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An Impulsive Delay Discrete Stochastic Neural Network Fractional-Order Model and Applications in Finance

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Abstract. In this paper, we propose a new tool for modeling and analysis in finance, introducing an impulsive discrete stochastic neural network (NN) fractional-order model. The main advantages of the proposed approach are: (i) Using NNs which can be trained without the restriction of a model to derive parameters and discover relationships, driven and shaped solely by the nature of the data; (ii) using fractional-order differences, whose nonlocal property makes the fractional calculus a suitable tool for modeling actual financial systems; (iii) using impulsive perturbations, which give an opportunity to control the dynamic behavior of the model; (iv) including a stochastic term, which allows to study the effect of noise disturbances generally existing in financial assets; (v) taking into account the existence of time delayed influences. The modeling approach proposed in this paper can be applied to investigate macroeconomic systems.

1. Introduction

During the past few decades, impressive applications of NNs have been proposed for various fields such as optimization, linear and nonlinear programming, associative memory, pattern recognition, fault diagnosis, image processing, and computer vision. Also, NNs have been successfully applied as an alternative modeling approach to statistical models in finance as they are data-driven which does not require any distributional assumption about the underlying data. Such an assumption has a drastic impact on the accuracy and stability of the model. Indeed, most financial asset returns exhibit a fat-tail distribution, but conventional value at risk (VaR) estimation models assume that returns are normally distributed. With the NN approach, no distributional assumption regarding the return distribution is required for estimating and forecasting the VaR using intra-daily data [38]. Another common advantage of an NN model is its ability to deal with uncertain and robust data. Different aspects of NNs applications in quantitative finance are given in [25, 29, 40].

As it is pointed out in [20], dynamics of financial assets demonstrate the stochastic behavior. A large number of stochastic financial models appeared in the literature, see, for example, [13, 15, 18] and the references therein. Also, the important effect of noise disturbances should be taken into account in studying the dynamics of a financial system by means of the neural network approach. Recently, some stochastic financial neural network models have been reported in [2, 10, 17].

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It is now recognized that introducing some kinds of time delays is inevitable in financial models. For example, the time delay represents a memory length of a moving average rule in discrete-time stochastic heterogeneous agent models of financial markets [12]. Also, it has been shown by many authors that the introduction of information delay into the dynamic models significantly changes their asymptotical properties [24]. Analysis on different market data shows that price fluctuations can be caused by time delayed influences. During the last couple of decades, integer-order delay economic models have attracted the attention of many mathematicians and economists, and various delay economic models have been examined [5, 26, 36].

On the other hand, the state of an economic system is often subject to instantaneous perturbations and experiences abrupt changes at certain instants which may be caused by population changes, technological and financial structural changes, that is, it exhibits impulsive effects. It is now well known that real-world phenomena, which are subject to short-term perturbations whose duration is negligible in comparison with the duration of the process, are more accurately described using impulsive systems. Among them, due to their numerous applications, delay impulsive systems and delay impulsive stochastic systems have received a great deal of attention, see, e.g., [9, 44–46]. In the paper [33], it is found that the studying of impulsive models in finance can give answers that a discrete model or pure continuous model cannot give. Recently, several impulsive financial delay models have been developed [27, 31, 43]. It has been argued that the impulsive perturbations have a control power which can be used to compensate the deviating trend in a financial model, i.e., by means of appropriate impulsive perturbations, we can control the dynamic behavior of the system. Since impulsive control arises naturally in a wide variety of financial models, some impulsive control laws are proposed in [4, 16, 19, 32, 37].

In the last decade, a great progress in studying fractional evolution models has been made. Indeed, fractional-order modeling has come to play a crucial rôle in many applications and real-world physical phenomena [28, 34, 35]. Recently, there have been several attempts to incorporate the fractional approach into some financial models [6, 14, 20, 41]. The empirical studies conducted in some of these papers suggest that a discrete fractional dynamical system can describe the actual economic data accurately and predict the future behavior more reasonably than an integer-order model. However, to the best of our knowledge, the fractional concept has not been applied previously to an NN financial stochastic model.

The goal of the present work is to develop a financial model that brings together the advantages of all of the following:

(i) The NN approach in finance;

(ii) the stochastic dynamic approach;

(iii) the greater flexibility in the model offered by a fractional-order difference;

(iv) the potential rôle of the memory in financial indicators;

(v) the control power of some impulsive perturbations.

Motivated by the above discussion, we propose to extend the impulsive stochastic delay NN concept in financial mathematics to the fractional-order case. The **main contribution** of our paper is in two aspects:

(a) We develop an impulsive discrete stochastic NN fractional-order model that can be used to determine qualitative properties of impulsive financial systems and all their subsystems, and

(b) we provide sufficient conditions for the global Mittag–Leffler stability in mean square of the zero solution of the model, extending the existing theory to the fractional order case.

Also, such knowledge could help scientists better understand the effects of some impulsive perturbations, and to design an appropriate control strategy.

2. Fractional-Order Impulsive Discrete Stochastic Neural Network Model with Delays

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space with norm $\|\cdot\|$, $\mathbb{R}^+ = [0, \infty)$,

$$\mathbb{N}_0 = \{0, 1, 2, \ldots\}, \quad \mathbb{N} = \mathbb{N}_0 \setminus \{0\}, \quad \mathbb{N}_{-\tau} = \{-\tau, -\tau + 1, \ldots, -1, 0\}$$

for $\tau \in \mathbb{N}$ denoting the upper bound of the time delay. Throughout the paper, we suppose that matrices, if not explicitly specified, have compatible dimensions. For any matrix A, $\|A\|$ denotes the matrix norm of A induced by the Euclidean vector norm, and A^T denotes the transpose of the matrix A. For square matrices X and Y, the notation $X \geq Y$ (respectively, $X > Y$, $X \leq Y$, $X < Y$) means that $X - Y$ is positive semi-definite (respectively, positive definite, negative semi-definite, negative definite). I is the identity matrix with compatible dimension.

