Filomat 32:18 (2018), 6327–6337
https://doi.org/10.2298/FIL1818327G

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

A Study of the Quasi Covering Dimension of Alexandroff Countable Spaces Using Matrices

D.N. Georgiou[a], A.C. Megaritis[b], F. Sereti[a]

[a] *University of Patras, Department of Mathematics, 26500 Patras, Greece*
[b] *Technological Educational Institute of Peloponnese, Department of Computer Engineering, 23100, Sparta, Greece*

Abstract. The notion of Alexandroff space was firstly appeared in [1]. Different types of the covering dimension in the set of all Alexandroff countable spaces have been studied (see [5]). Inspired by [9], where a new topological dimension, called quasi covering dimension was developed, in this paper we study this new dimension in the set of all Alexandroff countable topological spaces using the matrix algebra. Especially, we characterize the open and dense subsets of an arbitrary Alexandroff countable space X using matrices. Under certain additional requirements on X, we provide a computational procedure for the determination of the quasi covering dimension of X.

1. Introduction

There are three kinds of dimension of a topological space: the covering dimension, the small inductive dimension and the large inductive dimension which have been studied in details (see, for example [3, 13]). Moreover, the meaning of the matrix gave different characterizations of some dimensions, such as of the covering dimension and the small inductive dimension (see [2, 6–8]).

Inspired by Dimension Theory a new dimension-function in the set of topological spaces, called quasi covering dimension, was inserted and many of its properties were studied (see [9]). The study of this new dimension in the set of finite topological spaces was completed by its investigation under the view of matrices (see [10]).

The beginning of the study of Alexandroff spaces was a result of the important role of finite spaces in digital topology and the fact that these spaces have all the properties of finite topological spaces which are related to such theory (see [12]). In digital spaces, image synthesis, image analysis and computer graphics, it is necessary to describe topological properties of n-dimensional digital image arrays (see [11, 14]). Therefore, the search for models, especially topological models, of the supports of such images was necessary. Alexandroff spaces have been studied as such topological models. But a problem on which research has been focused is the notion of dimension for Alexandroff spaces. A dimension for Alexandroff spaces, called Alexandroff dimension, which is essentially the small inductive dimension of [13], was

2010 *Mathematics Subject Classification*. Primary 54F45; Secondary 54A05, 65F30
Keywords. Quasi covering dimension, Alexandroff countable space, matrix theory
Received: 21 May 2018; Revised; 05 November 2018; Accepted: 12 November 2018
Communicated by Ljubiša D.R. Kočinac
The third author of this paper F. Sereti (with Scholarship Code: 2547) would like to thank the General Secretariat for Research and Technology (GSRT) and the Hellenic Foundation for Research and Innovation (HFRI) for the financial support of this study
Email addresses: georgiou@math.upatras.gr (D.N. Georgiou), thanasismeg13@gmail.com (A.C. Megaritis), seretifot@gmail.com (F. Sereti)

studied in [4] and [15]. Also, the Krull dimension of Alexandroff T_0-spaces came to enrich this research (see [16]) and recently in [5] the so-called covering dimension of Alexandroff spaces was studied.

In this paper, we study the quasi covering dimension in the set of Alexandroff countable topological spaces using matrices. Especially, in Section 2, we remind basic definitions and notations which are used in the rest of this study. In Section 3, we characterize the open and dense subsets of Alexandroff countable spaces using matrices and in Section 4, we study the quasi covering dimensions of those spaces, adapting the results of Section 3.

2. Preliminaries

In this section, we recall basic terminology and notions which will be used in this study.

First of all, we mention that an *Alexandroff space* is a space such that every point has a minimal open neighborhood or equivalently, the intersection of every family of open sets is open (see [1]). Also, a topological space X is called *countable* if the set X is countable.

Definition 2.1. Let X be a topological space.

(1) A *cover* c of X is a non-empty set of subsets of X whose union is X. In particular, a cover c of X is called *open* if all elements of c are open in X.

(2) A *quasi cover* c of X is a non-empty set of subsets of X whose union is dense in X. Especially, a quasi cover c of X is called *open* if all elements of c are open in X. Also, two quasi covers c_1 and c_2 of X are called *similar* (in short, we write $c_1 \sim c_2$) if their unions are the same dense subset of X.

(3) A family r of subsets of X is called a *refinement* of a family c of subsets of X if each element of r is contained in an element of c.

In what follows we consider two symbols "-1" and "∞" for which we suppose that:

(a) $-1 < k < \infty$, for every $k \in \{0, 1, \ldots\}$ and

(b) $\infty + k = k + \infty = \infty$, $-1 + k = k + (-1) = k$, for every $k \in \{0, 1, \ldots\} \cup \{-1, \infty\}$.

Definition 2.2. The *order* of a family r of subsets of a space X is defined as follows:

(1) $\mathrm{ord}(r) = -1$ iff r consists of the empty set only.

(2) $\mathrm{ord}(r) = k$, where $k \in \{0, 1, \ldots\}$, iff the intersection of any $k + 2$ distinct elements of r is empty and there exist $k + 1$ distinct elements of r whose intersection is not empty.

