1) \in I$. Thus, in both cases $X \setminus A \in SO_I(X,\tau) \setminus \{\emptyset\}$. Hence, $SO_I(X,\tau) \setminus \{\emptyset\}$ is an ultrafilter on X.

By Theorem 5.8, $(X, SO_I(X,\tau))$ is hyperconnected modulo I and $\tau \subseteq SO_I(X,\tau)$. Since (X,τ) is maximal hyperconnected modulo I, $\tau = SO_I(X,\tau)$. □

Theorem 5.13. *Let (X, τ) be a space and I be an ideal in X. If $SO_I(X, \tau) \setminus \{\emptyset\}$ is an ultrafilter on X, then $(X, SO_I(X, \tau))$ is maximal hyperconnected modulo I.*

Proof. Obviously, $SO_I(X, \tau) \setminus \{\emptyset\}$ is filter on X. By Theorem 5.4, (X, τ) is hyperconnected modulo I. By Theorem 5.9, $(X, SO_I(X, \tau))$ is hyperconnected modulo I. Suppose that $(X, SO_I(X, \tau))$ is not maximal hyperconnected modulo I. Then there exists a hyperconnected modulo I space (X, τ_1) such that $SO_I(X, \tau) \subsetneq \tau_1$. Then $SO_I(X, \tau) \subsetneq SO_I(X, \tau_1)$. Since (X, τ_1) is hyperconnected modulo I, by Theorem 5.4, $SO_I(X, \tau_1) \setminus \{\emptyset\}$ is a filter on X, a contradiction. Hence, $(X, SO_I(X, \tau))$ is maximal hyperconnected modulo I. □

Definition 5.14. ([10]) A space X is said to be a door space if for each subset $A \subset X$, either A or $X \setminus A$ is open.

Theorem 5.15. *Let (X, τ) be a space and I be an ideal in X. Then (X, τ) is hyperconnected modulo I door space if and only if $\tau \setminus \{\emptyset\}$ is an ultrafilter on X.*

Proof. Let (X, τ) be hyperconnected modulo I door space. Then for each $A, B \in \tau \setminus \{\emptyset\}$, $A \cap B \in \tau \setminus \{\emptyset\}$. Let $A \in \tau \setminus \{\emptyset\}$ and $A \subset B$. If $B = X$, then $B \in \tau \setminus \{\emptyset\}$. Otherwise, $B \neq X$. Suppose that $B \notin \tau \setminus \{\emptyset\}$. Since X is door space, $X \setminus B \in \tau \setminus \{\emptyset\}$. Then $A \cap (X \setminus B) = \emptyset \in I$, a contradiction. Consider $\emptyset \neq A \subsetneq X$. Then, by the definition of door space, either A or $X \setminus A$ is open. Thus, $\tau \setminus \{\emptyset\}$ is an ultrafilter on X.

Conversely, let $\tau \setminus \{\emptyset\}$ be an ultrafilter on X. Then, by Theorem 5.13, (X, τ) is maximal hyperconnected modulo I. By the definition of an ultrafilter, X is a door space. □

Corollary 5.16. *Let (X, τ) be a space and I be an ideal in X. If X is hyperconnected modulo I door space, then X is maximal hyperconnected modulo I.*

Theorem 5.17. *Let (X, τ) be a space and I be an ideal in X. Then (X, τ) is maximal hyperconnected modulo I if and only if it is submaximal and hyperconnected modulo I.*

Proof. Let (X, τ) be maximal hyperconnected modulo I and $A \subsetneq X$. Then $X \setminus Cl(A) \in I$ implies $X = Cl(A)$. Then $SO_I(X, \tau) \setminus \{\emptyset\}$ is an ultrafilter on X and $SO_I(X, \tau) = \tau$. Since $A \subsetneq X$ and $\tau \setminus \{\emptyset\}$ is an ultrafilter on X, A or $X \setminus A \in \tau \setminus \{\emptyset\}$. If $X \setminus A \in \tau \setminus \{\emptyset\}$, then A is closed, a contradiction. Thus, $A \in \tau \setminus \{\emptyset\}$. Hence, X is submaximal.