To develop our fractional-order model, we first start with a classical integer-order discrete NN model defined by the system

$$x(k + 1) = Cx(k) + Af(x(k)), \quad k \in \mathbb{N}_0, \tag{2.1}$$

where $x(k) = [x_1(k), x_2(k), \ldots, x_n(k)]^T \in \mathbb{R}^n$ is the state vector associated with the n inputs at time k, the diagonal matrix $C = \operatorname{diag}(c_1, c_2, \ldots, c_n)$ has constant entries c_i, $i = 1, 2, \ldots, n$, the constant matrix

$$A = (a_{ij})_{1 \leq i,j \leq n}$$

is the connection weight matrix, and

$$f(x) = [f_1(x_1), f_2(x_2), \ldots, f_n(x_n)]^T$$

is the neuron activation function. In finance, depending on the activation function, NN models of type (2.1) have been used for various tasks [38, 40].

Denote by $(\Omega, \mathcal{F}, P)$ a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., the filtration contains all P-null sets and is right continuous). $\mathbb{E}\{\cdot\}$ stands for the mathematical expectation operator with respect to the given probability measure P.

The potential rôle of the memory in financial indicators has been explored in various contexts [21, 36]. It is well known that time delay can generate oscillations, divergence, or instability which may be harmful to the system. Therefore, the study of neural dynamics with the consideration of time delays has become extremely important in manufacturing high-quality neural networks.

If we incorporate jointly the time delay and the noise disturbances in the financial NN modeling, we then obtain the more general model

$$x(k + 1) = Cx(k) + Af(x(k)) + Bg(x(k - \tau(k))) + \sigma(k, x(k), x(k - \tau(k)))w(k), \quad k \in \mathbb{N}_0,$$

where

$$B = (b_{ij})_{1 \leq i,j \leq n}$$

is the delayed connection weight matrix,

$$g(x) = [g_1(x_1), g_2(x_2), \ldots, g_n(x_n)]^T$$

is the delayed neuron activation function, $\tau(k) \in (0, \tau)$ denotes the time delay depending on k, $\sigma(\cdot)$ represents the random perturbation weight, and

$$w = [w_1, w_2, \ldots, w_n]^T$$

is an n-dimensional Brownian motion on $(\Omega, \mathcal{F}, P)$ with

$$\mathbb{E}[w(k)] = 0, \quad \mathbb{E}[w^2(k)] = 1, \quad \mathbb{E}[w(i)w(j)] = 0 \quad \text{for} \quad i \neq j. \tag{2.2}$$

Impulsive effects exist widely in many financial processes in which the system's states change abruptly at certain moments of time, involving such fields as asset management, risk assessment, investment analysis, and so on. It is known that impulses can make unstable systems stable or, otherwise, stable systems can become unstable after impulsive effects. The aim in combined stochastic and impulse control is to maximize a certain functional, depending on a controlled financial process.

As an impulsive generalization of the model (2.1), we consider the n-neuron impulsive control discrete neural network with time-depended delays

$$\begin{aligned} x(k+1) &= Cx(k) + Af(x(k)) + Bg(x(k-\tau(k))) + \sigma(k, x(k), x(k-\tau(k)))w(k), \quad k \neq k_m - 1, \\ x(k_m) &= D_m x(k_m - 1), \quad m \in \mathbb{N}_0, \end{aligned} \tag{2.3}$$

where the impulsive moments satisfy $0 = k_0 < k_1 < k_2 < \ldots$ and $\lim_{m\to\infty} k_m = \infty$, and the matrix D_m is the $n \times n$ impulse gain matrix at the instants $k_m - 1$, $m \in \mathbb{N}_0$.

However, the model (2.3) has rather limited capabilities for modelling fractional-order systems. Thus, in this paper, we suggest the use of fractional-order calculus to built a model of an NN system.

Let $k \neq k_m - 1$, $m \in \mathbb{N}_0$. By subtracting x_k from both sides of the first equation in the model (2.3), we can rewrite (2.3) as

$$\begin{aligned} \nabla x(k+1) &= C_d x(k) + Af(x(k)) + Bg(x(k-\tau(k))) + \sigma(k, x(k), x(k-\tau(k)))w(k), \quad k \neq k_m - 1, \\ x(k+1) &= \nabla x(k+1) + x(k), \\ x(k_m) &= D_m x(k_m - 1), \quad m \in \mathbb{N}_0, \end{aligned} \tag{2.4}$$

where $C_d = C - I$, and ∇ is the first-order backward difference.

To obtain the fractional-order generalization of the model (2.4), we shall use the following definition of the fractional-order difference (see e.g., [28, Formula (2.6)] or [8, 11]):

Definition 2.1. The *fractional-order backward difference* of order $q \in \mathbb{R}$ is given by

$$\nabla^q x(k) = \sum_{j=0}^{k} (-1)^j \binom{q}{j} x(k-j),$$

where

$$\binom{q}{j} = \begin{cases} 1 & \text{for} \quad j = 0, \\ \frac{q(q-1)\cdots(q-j+1)}{j!} & \text{for} \quad j \in \mathbb{N}, \end{cases}$$

and $k \in \mathbb{N}$ is the number of the sample for which the difference is obtained.

Using Definition 2.1, for $q \in \mathbb{R}$, we propose the fractional-order impulsive discrete NN model with time-varying delays

$$\begin{aligned} \nabla^q x(k+1) &= C_d x(k) + Af(x(k)) + Bg(x(k-\tau(k))) + \sigma(k, x(k), x(k-\tau(k)))w(k), \quad k \neq k_m - 1, \\ x(k+1) &= \nabla^q x(k+1) - \sum_{j=1}^{k+1} (-1)^j \binom{q}{j} x(k-j+1), \\ x(k_m) &= D_m x(k_m - 1), \quad m \in \mathbb{N}_0. \end{aligned} \tag{2.5}$$

In this model, the order of differentiation q is taken the same for all state variables $x_i(k)$, $i = 1, \ldots, n$. This is referred to as a commensurate order.