(3) $\mathrm{ord}(r) = \infty$ iff $\mathrm{ord}(r) \neq k$, for every $k \in \{-1, 0, 1, \ldots\}$.

Definition 2.3. The function dim, called *covering dimension*, with domain the set of all topological spaces and range the set $\{0, 1, \ldots\} \cup \{-1, \infty\}$, is defined as follows:

(1) $\dim(X) \leqslant k$, where $k \in \{-1, 0, 1, \ldots\}$, iff for every finite open cover c of X, there exists a finite open cover r of X, which is a refinement of c and satisfies the relation $\mathrm{ord}(r) \leqslant k$.

(2) $\dim(X) = k$, where $k \in \{0, 1, \ldots\}$, iff $\dim(X) \leqslant k$ and $\dim(X) \nleqslant k - 1$.

(3) $\dim(X) = \infty$ iff $\dim(X) \leqslant k$ does not hold for every $k \in \{-1, 0, 1, \ldots\}$.

Definition 2.4. ([9]) The function $\dim_q$, called *quasi covering dimension*, with domain the set of all topological spaces and range the set $\{0, 1, \ldots\} \cup \{-1, \infty\}$, is defined as follows:

(1) $\dim_q(X) \leqslant k$, where $k \in \{-1, 0, 1, \ldots\}$, iff for every finite open quasi cover c of X, there exists a finite open quasi cover r of X, which is a refinement of c, is similar to c and satisfies the relation $\mathrm{ord}(r) \leqslant k$.

(2) $\dim_q(X) = k$, where $k \in \{0, 1, \ldots\}$, iff $\dim_q(X) \leqslant k$ and $\dim_q(X) \nleqslant k - 1$.

(3) $\dim_q(X) = \infty$ iff $\dim_q(X) \leqslant k$ does not hold for every $k \in \{-1, 0, 1, \ldots\}$.

In the rest of this paper, we denote by $X = \{x_1, x_2, \dots\}$ an Alexandroff countable space and by U_i the smallest open set of X which contains the point x_i, $i = 1, 2, \dots$. Also, by ω we denote the first infinite cardinal and by $T = (t_{ij})$ we denote the $\omega \times \omega$ matrix, where

$$t_{ij} = \begin{cases} 1, & \text{if } x_i \in U_j \\ 0, & \text{otherwise} \end{cases}$$

which is called *incidence matrix* of X. We denote by $c_1, c_2, \dots$ the columns of the matrix T and by $\mathbf{1}$ the $\omega \times 1$ matrix

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \end{pmatrix}$$

which has all elements equal to 1. Finally, if A and B are two matrices of the same type, we denote by $\max(A + B)$ the largest element of the matrix $A + B$.

3. Open and Dense Subsets of Alexandroff Countable Spaces and Matrices

In this section we characterize the open and dense subsets D of an arbitrary Alexandroff countable space $X = \{x_1, x_2, \dots\}$ using matrices. Firstly, we observe that in an Alexandroff countable space X, each open subset D of X is either finite or countable, and the following example verifies this claim.

Example 3.1. We consider the Alexandroff countable space $X = \{x_1, x_2, x_3, \dots\}$ with the topology $\tau = \{\emptyset, \{x_1\}, \{x_1, x_2\}, \{x_1, x_3, x_4, \dots\}, X\}$. Then, the open set $\{x_1\}$ is finite and the open set $\{x_1, x_3, x_4, \dots\}$ is countable.

Thus, in order to study the open subsets D of an arbitrary Alexandroff countable topological space X which are, in parallel, and dense subsets in X, using matrices, we examine the following cases:
Case 1: D is finite
Case 2: D is countable

Case 1: We suppose that D is finite, writing $D = \{x_{i_1}, \dots x_{i_m}\}$, where $i_1, \dots i_m$ are distinct elements of the set $\{1, 2, \dots\}$.

Notation 3.2. We denote by $a_{i_1 \cdots i_m}$ and $b_{j_1 \cdots j_l}$, for some distinct elements $i_1, \dots i_m$ and $j_1, \dots, j_l$ of $\{1, 2, \dots\}$ the $\omega \times 1$ matrices

$$a_{i_1 \cdots i_m} = \begin{pmatrix} a^1_{i_1 \cdots i_m} \\ a^2_{i_1 \cdots i_m} \\ a^3_{i_1 \cdots i_m} \\ \vdots \end{pmatrix} \quad \text{and} \quad b_{j_1 \cdots j_l} = \begin{pmatrix} b^1_{j_1 \cdots j_l} \\ b^2_{j_1 \cdots j_l} \\ b^3_{j_1 \cdots j_l} \\ \vdots \end{pmatrix},$$

where

$$a^i_{i_1 \cdots i_m} = \begin{cases} 1, & \text{if } i \in \{i_1, \dots, i_m\} \\ 0, & \text{otherwise} \end{cases}$$

and

$$b^i_{j_1 \cdots j_l} = \begin{cases} 0, & \text{if } t_{ij_1} = \dots = t_{ij_l} = 0 \\ 1, & \text{otherwise.} \end{cases}$$