Conversely, let (X, τ) be submaximal. Suppose that (X, τ) is not maximal hyperconnected modulo I. Then there is a topology τ^1 containing τ such that (X, τ^1) is hyperconnected modulo I. let $\emptyset \neq U \in \tau^1$. Then $X \setminus Cl_\tau(U) \subseteq X \setminus Cl_{\tau^1}(U) \in I$. Since (X, τ) is submaximal, then $U \in \tau$ implies $\tau^1 = \tau$. Hence, (X, τ) is maximal hyperconnected modulo I. □

References

[1] M.E. Abd E1-Monsef, S.N. E1-Deeb, R.A. Mahmoud, β-open sets and β-continuous mapping, Bull. Fac. Sci. Assiut Univ. 12 (1983) 77–90.
[2] N. Ajmal, J.K. Kohli, Properties of hyperconnected spaces, their mappings into Hausdorff spaces and embeddings into hyperconnected spaces, Acta Math. Hungar. 60 (1992) 41–49.
[3] D. Andrijević, Semi-preopen sets, Mat. Vesnik 38 (1986) 24–32.
[4] N. Bourbaki, General Topology, Springer-Verlag Berlin, (1998).
[5] S.G. Crossley, S.K. Hildebrand, Semi-closure, Texas J. Sci. 22 (1971) 99–112.
[6] J. Dontchev, Ideal resolvability, Topology Appl. 93 (1999) 1–16.
[7] N. E1-Deeb, I.A. Hasanein, A.S. Mashhour, T. Noiri, On p-regular spaces, Bull. Math. Soc. Sci. Math. R. S. Roumanie (N.S.) 27(75) (1983) 311–315.
[8] B. Garai, C. Bandyopadhyay, Nowhere dense sets and hyperconnected s-topological spaces, Bull. Calcutta Math. Soc. 92 (2000) 55–58.
[9] B. Garai, C. Bandyopadhyay, I-dense sets and I-o.h.i. spaces, Kyungpook Math. J. 41 (2001) 351–359.
[10] J.L. Kelley, General Topology, Springer (1991).
[11] M.R. Koushesh, Connectedness modulo an ideal, Topology Appl. 214 (2016) 150–179.
[12] N. Levine, Semi-open sets and semi-continuity in topological spaces, Amer. Math. Monthly 70 (1963) 36–41.
[13] N. Levine, Dense topologies, Amer. Math. Monthly 75 (1968) 847–852.
[14] S.N. Maheshwari, U. Tapi, Connectedness of a stronger type in topological spaces, Nanta Math. 12 (1979) 102–109.

[15] A.S. Mashhour, M.E. Abd E1-Monsef, S.N. El-Deep, On preeontinuous and weak precontinuous mappings, Proc. Math. Phys. Soc. Egypt 53 (1982) 47–53.
[16] P.M. Mathew, On hyperconnected spaces, Indian J. Pure Appl. Math. 19 (1988) 1180–1184.
[17] O. Njástad, On some classes of nearly open sets, Pacific J. Math. 15 (1965) 961–970.
[18] T. Noiri, A note on hyperconnected sets, Mat. Vesnik 3 (16)(31) (1979) 53–60.
[19] T. Noiri, Properties of hyperconnected spaces, Acta Math. Hungar. 66 (1995) 147–154.
[20] V. Pipitone, G. Russo, Spazi semiconnessi e spazi semiaperti, Rend. Circ. Mat. Palermo 24 (1975) 273–285.
[21] D.A. Rose, T.R. Hamlet, Ideally equivalent topologies, Math. Chronicle 20 (1991) 149–156.
[22] L.A. Steen, J.A. Seebach, Jr., Counterexamples in Topology, Holt, Rinehart and Winster (New York, 1970).
[23] T. Thompson, Characterizations of irreducible spaces, Kyungpook Math. J. 21 (1981) 191–194.