Let us consider the first two equations in the model (2.5) which represent the continuous part. Substituting the first equation in the problem (2.5) into the second one yields

$$\begin{aligned} x(k+1) =& C_d x(k) - \sum_{j=1}^{k+1} (-1)^j \binom{q}{j} x(k-j+1) \\ &+ Af(x(k)) + Bg(x(k-\tau(k))) + \sigma(k, x(k), x(k-\tau(k)))w(k), \quad k \neq k_m - 1, \quad m \in \mathbb{N}_0. \end{aligned}$$

From here, by setting $\lambda_j = (-1)^j \binom{q}{j}$, we obtain

$$\begin{aligned} x(k+1) =& (C_d - \lambda_1 I)x(k) - \sum_{j=2}^{k+1} \lambda_j x(k-j+1) \\ &+ Af(x(k)) + Bg(x(k-\tau(k))) + \sigma(k, x(k), x(k-\tau(k)))w(k), \quad k \neq k_m - 1, \quad m \in \mathbb{N}_0. \end{aligned} \tag{2.6}$$

Let us put in (2.6)

$$C_0 = C_d - \lambda_1 I = C + (q-1)I,$$

and, for all $j \in \mathbb{N}$

$$C_j = -\lambda_{j+1} I.$$

This leads to

$$\begin{aligned} x(k+1) =& C_0 x(k) + C_1 x(k-1) + C_2 x(k-2) + \ldots + C_k x(0) \\ &+ Af(x(k)) + Bg(x(k-\tau(k))) + \sigma(k, x(k), x(k-\tau(k)))w(k), \quad k \neq k_m - 1, \quad m \in \mathbb{N}_0. \end{aligned} \tag{2.7}$$

Practical implementation needs to reduce the number of samples taken into consideration, and therefore we shall consider the equation

$$\begin{aligned} x(k+1) =& C_0 x(k) + \sum_{j=1}^{\eta} C_j x(k-j) + Af(x(k)) + Bg(x(k-\tau(k))) \\ &+ \sigma(k, x(k), x(k-\tau(k)))w(k), \quad k \neq k_m - 1, \quad m \in \mathbb{N}_0, \end{aligned} \tag{2.8}$$

where $0 \leq \eta \leq k-1$. Using (2.8), we obtain the impulsive model

$$\begin{aligned} x(k+1) =& C_0 x(k) + \sum_{j=1}^{\eta} C_j x(k-j) + Af(x(k)) + Bg(x(k-\tau(k))) \\ &+ \sigma(k, x(k), x(k-\tau(k)))w(k), \quad k \neq k_m - 1, \\ x(k_m) =& D_m x(k_m - 1), \quad m \in \mathbb{N}_0. \end{aligned} \tag{2.9}$$

Let $\tilde{\tau} = \max\{\eta, \tau\}$ and $\varphi_0 \in C(\mathbb{N}_{-\tilde{\tau}}, \mathbb{R}^n)$. Denote by $x(k) = x(k; k_0, \varphi_0)$ the solution of (2.7) such that

$$x(\theta) = \varphi_0(\theta) \quad \text{for} \quad \theta \in \mathbb{N}_{-\tilde{\tau}} \quad \text{and} \quad x(k_0) = \varphi_0(0). \tag{2.10}$$

The impulsive control of (2.9) is performed in the following way: The point $P_k = (k, x(k))$ begins its motion from the point $(k_0, x(k_0))$ and moves along $(k, x(k))$ described by the solution x of the system (2.8) until time $k_1 > k_0$, at which point P_k is "instantly" transferred from the position $P_{k_1} = (k_1, x(k_1))$ into the position (k_1, x_1^+), $x_1^+ = D_1 x(k_1)$. Then the point P_k continues to move further along $x(k) = x(k; k_1, x_1^+)$ as the solution of (2.8) starting at (k_1, x_1^+) until it triggers a second transfer at $k_2 > k_1$, and so on. Clearly, this process continues as long as the solution of (2.8) exists. More precisely, the solution $x(k) = x(k; k_0, \varphi)$ of the initial value problem (2.9), (2.10) is characterized by the following:

1. For $\theta \in \mathbb{N}_{-\tilde{\tau}}$, the solution $x(\theta) = x(k_0 + \theta)$ satisfies the initial conditions (2.10).

2. For $k_0 < k \leq k_1$, $x(k)$ coincides with the solution of the problem (2.8), (2.10) (see [8, 11]). At the moment $k = k_1$, the mapping point $(k, x(k; k_0, \varphi_0))$ is "instantly" transferred from the position $P_{k_1} = (k_1, x(k_1))$ into the position (k_1, x_1^+), $x_1^+ = D_1 x(k_1)$.

3. For $k_1 < k \leq k_2$, the solution $x(k)$ coincides with the solution of the equation (2.6) with the initial condition

$$x_{k_1} = x(k_1 + \theta) = \varphi_1(\theta), \quad \theta \in \mathbb{N}_{-\tilde{\tau}},$$

where

$$\varphi_1(k - k_1) = \begin{cases} \varphi_0(k - k_1) & \text{if} \quad k \in [k_0 - \tau, k_0] \cap [k_1 - \tilde{\tau}, k_1], \\ x(k; k_0, \varphi_0) & \text{if} \quad k \in (k_0, k_1) \cap [k_1 - \tilde{\tau}, k_1], \\ D_1 x(k; k_0, \varphi_0) & \text{if} \quad k = k_1. \end{cases}$$

At the moment $k = k_2$, the mapping point $(k, x(k))$ jumps momentarily, and so on.

3. Some Definitions and Lemmas

In this section, we provide the reader the necessary background on fractional discrete calculus and a discrete fractional comparison principle. We define the notion of Mittag–Leffler stability for our model under consideration. Let us note that this notion has been introduced in [22] for nonautonomous ordinary differential equations of fractional order.