Proposition 3.3. *Let $i_1, \dots, i_m$ be distinct elements of the set $\{1, 2, \dots\}$. Then, $\{x_{i_1}, \dots, x_{i_m}\} = U_{j_1} \cup \dots \cup U_{j_l}$, for some $j_1, \dots, j_l \in \{i_1, \dots, i_m\}$ if and only if $a_{i_1 \cdots i_m} = b_{j_1 \cdots j_l}$.*

Proof. Let $\{x_{i_1}, \ldots, x_{i_m}\} = U_{j_1} \cup \ldots \cup U_{j_l}$, for some $j_1, \ldots, j_l \in \{i_1, \ldots, i_m\}$. We prove that $a_{i_1\cdots i_m} = b_{j_1\cdots j_l}$. For every $i \in \{1, 2, \ldots\}$, in the i-row of those matrices we have the following cases:

(1) $a^i_{i_1\cdots i_m} = 1 \Leftrightarrow i \in \{i_1, \ldots, i_m\} \Leftrightarrow x_i \in \{x_{i_1}, \ldots, x_{i_m}\}$
$\Leftrightarrow$ there exists $r \in \{1, \ldots, l\}$ such that $x_i \in U_{j_r}$
$\Leftrightarrow t_{ij_r} = 1 \Leftrightarrow b^i_{j_1\cdots j_l} = 1,$

(2) $a^i_{i_1\cdots i_m} = 0 \Leftrightarrow i \notin \{i_1, \ldots, i_m\} \Leftrightarrow x_i \notin \{x_{i_1}, \ldots, x_{i_m}\}$
$\Leftrightarrow x_i \notin U_{j_r}$, for each $r \in \{1, \ldots, l\}$
$\Leftrightarrow t_{ij_r} = 0$, for each $r \in \{1, \ldots, l\} \Leftrightarrow b^i_{j_1\cdots j_l} = 0.$

Therefore, $a_{i_1\cdots i_m} = b_{j_1\cdots j_l}$.

Conversely, we suppose that $a_{i_1\cdots i_m} = b_{j_1\cdots j_l}$, for some $j_1, \ldots, j_l \in \{i_1, \ldots, i_m\}$ and we prove that $\{x_{i_1}, \ldots, x_{i_m}\} = U_{j_1} \cup \ldots \cup U_{j_l}$. Let $x_i \in \{x_{i_1}, \ldots, x_{i_m}\}$. Then, $a^i_{i_1\cdots i_m} = 1$ and by assumption $b^i_{j_1\cdots j_l} = 1$. That is, there exists $r \in \{1, \ldots, l\}$ such that $t_{ij_r} = 1$ or equivalently $x_i \in U_{j_r}$ and thus $x_i \in U_{j_1} \cup \ldots \cup U_{j_l}$. Therefore, $\{x_{i_1}, \ldots, x_{i_m}\} \subseteq U_{j_1} \cup \ldots \cup U_{j_r}$. For the opposite side, let $x_i \in U_{j_1} \cup \ldots \cup U_{j_l}$. Then, there exists $r \in \{1, \ldots, l\}$ such that $x_i \in U_{j_r}$ or equivalently $t_{ij_r} = 1$. Thus, $b^i_{j_1\cdots j_l} = 1$ and by assumption $a^i_{i_1\cdots i_m} = 1$. Therefore, $x_i \in \{x_{i_1}, \ldots, x_{i_m}\}$ and so $U_{j_1} \cup \ldots \cup U_{j_l} \subseteq \{x_{i_1}, \ldots, x_{i_m}\}$. Thus, $\{x_{i_1}, \ldots, x_{i_m}\} = U_{j_1} \cup \ldots \cup U_{j_l}$. □

Corollary 3.4. *Let $i_1, \ldots, i_m$ be distinct elements of the set $\{1, 2, \ldots\}$. Then, we have $\{x_{i_1}, \ldots, x_{i_m}\} = U_{i_r}$, for some $r \in \{1, \ldots, m\}$ if and only if $a_{i_1\cdots i_m} = c_{i_r}$.*

Proof. By Proposition 3.3 we have $\{x_{i_1}, \ldots, x_{i_m}\} = U_{i_r}$, for some $r \in \{1, \ldots, m\}$ if and only if $a_{i_1\cdots i_m} = b_{i_r}$. Since $b_{i_r} = c_{i_r}$, we get the desired result. □

Proposition 3.5. *Let $j_1, \ldots, j_l$ be distinct elements of the set $\{1, 2, \ldots\}$. Then, the set $U_{j_1} \cup \ldots \cup U_{j_l}$ is dense in X if and only if $\max(b_{j_1\cdots j_l} + c_j) = 2$, for each $j \in \{1, 2, \ldots\} \setminus \{j_1, \ldots, j_l\}$.*

Proof. We suppose that the union $U_{j_1} \cup \ldots \cup U_{j_l}$ is a dense subset of X and we prove that $\max(b_{j_1\cdots j_l} + c_j) = 2$, for each $j \in \{1, 2, \ldots\} \setminus \{j_1, \ldots, j_l\}$. Let $j \in \{1, 2, \ldots\} \setminus \{j_1, \ldots, j_l\}$ and $k = \max(b_{j_1\cdots j_l} + c_j)$. By definitions of the matrices $b_{j_1\cdots j_l}$ and T we have either $k = 1$ or $k = 2$. Since $U_{j_1} \cup \ldots \cup U_{j_l}$ is dense in X, there exists $q \in \{1, \ldots, l\}$ such that $U_{j_q} \cap U_j \neq \emptyset$. Therefore, there exists $i_0 \in \{1, 2, \ldots\}$ such that $t_{i_0 j_q} = t_{i_0 j} = 1$. Thus, $b^{i_0}_{j_1\cdots j_l} + t_{i_0 j} = 2$ and therefore $k = 2$.

Now, we suppose that $\max(b_{j_1\cdots j_l} + c_j) = 2$, for each $j \in \{1, 2, \ldots\} \setminus \{j_1, \ldots, j_l\}$ and we prove that the set $U_{j_1} \cup \ldots \cup U_{j_l}$ is dense in X. In contrast, we suppose that it is not dense in X. Then, there exists an open set U of X such that

$$U \cap (U_{j_1} \cup \ldots \cup U_{j_l}) = \emptyset. \tag{3.1}$$

Therefore, there exists $\mu \in \{1, 2, \ldots\}$ such that $U_\mu \subseteq U$ and $x_\mu \notin U_{j_1} \cup \ldots \cup U_{j_l}$. By assumption, $\max(b_{j_1\cdots j_l} + c_\mu) = 2$. Thus, there exists $i_0 \in \{1, 2, \ldots\}$ such that $b^{i_0}_{j_1\cdots j_l} = t_{i_0\mu} = 1$. Therefore, $x_{i_0} \in U_{j_q} \cap U_\mu$, for some $q \in \{1, \ldots, l\}$ which contradicts the relation (3.1). Thus, the set $U_{j_1} \cup \ldots \cup U_{j_l}$ is dense in X. □

Since for every open subset $D = \{x_{i_1}, \ldots, x_{i_m}\}$ of an Alexandroff countable space X there exist elements $j_1, \ldots, j_l \in \{i_1, \ldots, i_m\}$ such that $D = U_{j_1} \cup \ldots \cup U_{j_l}$, from the above results we can get a characterization of the finite open and dense subsets D of an Alexandroff countable space X as follows:

Proposition 3.6. *Let $i_1, \ldots, i_m$ be distinct elements of the set $\{1, 2, \ldots\}$. Then, the set $D = \{x_{i_1}, \ldots, x_{i_m}\}$ is open and dense in X if and only if there exist elements $j_1, \ldots, j_l \in \{i_1, \ldots, i_m\}$ satisfying the following conditions:*
(1) $a_{i_1\cdots i_m} = b_{j_1\cdots j_l}$,

(2) $\max(b_{j_1\cdots j_l} + c_j) = 2$, *for each* $j \in \{1, 2, \ldots\} \setminus \{j_1, \ldots, j_l\}$.

Proof. We suppose that the set $D = \{x_{i_1}, \ldots, x_{i_m}\}$ is open and dense in X and we prove that there exist elements of $\{i_1, \ldots, i_m\}$ satisfying the conditions (1) and (2) of the proposition. As we have observed above, there exist $j_1, \ldots, j_l \in \{i_1, \ldots, i_m\}$ such that $D = U_{j_1} \cup \ldots \cup U_{j_l}$. By Proposition 3.3 we have $a_{i_1\cdots i_m} = b_{j_1\cdots j_l}$ and the condition (1) is satisfied. Also, since D is dense in X, by Proposition 3.5 $\max(b_{j_1\cdots j_l} + c_j) = 2$, for every $j_1, \ldots, j_l \in \{i_1, \ldots, i_m\}$ and the condition (2) is satisfied.

Conversely, we suppose that there exist elements $j_1, \ldots, j_l$ of $\{i_1, \ldots, i_m\}$ satisfying the conditions (1) and (2). We prove that the set D is open and dense in X. By Proposition 3.3, since the condition (1) is satisfied, we have $D = U_{j_1} \cup \ldots \cup U_{j_l}$ and thus D is open. Moreover, by Proposition 3.5, since the condition (2) is satisfied, the set D is dense in X. □