In what follows, we shall use a discrete analogue of the Mittag–Leffler function, which plays an important rôle in the solution of noninteger-order differential equations. The standard Mittag–Leffler function (see [22]) is given as

$$E_q(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(qj + 1)},$$

where $q > 0$. It is also common to represent the Mittag–Leffler function in two parameters $p, q > 0$ such that

$$E_{q,p}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(qj + p)}. \tag{3.1}$$

For $p = 1$, we have $E_q = E_{q,1}$. Also, $E_{1,1}(z) = e^z$. The discrete analogue of the Mittag–Leffler function (3.1), called *nabla discrete Mittag–Leffler function*, is defined in [1] by

$$F_{q,p}(\lambda, z) = \sum_{j=0}^{\infty} \lambda^j \frac{\Gamma(z + jq + p - 1)}{\Gamma(z)\Gamma(qj + p)},$$

where $\lambda \in \mathbb{R} \setminus \{1\}$, $p, q, z \in \mathbb{C}$, and $\mathrm{Re}(q) > 0$. For $p = 1$, we have

$$F_q(\lambda, z) = \sum_{j=0}^{\infty} \lambda^j \frac{\Gamma(z + jq)}{\Gamma(z)\Gamma(qj + 1)}. \tag{3.2}$$

Note that, for $0 < q \leq 1$, the solution of the discrete fractional initial value problem (IVP)

$$\nabla^q v(k) = \lambda v(k), \quad v(k_0) = v_0 \in \mathbb{R}$$

is given by

$$v(k) = v_0 E_q(\lambda, z),$$

i.e., it is a discrete analogue of the exponential function. Indeed, putting $q = 1$ in (3.2), we obtain a discrete exponential function

$$F_1(\lambda, z) = \sum_{j=0}^{\infty} \lambda^j \frac{\Gamma(z + j)}{\Gamma(z)\Gamma(j + 1)} = \sum_{j=0}^{\infty} \lambda^j \binom{z + j - 1}{j} = \sum_{j=0}^{\infty} (-\lambda)^j \binom{-z}{j} = (1 - \lambda)^{-z}.$$

For some basic concepts and theorems on discrete Mittag–Leffler functions, we refer the reader to [1, 3].

We shall introduce the following discrete analogue of Mittag–Leffler stability defined by [22].

Definition 3.1. The zero solution $x(k) \equiv 0$ of (2.9) is said to be *globally Mittag–Leffler stable in mean square* if there exist constants $\mu > 0$ and $d > 0$ such that

$$\mathbb{E}\left[\|x(k)\|^2\right] \leq \left\{ \sup_{\theta \in \mathbb{N}_{-\tilde{\tau}}} \mathbb{E}\left[\|x(\theta)\|^2\right] F_q(-\mu, (k-k_0)) \right\}^d, \quad k \geq k_0,$$

where $q \in (0, 1]$.

We make the following assumptions:

(H$_1$) $f(0) = g(0) = 0$, $\sigma^T(k, 0, 0) = 0$, $k \in \mathbb{N}_0$.

(H$_2$) There exist positive numbers l_i^f and l_i^g $(i = 1, 2, \ldots, n)$, such that for any $u, v \in \mathbb{R}$,

$$|f_i(u) - f_i(v)| \leq l_i^f |u - v| \quad \text{and} \quad |g_i(u) - g_i(v)| \leq l_i^g |u - v|.$$

For the sake of simplicity, we let

$$L_f = \mathrm{diag}(l_1^f, l_2{}^f, \ldots, l_n^f) \quad \text{and} \quad L_g = \mathrm{diag}(l_1^g, l_2^g, \ldots, l_n^g).$$

(H$_3$) There exist two positive definite matrices θ_1 and θ_2 such that

$$\sigma^T(k, x, y)\sigma(k, x, y) \leq x^T \theta_1 x + y^T \theta_2 y, \quad k \in \mathbb{N}_0, \quad x, y \in \mathbb{R}^n.$$

Together with the model (2.8), we consider the comparison impulsive fractional-order difference equation

$$\begin{aligned} \nabla^q u(k+1) &= g(k, u(k)), \quad k \neq k_m - 1, \\ u(k_m) &= B_m u(k_m - 1), \quad m \in \mathbb{N}_0, \end{aligned} \tag{3.3}$$

where $0 < q \leq 1$, $g : \mathbb{N}_0 \times \mathbb{R}^+ \to \mathbb{R}^+$, $B_m : \mathbb{N}_0 \to \mathbb{R}^+$.

According to [7], we denote by $u^+(t) = u^+(k; k_0, u_0)$ the *under function* of equation (3.3), which satisfies the initial condition

$$u^+(k_0; k_0, u_0) = u_0 \in \mathbb{R}^+. \tag{3.4}$$

In the next section, the problem of global Mittag–Leffler stability in mean square of the trivial solution of (2.9) is investigated for $0 < q \leq 1$. To this end, we shall use Lyapunov functions from the class ϕ_0 (see Appendix A). Moreover, the technique of investigation essentially depends on the choice of minimal subsets of a suitable space of functions, by the elements of which the derivatives of Lyapunov functions are estimated. It is well known that this method (known as the Lyapunov–Razumikhin function method) has been widely used in the treatment of stability of functional differential and difference equations [23, 31, 32, 42].

In the sequel, we shall use the following auxiliary results.

Lemma 3.2. *Assume the following:*

1. *The function $g : \mathbb{N}_0 \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous in each of the sets $(k_m - 1, k_m] \times \mathbb{R}^+$, $m \in \mathbb{N}_0$.*
2. *The functions $\psi_m(u) = u + B_m u \geq 0$, $m \in \mathbb{N}_0$ are nondecreasing with respect to u.*
3. *The under function $u^+(k; k_0, u_0)$ of IVP (3.3), (3.4) is defined on $\mathbb{R}^+$.*
4. *The function $V \in \phi_0$ is such that for $t \in [t_0, \infty)$, $\varphi \in \mathcal{PC}$,*

   $$V(k_m, x(k_m)) \leq \psi_k(V(k_m - 1, x(k_m - 1)), \quad m \in \mathbb{N}_0,$$

   and the inequality

   $$\nabla^q V(k+1, x(k+1)) \leq g(k, V(k, x(k))), \quad k \neq k_m - 1, \quad m \in \mathbb{N}_0$$

   is valid whenever $V(k + \theta, x(k + \theta)) \leq V(k, x(k))$ for $-\tilde{\tau} \leq \theta \leq 0$ and $0 < q \leq 1$.