Example 3.7. We consider the Alexandroff countable space $X = \{x_1, x_2, x_3, \ldots\}$ with the topology $\tau = \{\emptyset, \{x_1\}, \{x_1, x_2\}, \{x_1, x_3, x_4, \ldots\}, X\}$. The incidence matrix T of X is the following matrix

$$T = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \ldots \\ 0 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 1 & 1 & 1 & 1 & \ldots \\ 0 & 0 & 1 & 1 & 1 & 1 & \ldots \\ 0 & 0 & 1 & 1 & 1 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix},$$

where $U_1 = \{x_1\}$, $U_2 = \{x_1, x_2\}$ and $U_3 = U_4 = \ldots = \{x_1, x_3, x_4, \ldots\}$. The finite set $\{x_1, x_2\}$ is open in X since

$$a_{12} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix} = b_2,$$

that is $\{x_1, x_2\} = U_2$. Also, it is dense in X since

$$b_2 + c_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix} \quad \text{and} \quad b_2 + c_j = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ \vdots \end{pmatrix},$$

for every $j = 3, 4, 5, \ldots$. Therefore, $\max(b_2 + c_j) = 2$, for every $j = 1, 3, 4, 5, \ldots$. But, the finite set $\{x_4, x_5\}$ is not open in X since

$$a_{45} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ \vdots \end{pmatrix} \neq b_4 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ \vdots \end{pmatrix}.$$

In similar way, since $b_4 = b_5 = b_{45}$, we have $a_{45} \neq b_5$ and $a_{45} \neq b_{45}$.

Case 2: We suppose that D is countable, writing $D = \{x_{i_1}, x_{i_2}, \ldots\}$, where $i_1, i_2, \ldots$ are distinct elements of the set $\{1, 2, \ldots\}$. In this case we must state that since X is Alexandroff, the set D is also Alexandroff with the corresponding subspace topology.

Notation 3.8. We put $I = \{i_1, i_2, \ldots\}$ and let $J = \{j_1, j_2, \ldots\}$ be a countable subset of $\{1, 2, \ldots\}$. We denote by a_I and b_J the $\omega \times 1$ matrices

$$a_I = \begin{pmatrix} a_I^1 \\ a_I^2 \\ a_I^3 \\ \vdots \end{pmatrix} \quad \text{and} \quad b_J = \begin{pmatrix} b_J^1 \\ b_J^2 \\ b_J^3 \\ \vdots \end{pmatrix},$$

where

$$a_I^i = \begin{cases} 1, & \text{if } i \in \{i_1, i_2, \ldots\} \\ 0, & \text{otherwise} \end{cases}$$

and

$$b_J^i = \begin{cases} 0, & \text{if } t_{ij_1} = t_{ij_2} = \ldots = 0, \\ 1, & \text{otherwise.} \end{cases}$$

Proposition 3.9. *Let $I = \{i_1, i_2, \ldots\} \subseteq \{1, 2, \ldots\}$. Then, the following conditions are satisfied:*
(1) $\{x_{i_1}, x_{i_2}, \ldots\} = U_{j_1} \cup \ldots \cup U_{j_l}$, for some $j_1, \ldots, j_l \in I$ if and only if $a_I = b_{j_1 \cdots j_l}$.
(2) $\{x_{i_1}, x_{i_2}, \ldots\} = \bigcup_{r=1}^{\infty} U_{j_r}$, for some countable subset $J = \{j_1, j_2, \ldots\}$ of I if and only if $a_I = b_J$.

Proof. (1) Firstly, we suppose that $\{x_{i_1}, x_{i_2}, \ldots\} = U_{j_1} \cup \ldots \cup U_{j_l}$, for some elements $j_1, \ldots, j_l \in I$. We prove that $a_I = b_{j_1 \cdots j_l}$. For every $i \in \{1, 2, \ldots\}$, in the i-row of these matrices we have:

a) $a_I^i = 1 \Leftrightarrow i \in \{i_1, i_2, \ldots\} \Leftrightarrow x_i \in \{x_{i_1}, x_{i_2}, \ldots\}$
$\Leftrightarrow$ there exists $r \in \{1, \ldots, l\}$ such that $x_i \in U_{j_r}$
$\Leftrightarrow t_{ij_r} = 1 \Leftrightarrow b^i_{j_1 \cdots j_l} = 1,$

b) $a_I^i = 0 \Leftrightarrow i \notin \{i_1, i_2, \ldots\} \Leftrightarrow x_i \notin \{x_{i_1}, x_{i_2}, \ldots\}$
$\Leftrightarrow x_i \notin U_{j_r}$, for each $r \in \{1, \ldots, l\}$
$\Leftrightarrow t_{ij_r} = 0$, for each $r \in \{1, \ldots, l\} \Leftrightarrow b^i_{j_1 \cdots j_l} = 0.$

Thus, $a_I = b_{j_1 \cdots j_l}$.