Then

$$\sup_{-\tilde{\tau}\leq\theta\leq 0} V(k_0+\theta,\varphi_0(\theta)) \leq u_0$$

implies

$$V(k,x(k;k_0,\varphi_0)) \leq u^+(k;k_0,u_0) \quad \text{for all} \quad k \geq k_0.$$

Lemma 3.2 is the discrete analogue of [35, Theorem 3.1]. The proof is essentially a repetition of the arguments used there, and we omit the details here.

Remark 3.3. Efficient comparison results for discrete fractional-order systems without delays and impulses are given in [7].

In the case when $g(k,u) = \lambda u$ for $(k,u) \in \mathbb{N}_0 \times \mathbb{R}^+$, where $\lambda \in \mathbb{R} \setminus \{1\}$ and $\psi_k(u) = u$ for $u \in \mathbb{R}^+$, $k \in \mathbb{N}$, we deduce the following corollary from Lemma 3.2.

Corollary 3.4. *Assume that the function $V \in \phi_0$ is such that*

$$\begin{aligned} &V(k_m,x(k_m)) \leq V(k_m-1,x(k_m-1)), \quad x \in \mathbb{R}^n, \quad m \in \mathbb{N}_0,\\ &\nabla^q V(k+1,x(k+1)) \leq \lambda V(k,x(k)), \quad k \neq k_m-1, \quad m \in \mathbb{N}_0 \end{aligned}$$

is valid whenever $V(k+\theta,x(k+\theta)) \leq V(k,x(k))$ for $-\tilde{\tau} \leq \theta \leq 0$ and $0 < q \leq 1$. Then

$$V(k,x(k;k_0,\varphi_0)) \leq \sup_{-\tilde{\tau}\leq\theta\leq 0} V(k_0+\theta,\varphi_0(\theta)) F_q(\lambda,(k-k_0)) \quad \text{for all} \quad k \geq k_0.$$

Lemma 3.5 (See [39]). *Given any $X, Y \in \mathbb{R}^n$ and a positive definite matrix $Q \in \mathbb{R}^{n\times n}$, we have*

$$2X^T Y \leq X^T Q^{-1} X + Y^T Q Y.$$

Lemma 3.6 (See [8]). *For $j \geq 2$, the factors $\lambda_j = (-1)^j \binom{q}{j}$ are negative for $q \in (0,1)$, positive for $q \in (1,2)$ and zero for $q \in \{0,1\}$.*

4. Impulsive Control for $0 < q \leq 1$

In this section, by applying Lyapunov functions and Razumikhin's technique combined with an impulsive feedback control, we establish some criteria for global Mittag–Leffler stability in mean square of the trivial solution of (2.9) for $0 < q \leq 1$.

Theorem 4.1. *Assume (H_1)–(H_3) and that there exist real numbers α, β, γ, positive numbers $\tilde{\gamma}, \nu, \theta_1, \theta_2$, positive diagonal matrices Q_l, $1 \leq l \leq 6$, and $Q_{7_{ij}}$, $1 \leq i, j \leq \eta$, such that*

(i) $$\begin{aligned} &C_0^T\left(I + AQ_1^{-1}A^T + C_1Q_2^{-1}C_1^T + BQ_3^{-1}B^T\right)C_0\\ &+L_f\left(Q_1 + A^TA + A^TC_1Q_4^{-1}C_1^TA + A^TBQ_5^{-1}B^TA\right)L_f + \theta_1 - \alpha I \leq 0; \end{aligned}$$

(ii) $L_g(Q_3 + Q_5 + Q_6 + B^TB)L_g + \theta_2 - \beta I \leq 0;$

(iii) $Q_2 + Q_4 + C_1^TC_1 + C_1^TBQ_6^{-1}B^TC_1 - \gamma I \leq 0;$

(iv) $$\sum_{\substack{j=1\\ j\neq i}}^{\eta}\left(Q_{7_{ji}}^{-1} + Q_{7_{ij}}\right) - \tilde{\gamma} I \leq 0;$$

(v) $\alpha + \beta + \gamma\eta(1+\tilde{\gamma}) \leq \nu < q;$

(vi) $D_m = \mathrm{diag}(d_{1m}, \ldots, d_{nm})$ *and* $\kappa = \max\{d_{jm}^2\} \le 1$, $m \in \mathbb{N}_0$, $j = 1, 2, \ldots, n$.

Then the zero solution of (2.9) *is globally Mittag–Leffler stable in mean square.*

Proof. Define the Lyapunov function

$$V(x) = x^T x = \|x\|^2 .$$

For the sake of convenience, let

$$u(\cdot) = \sum_{j=1}^{\eta} C_j x(k-j) + Bg(x(k-\tau(k))),$$
$$\sigma(\cdot) = \sigma(k, x(k), x(k-\tau(k))).$$

Then we rewrite the impulse-free component of the neural network (2.9) as

$$x(k+1) = C_0 x(k) + Af(x(k)) + u(\cdot) + \sigma(\cdot)w(k).$$

From the above representation, for all $k \in [k_m, k_{m+1}-1]$, we have

$$\begin{aligned} V(k+1, x(k+1)) &= x^T(k+1)x(k+1) \\ &= x^T(k)C_0^T C_0 x(k) + x^T(k)C_0^T Af(x(k)) + x^T(k)C_0^T u(\cdot) + x^T(k)C_0^T \sigma(\cdot)w(k) \\ &\quad + f^T(x(k))A^T C_0 x(k) + f^T(x(k))A^T Af(x(k))) + f^T(x(k))A^T u(\cdot) + f^T(x(k))A^T \sigma(\cdot)w(k) \\ &\quad + u^T(\cdot)C_0 x(k) + u^T(\cdot)Af(x(k)) + u^T(\cdot)u(\cdot) + u^T(\cdot)\sigma(\cdot)w(k) \\ &\quad + \sigma^T(\cdot)C_0 x(k)w(k) + \sigma^T(\cdot)TAf(x(k))w(k) + \sigma^T(\cdot)u(\cdot)w(k) + w^T(k)\sigma^T(\cdot)\sigma(\cdot)w(k). \end{aligned}$$