Conversely, we suppose that $a_I = b_{j_1 \cdots j_l}$, for some elements $j_1, \ldots, j_l \in I$ and we prove that $\{x_{i_1}, x_{i_2}, \ldots\} = U_{j_1} \cup \ldots \cup U_{j_l}$. Let $x_i \in \{x_{i_1}, x_{i_2}, \ldots\}$. Then, $a_I^i = 1$ and by assumption $b^i_{j_1 \cdots j_l} = 1$. That is, there exists $r \in \{1, \ldots, l\}$ such that $t_{ij_r} = 1$ or equivalently $x_i \in U_{j_r}$. Thus, $x_i \in U_{j_1} \cup \ldots \cup U_{j_l}$ and hence $\{x_{i_1}, x_{i_2}, \ldots\} \subseteq U_{j_1} \cup \ldots \cup U_{j_l}$. In the opposite side, let $x_i \in U_{j_1} \cup \ldots \cup U_{j_l}$. Then, there exists $r \in \{1, \ldots, l\}$ such that $x_i \in U_{j_r}$ or equivalently $t_{ij_r} = 1$. That is, $b^i_{j_1 \cdots j_l} = 1$ and by assumption $a_I^i = 1$. Therefore, $x_i \in \{x_{i_1}, x_{i_2}, \ldots\}$ and hence $U_{j_1} \cup \ldots \cup U_{j_l} \subseteq \{x_{i_1}, x_{i_2}, \ldots\}$. Thus, $\{x_{i_1}, x_{i_2}, \ldots\} = U_{j_1} \cup \ldots \cup U_{j_l}$.

(2) Let $\{x_{i_1}, x_{i_2}, \ldots\} = \bigcup_{r=1}^{\infty} U_{j_r}$, for some countable subset $J = \{j_1, j_2, \ldots\}$ of I. We prove that $a_I = b_J$. For every $i \in \{1, 2, \ldots\}$, in the i-row of these matrices we have:

a) $a_I^i = 1 \Leftrightarrow i \in \{i_1, i_2, \ldots\} \Leftrightarrow x_i \in \{x_{i_1}, x_{i_2}, \ldots\}$
$\Leftrightarrow$ there exists $r \in \{1, 2, \ldots\}$ such that $x_i \in U_{j_r}$
$\Leftrightarrow t_{ij_r} = 1 \Leftrightarrow b_J^i = 1,$

b) $a_I^i = 0 \Leftrightarrow i \notin \{i_1, i_2, \ldots\} \Leftrightarrow x_i \notin \{x_{i_1}, x_{i_2}, \ldots\}$

$$\Leftrightarrow x_i \notin U_{j_r}, \text{ for each } r \in \{1, 2, \ldots\}$$
$$\Leftrightarrow t_{ij_r} = 0, \text{ for each } r \in \{1, \ldots, l\} \Leftrightarrow b_J^i = 0.$$

Therefore, $a_I = b_J$.

Conversely, let $a_I = b_J$, for some countable subset $J = \{j_1, j_2, \ldots\}$ of I. We prove that $\{x_{i_1}, x_{i_2}, \ldots\} = \bigcup_{r=1}^{\infty} U_{j_r}$. Let $x_i \in \{x_{i_1}, x_{i_2}, \ldots\}$. Then, $a_I^i = 1$ and by assumption $b_J^i = 1$. That is, there exists $r \in \{1, 2, \ldots\}$ such that $t_{ij_r} = 1$ or equivalently $x_i \in U_{j_r}$. Thus, $\{x_{i_1}, x_{i_2}, \ldots\} \subseteq \bigcup_{r=1}^{\infty} U_{j_r}$. In the opposite side, let $x_i \in \bigcup_{r=1}^{\infty} U_{j_r}$. Then, there exists $r \in \{1, 2, \ldots\}$ such that $x_i \in U_{j_r}$ or equivalently $t_{ij_r} = 1$. Therefore, $b_J^i = 1$ and by assumption $a_I^i = 1$, that is $x_i \in \{x_{i_1}, x_{i_2}, \ldots\}$ and so $\bigcup_{r=1}^{\infty} U_{j_r} \subseteq \{x_{i_1}, x_{i_2}, \ldots\}$. Thus, $\{x_{i_1}, x_{i_2}, \ldots\} = \bigcup_{r=1}^{\infty} U_{j_r}$. □

Corollary 3.10. *Let $I = \{i_1, i_2, \ldots\} \subseteq \{1, 2, \ldots\}$. Then, we have $\{x_{i_1}, x_{i_2}, \ldots\} = U_{i_r}$, for some $r \in \{1, 2, \ldots\}$ if and only if $a_I = c_{i_r}$.*

Proof. By Proposition 3.9 $\{x_{i_1}, x_{i_2}, \ldots\} = U_{i_r}$, for some $r \in \{1, 2, \ldots\}$ if and only if $a_I = b_{i_r}$. Since $b_{i_r} = c_{i_r}$, we have the desired result. □

Proposition 3.11. *Let $J = \{j_1, j_2, \ldots\} \subseteq \{1, 2, \ldots\}$. Then, the set $\bigcup_{r=1}^{\infty} U_{j_r}$ is dense in X if and only if* $\max(b_J + c_j) = 2$, *for every $j \in \{1, 2, \ldots\} \setminus J$.*

Proof. Let $\bigcup_{r=1}^{\infty} U_{j_r}$ be a dense subset of X, $j \in \{1, 2, \ldots\} \setminus J$ and $\max(b_J + c_j) = k$. We prove that $k = 2$. By definitions of the matrices b_J and T we have either $k = 1$ or $k = 2$. Since the union is dense in X, there exists $r \in \{1, 2, \ldots\}$ such that $U_{j_r} \cap U_j \neq \emptyset$. Thus, there exists $i_0 \in \{1, 2, \ldots\}$ such that $t_{i_0 j_r} = t_{i_0 j} = 1$. Therefore, $b_J^{i_0} + t_{i_0 j} = 2$ and so $k = 2$.

Conversely, we prove that the union $\bigcup_{r=1}^{\infty} U_{j_r}$ is a dense subset of X. We suppose that it is not dense in X. Then, there exists an open set U of X such that $U \cap \bigcup_{r=1}^{\infty} U_{j_r} = \emptyset$. Thus, there exists $x_j \in U_j \subseteq U$ such that $x_j \notin \bigcup_{r=1}^{\infty} U_{j_r}$. By assumption $\max(b_J + c_j) = 2$. That is, there exists $i_0 \in \{1, 2, \ldots\}$ such that $b_J^{i_0} = t_{i_0 j} = 1$ or equivalently $x_{i_0} \in U_j \cap \bigcup_{r=1}^{\infty} U_{j_r}$ which is a contradiction. Thus, the union is dense in X. □

Since each open countable subset $D = \{x_{j_1}, x_{j_2}, \ldots\}$ of X can be written as $D = U_{j_1} \cup \ldots \cup U_{j_l}$, for some $j_1, \ldots, j_l \in \{i_1, i_2, \ldots\}$ or as $D = \bigcup_{r=1}^{\infty} U_{j_r}$, for some countable subset $\{j_1, j_2, \ldots\}$ of $\{i_1, i_2, \ldots\}$, we can have the following characterization of the countable open and dense subsets D of X.

Proposition 3.12. *Let $I = \{i_1, i_2, \ldots\} \subseteq \{1, 2, \ldots\}$. Then, the set $D = \{x_{i_1}, x_{i_2}, \ldots\}$ is open and dense in X if and only if one of the following conditions is satisfied:*

(1) $a_I = b_{j_1 \cdots j_l}$, for some $j_1, \ldots, j_l \in I$ and $\max(b_{j_1 \cdots j_l} + c_j) = 2$, *for every $j \in \{1, 2, \ldots\} \setminus \{j_1, \ldots, j_l\}$ or*

(2) $a_I = b_J$, for some countable subset $J = \{j_1, j_2, \ldots\}$ of I and $\max(b_J + c_j) = 2$, *for every $j \in \{1, 2, \ldots\} \setminus J$.*

Proof. We suppose that the set D is open and dense in X. Then, we have $D = U_{j_1} \cup \ldots \cup U_{j_l}$, for some $j_1, \ldots, j_l \in I$ or $D = \bigcup_{r=1}^{\infty} U_{j_r}$, for some countable subset $J = \{j_1, j_2, \ldots\}$ of I. Therefore, by Proposition 3.9 we have $a_I = b_{j_1 \cdots j_l}$ or $a_I = b_J$, respectively. Also, since D is dense in X, by Propositions 3.5 and 3.11 $\max(b_{j_1 \cdots j_l} + c_j) = 2$, for every $j \in \{1, 2, \ldots\} \setminus \{j_1, \ldots, j_l\}$ or $\max(b_J + c_j) = 2$, for every $j \in \{1, 2, \ldots\} \setminus J$, respectively.

Conversely, if the condition (1) of the proposition holds, then by Proposition 3.9 we have $D = U_{j_1} \cup \ldots \cup U_{j_l}$ and thus D is open in X and by Proposition 3.5 D is dense in X. If the condition (2) of the proposition holds, then by Proposition 3.9 we have $D = \bigcup_{r=1}^{\infty} U_{j_r}$ and thus D is open in X and by Proposition 3.11 D is dense in X. □

Example 3.13. We consider the Alexandroff countable space $X = \{x_1, x_2, x_3, \ldots\}$ with the topology $\tau = \{\emptyset, \{x_2\}, \{x_1, x_2\}, \{x_2, x_3, x_4, \ldots\}, X\}$. The incidence matrix T of X is the following matrix

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 1 & 1 & 1 & 1 & \ldots \\ 0 & 0 & 1 & 1 & 1 & 1 & \ldots \\ 0 & 0 & 1 & 1 & 1 & 1 & \ldots \\ 0 & 0 & 1 & 1 & 1 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix},$$

where $U_1 = \{x_1, x_2\}$, $U_2 = \{x_2\}$, $U_3 = U_4 = U_5 = \ldots = \{x_2, x_3, x_4, \ldots\}$. Let $D = \{x_2, x_3, x_4, \ldots\}$ and $I = \{2, 3, 4, \ldots\}$. Following Proposition 3.12 we observe that the set D is open and dense in X. Indeed, since

$$a_I = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ \vdots \end{pmatrix} = b_{23},$$

the set D is open in X. Moreover, it is dense in X since

$$b_{23} + c_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \\ 1 \\ 1 \\ \vdots \end{pmatrix} \quad \text{and} \quad b_{23} + c_j = \begin{pmatrix} 0 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ \vdots \end{pmatrix},$$

for every $j \in \{4, 5, \ldots\}$. Thus, $\max(b_{23} + c_j) = 2$, for every $j \in \{1, 2, \ldots\} \setminus \{2, 3\}$.