Now, taking the expectation on both sides, and by virtue of (2.2), we obtain

$$\begin{aligned} \mathbb{E}[V(k+1, x(k+1))] \le \mathbb{E}\Big[& x^T(k)C_0^T C_0 x(k) + 2x^T(k)C_0^T Af(x(k)) + 2x^T(k)C_0^T u(\cdot) \\ & + f^T(x(k)A^T Af(x(k))) + 2f^T(x(k))A^T u(\cdot) + u^T(\cdot)u(\cdot) + \sigma^T(\cdot)\sigma(\cdot)\Big]. \end{aligned} \tag{4.1}$$

From Lemma 3.5 and assumption (H_2), it follows that

$$\begin{aligned} 2x^T(k)C_0^T Af(x(k)) &\le x^T(k)C_0^T A Q_1^{-1} A^T C_0 x(k) + f^T(x(k))Q_1 f(x(k)) \\ &\le x^T(k)\left[C_0^T A Q_1^{-1} A^T C_0 + L_f Q_1 L_f\right] x(k); \end{aligned}$$

$$\begin{aligned} 2x^T(k)C_0^T u(\cdot) &= 2x^T(k)C_0^T \left(\sum_{j=1}^{\eta} C_j x(k-j) + Bg(x(k-\tau(k))) \right) \\ &\le x^T(k)C_0^T C_1 Q_2^{-1} C_1^T C_0 x(k) + \tilde{x}^T(\cdot)Q_2\tilde{x}(\cdot) \\ &\quad + x^T(k)C_0^T B Q_3^{-1} B^T C_0 x(k) + g^T(x(k-\tau(k)))Q_3 g(x(k-\tau(k))) \\ &\le x^T(k)\left[C_0^T C_1 Q_2^{-1} C_1^T C_0 + C_0^T B Q_3^{-1} B^T C_0\right] x(k) \\ &\quad + \tilde{x}^T(\cdot)Q_2\tilde{x}(\cdot) + x^T(k-\tau(k))L_g Q_3 L_g x(k-\tau(k)); \end{aligned}$$

$$f^T(x(k))A^T Af(x(k)) \le x^T(k)L_f A^T A L_f x(k);$$

$$\begin{aligned} 2f^T(x(k))A^T u(\cdot) &= 2f^T(x(k))A^T \left(\sum_{j=1}^{\eta} C_j x(k-j) + Bg(x(k-\tau(k))) \right) \\ &\le f^T(x(k))A^T C_1 Q_4^{-1} C_1^T Af(x(k)) + \tilde{x}^T(\cdot)Q_4\tilde{x}(\cdot) \end{aligned}$$

$$\begin{aligned}
&+f^T(x(k))A^TBQ_5^{-1}B^TAf(x(k)) + g^T(x(k-\tau(k)))Q_5g(x(k-\tau(k)))\\
\leq\ & x^T(k)L_f\left[A^TC_1Q_4^{-1}C_1^TA + A^TBQ_5^{-1}B^TA\right]L_fx(k)\\
&+\tilde{x}^T(\cdot)Q_4\tilde{x}(\cdot) + x^T(k-\tau(k))L_gQ_5L_gx(k-\tau(k));
\end{aligned}$$

$$\begin{aligned}
u^T(\cdot)u(\cdot) &\leq \tilde{x}^T(\cdot)C_1^TC_1\tilde{x}(\cdot) + 2\tilde{x}^T(\cdot)C_1^TBg(x(k-\tau(k))) + g^T(x(k-\tau(k)))B^TBg(x(k-\tau(k)))\\
&\leq \tilde{x}^T(\cdot)C_1^TC_1\tilde{x}(\cdot) + \tilde{x}^T(\cdot)C_1^TBQ_6^{-1}B^TC_1\tilde{x}(\cdot) + g^T(x(k-\tau(k)))(Q_6+B^TB)g(x(k-\tau(k)))\\
&\leq \tilde{x}^T(\cdot)\left[C_1^TC_1 + C_1^TBQ_6^{-1}B^TC_1\right]\tilde{x}(\cdot) + x^T(k-\tau(k))L_g\left[Q_6+B^TB\right]L_gx(k-\tau(k)),
\end{aligned}$$

where

$$\tilde{x}(\cdot) = \sum_{j=1}^{\eta} x(k-j).$$

Also, from condition (H_3), we have that

$$\sigma^T(k,x(k),x(k-\tau(k)))\sigma(k,x(k),x(k-\tau(k))) \leq x^T(k)\theta_1x(k) + x^T(k-\tau(k))\theta_2x(k-\tau(k)).$$

Substituting all these into (4.1) and using (i)–(iii), we obtain

$$\begin{aligned}
\mathbb{E}[V(k+1,x(k+1))] \leq\ & \mathbb{E}\Big[x^T(k)\Big[C_0^T\left(I + AQ_1^{-1}A^T + C_1Q_2^{-1}C_1^T + BQ_3^{-1}B^T\right)C_0\\
&+L_f\left(Q_1 + A^TA + A^TC_1Q_4^{-1}C_1^TA + A^TBQ_5^{-1}B^TA\right)L_f + \theta_1 - \alpha I\Big]x(k)\\
&+x^T(k-\tau(k))\left[L_g(Q_3+Q_5+Q_6+B^TB)L_g + \theta_2 - \beta I\right]x(k-\tau(k))\\
&+\tilde{x}^T(\cdot)\left(Q_2 + Q_4 + C_1^TC_1 + C_1^TBQ_6^{-1}B^TC_1 - \gamma I\right)\tilde{x}(\cdot)\\
&+\alpha x^T(k)x(k) + \beta x^T(k-\tau(k))x(k-\tau(k)) + \gamma\tilde{x}^T(\cdot)\tilde{x}(\cdot)\Big]\\
\leq\ & \mathbb{E}\Big[x^T(k)\Omega_1x(k) + x^T(k-\tau(k))\Omega_2x(k-\tau(k))\\
&+\tilde{x}^T(\cdot)\Omega_3\tilde{x}(\cdot) + \alpha x^T(k)x(k) + \beta x^T(k-\tau(k))(k-\tau(k)) + \gamma\tilde{x}^T(\cdot)\tilde{x}(\cdot)\Big]\\
\leq\ & \alpha\mathbb{E}[V(k,x(k))] + \gamma\mathbb{E}[V(\cdot,\tilde{x}(\cdot))] + \beta\mathbb{E}[V(k-\tau(k),x(k-\tau(k)))],
\end{aligned}$$

where, by virtue of (i)–(iii)

$$\begin{aligned}
\Omega_1 &= C_0^T\left(I + AQ_1^{-1}A^T + C_1Q_2^{-1}C_1^T + BQ_3^{-1}B^T\right)C_0\\
&\quad +L_f\left(Q_1 + A^TA + A^TC_1Q_4^{-1}C_1^TA + A^TBQ_5^{-1}B^TA\right)L_f + \theta_1 - \alpha I \leq 0,\\
\Omega_2 &= L_g(Q_3+Q_5+Q_6+B^TB)L_g + \theta_2 - \beta I \leq 0,\\
\Omega_3 &= Q_2 + Q_4 + C_1^TC_1 + C_1^TBQ_6^{-1}B^TC_1 - \gamma I \leq 0.
\end{aligned}$$

For $V(\cdot,\tilde{x}(\cdot))$, using the matrices $Q_{7_{ij}}$, $1 \leq i,j \leq \eta$, from Lemma 3.5 and (iv), we have

$$\begin{aligned}
V(\cdot,\tilde{x}(\cdot)) &= \left(\sum_{j=1}^{\eta} x^T(k-j)\right)\left(\sum_{j=1}^{\eta} x(k-j)\right)\\
&\leq \sum_{j=1}^{\eta} x^T(k-j)x(k-j) + \sum_{j=1}^{\eta} x^T(k-j)\left(\sum_{\substack{j=1\\ j\neq i}}^{\eta}\left(Q_{7_{ji}}^{-1} + Q_{7_{ij}}\right)\right)x(k-j)\\
&\leq (1+\tilde{\gamma})\sum_{j=1}^{\eta} x^T(k-j)x(k-j).
\end{aligned}$$

From here and using the Razumikhin condition $V(k+\theta, x(k+\theta)) < V(k, x(k))$, $\theta \in \mathbb{N}_{-\tilde{\tau}}$, we get

$$\mathbb{E}[V(k+1, x(k+1))] \leq \mathbb{E}\left[(\alpha + \beta + \gamma\eta(1+\tilde{\gamma}))V(k, x(k))\right],$$

and hence by (v)

$$V(k+1, x(k+1)) \leq \nu V(k, x(k)), \quad k \in [k_m, k_{m+1} - 1].$$

Therefore, for $k \neq k_m - 1$, $m \in \mathbb{N}_0$, we have

$$\nabla^q V(k+1, x(k+1)) - \sum_{j=1}^{k+1} (-1)^j \binom{q}{j} V(k-j+1, x(k-j+1)) \leq \nu V(k, x(k)), \quad \in [k_m, k_{m+1} - 1].$$

From the properties of the functions $V \in \phi_0$ and Lemma 3.3, we have

$$\begin{aligned} \nabla^q V(k+1, x(k+1)) &\leq (\nu - q)V(k, x(k)) + \sum_{j=2}^{k+1} (-1)^j \binom{q}{j} V(k-j+1, x(k-j+1)) \\ &\leq (\nu - q)V(k, x(k)), \quad k \in [k_m, k_{m+1} - 1]. \end{aligned}$$

By virtue of condition (v), there exits a real number $\mu > 0$ such that

$$\nabla^q V(k+1, x(k+1)) \leq -\mu V(k, x(k)), \quad k \in [k_m, k_{m+1} - 1].$$

Also, for $m \in \mathbb{N}$ and $k = k_m$, from (vi), we get

$$\begin{aligned} V(k_m, x(k_m)) &= \|x(k_m)\|^2 = x_1^2(k_m) + x_2^2(k_m) + \ldots + x_n^2(k_m) \\ &= d_{1m}^2 x_1^2(k_m - 1) + d_{2m}^2 x_2^2(k_m - 1) + \ldots + d_{nm}^2 x_n^2(k_m - 1) \\ &\leq \kappa \|x(k_m - 1\|^2 = \kappa V(k_m - 1, x(k_m - 1)) \\ &\leq V(k_m - 1, x(k_m - 1)). \end{aligned}$$

Since all conditions of Corollary 3.4 are met, we obtain

$$V(k, x(k; k_0, \varphi_0)) \leq \sup_{-\tilde{\tau} \leq \theta \leq 0} V(k_0 + \theta, \varphi_0(\theta)) F_q(-\mu, (k - k_0)) \quad \text{for all} \quad k \geq k_0,$$

and thus

$$\mathbb{E}\left[\|x(k)\|^2\right] \leq \sup_{\theta \in \mathbb{N}_{-\tilde{\tau}}} \mathbb{E}\left[\|x(\theta)\|^2\right] F_q(-\mu, (k - k_0)) \quad \text{for all} \quad k \geq k_0,$$

which implies that the trivial solution of (2.9) is globally Mittag–Leffler stable in mean square. The proof is complete. □

Remark 4.2. When $q = 1$, then system (2.9) is reduced to the integer-order system of stochastic discrete-time Hopfield neural networks with impulsive effects (2.3), which has been studied in [42]. Therefore, our results improve and generalize some known results.

Remark 4.3. It is worth mentioning that, in contrast to some earlier results [42], the presented impulsive control law does not depend on the lengths of the impulsive intervals. Thus, our approach is less restrictive and conservative.

5. An Illustrative Example and Applications in Finance

Let $q = 0.6$. In order to demonstrate the effectiveness of the obtained stability criteria, we consider the NN fractional-order model

$$\begin{aligned} x(k+1) =& C_0 x(k) + \sum_{j=1}^{2} C_j x(k-j) + Af(x(k)) + Bg(x(k-\tau(k))) \\ &+ \sigma(k, x(k), x(k-\tau(k)))w(k), \quad k \neq k_m - 1, \\ x(k_m) =& D_m x(k_m - 1), \quad m \in \mathbb{N}_0, \end{aligned} \tag{5.1}$$

where

$$x(k) = (x_1(k), x_2(k))^T, \quad C_0 = \operatorname{diag}(-0.042, -0.033),$$
$$\theta_1 = 0.02I, \quad \theta_2 = 0.007I,$$
$$A = \begin{pmatrix} -0.03 & 0.01 \\ 0.02 & 0.01 \end{pmatrix}, \quad B = \begin{pmatrix} -0.3 & 0.2 \\ 0.3 & 0.2 \end{pmatrix}, \quad D_m = \operatorname{diag}(-0.2, -0.5),$$

and the impulsive moments satisfy $0 = k_0 < k_1 < k_2 < \ldots$ and $\lim_{m\to\infty} k_m = \infty$. Since $q = 0.6$, we have

$$C_1 = \operatorname{diag}(0.12, 0.12) \quad \text{and} \quad C_2 = \operatorname{diag}(0.056, 0.056).$$

Let

$$f_i(x_i) = 0.01 \tanh(x_i) \quad \text{and} \quad g_i(x_i) = 0.1\left(|x_i + 1| - |x_i - 1|\right).$$

Hence, the assumptions (H_1) and (H_2) are satisfied for

$$L_f = \operatorname{diag}(0.01, 0.01) \quad \text{and} \quad L_g = \operatorname{diag}(0.1, 0.1).$$

It is easy to check that all conditions of Theorem 4.1 are satisfied for

$$Q_1 = \begin{pmatrix} 0.01 & 0 \\ 0 & 0.01 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 0.03 & 0 \\ 0 & 0.04 \end{pmatrix}, \quad Q_3 = \begin{pmatrix} 0.03 & 0 \\ 0 & 0.07 \end{pmatrix},$$
$$Q_4 = \begin{pmatrix} 0.03 & 0 \\ 0 & 0.04 \end{pmatrix}, \quad Q_5 = \begin{pmatrix} 0.03 & 0 \\ 0 & 0.06 \end{pmatrix}, \quad Q_6 = \begin{pmatrix} 0.04 & 0 \\ 0 & 0.07 \end{pmatrix},$$
$$Q_{7_{12}} = Q_{7_{21}} = \begin{pmatrix} 0.001 & 0 \\ 0 & 0.0001 \end{pmatrix}, \quad \alpha = 0.29, \quad \beta = 0.1, \quad \gamma = 0.12,$$
$$\tilde{\gamma} = 0.0002, \quad \kappa = 0.25, \quad \nu = 0.55.$$

Therefore, the zero solution of the model (5.1) is globally Mittag–Leffler stable in mean square.

The advanced properties of system (2.9) make it a potential candidate to model in a variety of financial assets related problems. The proposed approach extends some existing NN integer-order models in finance reported in the literature and expands recently presented impulsive control results of fractional order. For example, the models of the type (2.9) can be considered as

- a fractional order generalization the NN model applied in *VaR analysis*, proposed by [38]. In addition to the ability to consider non-normality of returns our fractional-order approach is able to (i) allow greater flexibility in the model, (ii) take into account external impulsive effects, and (iii) deal with events that are relatively infrequent e.g., some changes in the level of volatility;
- a discrete fractional order generalization with mixed delays of the impulsive *investment model* introduced in [27];
- a discrete fractional order generalization with mixed delays of the impulsive *advertising model* introduced in [37, 43];
- a discrete impulsive generalization with mixed delays of the model introduced by [41] for *modeling of the national economics in state-space*.

6. Conclusions and Future Directions

In this paper, we set up a new NN approach for financial modeling using a discrete fractional order impulsive model with delays. The proposed model outperforms existing ones by offering a fractional-order difference operator, impulsive control strategy and time delays. By using the fractional Lyapunov method, sufficient conditions for global Mittag–Leffler stability in mean square of the zero solution of the model are obtained. Our results can be used to design an impulsive control law under which to stabilize the behavior of different types of impulsive fractional-order NN models of diverse interest.

An objective of our future investigations is to construct an extension of the proposed model with Markovian jumping parameters and mode-dependent delays which will improve and generalize some switching regime models [30]. Fractional Brownian motion which has shown promise in other financial models (see [20]) may also be taken under our future consideration.

Appendix A

The second method of Lyapunov is one of the universal methods for investigating the dynamical systems of a different type. The method is also known as a direct method of Lyapunov or a method of the Lyapunov functions. Also, the Lyapunov–Razumikhin technique has been applied successfully by various authors to study of stability problems for discrete delay systems. See, for example [23, 42] and the references therein. For impulsive discrete systems the following generalization of Lyapunov functions is used.

A function $V : \mathbb{N}_0 \times \mathbb{R}^n \to \mathbb{R}^+$ *belongs to class* ϕ_0 if

1. V is continuous on each of the sets $(k_{m-1}, k_m) \times \mathbb{R}^n$, and $V(k_m, 0) = 0$ for all $m \in \mathbb{N}_0$.

2. V is locally Lipschitz continuous with respect to its second argument $x \in \mathbb{R}^n$.

3. For each $m = 1, 2, \ldots$ and $x \in \mathbb{R}^n$, there exist the finite limits

$$V(k_m - 0, x) = \lim_{\substack{k \to k_m \\ k < k_m}} V(k, x), \quad V(k_m + 0, x) = \lim_{\substack{k \to k_m \\ k > k_m}} V(k, x),$$

and the equalities

$$V(k_m - 0, x) = V(k_m, x).$$

are valid.

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