Now, let $D = \{x_3, x_4, x_5, \ldots\}$ and $I = \{3, 4, 5, \ldots\}$. The set D is not open in X since

$$a_I = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ \vdots \end{pmatrix} \neq \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ \vdots \end{pmatrix} = b_{j_1 \cdots j_l} = b_J,$$

for every $j_1, \ldots, j_l \in I$ and for every countable subset J of I.

4. Alexandroff Countable Spaces and Quasi Covering Dimension with Matrices

In this section, based on the results of Section 3, we compute the quasi covering dimension of an Alexandroff countable space $X = \{x_1, x_2, \ldots\}$ which satisfies certain additional requirements using matrices. For that we remind the following results:

Proposition 4.1. ([6]) *Let X be a finite space.*

(1) If some column of the matrix T is equal to $\mathbf{1}$, then $\dim(X) = 0$.

(2) If c_{j_i}, $i = 1, \ldots, m$, are m columns of the matrix T such that $c_{j_1} + \ldots + c_{j_m} \geqslant \mathbf{1}$ and $c_{r_1} + \ldots + c_{r_q} \ngeqslant \mathbf{1}$, for every $q < m$, then $\dim(X) = \max(c_{j_1} + \ldots + c_{j_m}) - 1$.

Proposition 4.2. ([5]) *Let X be an Alexandroff countable space which has at least an open finite cover $\{U_{i_1}, \ldots, U_{i_\mu}\}$.*

(1) If some column of the matrix T is equal to $\mathbf{1}$, then $\dim(X) = 0$.

(2) If c_{j_i}, $i = 1, \ldots, m$, are m columns of the matrix T such that $c_{j_1} + \ldots + c_{j_m} \geqslant \mathbf{1}$ and $c_{r_1} + \ldots + c_{r_q} \ngeqslant \mathbf{1}$, for every $q < m$, then $\dim(X) = \max(c_{j_1} + \ldots + c_{j_m}) - 1$.

Proposition 4.3. ([9]) *For every topological space X,* $\dim_q(X) = \sup\{\dim(D) : D$ *is open and dense in* $X\}$.

Agreement. In what follows, we consider an Alexandroff countable space $X = \{x_1, x_2, \ldots\}$ such that every open and dense subset $D = \{x_{i_1}, x_{i_2}, \ldots\}$ of X (in particular, the whole space X) can be written as $D = U_{j_1} \cup \ldots \cup U_{j_l}$, for some $j_1, \ldots, j_l \in \{i_1, i_2, \ldots\}$ (see [5, Proposition 2.1]).

Following Propositions 3.6 and 3.12 we can recognize the open and dense subsets of X through the matrix theory and from Propositions 4.1 and 4.2 we can compute each of their covering dimension by matrices. Therefore, from Proposition 4.3, the matrix-computation of the quasi covering dimension of X is succeeded.

Undoubtedly, the topological space X is defined by its topology or its base, each of which consists of open sets of X and their unions. In the following process, using matrices, we verify that these sets, finite or countable, are open, we examine which of these sets are dense in X and we see that any other subset of X is not open and dense in X. Therefore, the choice of $\mathcal{F}$ and C of all finite and countable subsets of X, respectively, in the following proceeding is absolutely determined by the topology of X. Thus, the proceeding of computing the quasi covering dimension $\dim_q$ of X is described as follows:

Step 1: Find the incidence matrix T of X and go to Step 2.

Step 2: Find $d = \dim(X)$ (use Proposition 4.2) and go to Step 3.

Step 3: Find the set $\mathcal{F}$ of all finite subsets $\{i_1, \ldots, i_m\}$ of $\{1, 2, \ldots\}$ for which there exist $j_1, \ldots, j_l \in \{i_1, \ldots, i_m\}$ satisfying the following properties:

(1) $a_{i_1 \cdots i_m} = b_{j_1 \cdots j_l}$,

(2) $\max(b_{j_1 \cdots j_l} + c_j) = 2$, for each $j \in \{1, 2, \ldots\} \setminus \{j_1, \ldots, j_l\}$.

If $\mathcal{F} = \emptyset$, then put $f = 0$ and go to Step 4. Otherwise, use Proposition 4.1 to print

$$f = \max\{\dim(\{x_{i_1}, \ldots, x_{i_m}\}) : \{i_1, \ldots, i_m\} \in \mathcal{F}\}$$

and go to Step 4.

Step 4: Find the set C of all countable proper subsets $I = \{i_1, i_2, \ldots\}$ of $\{1, 2, \ldots\}$ for which there exist $j_1, \ldots, j_l \in I$ satisfying the following properties:

(1) $a_I = b_{j_1 \cdots j_l}$,

(2) $\max(b_{j_1 \cdots j_l} + c_j) = 2$, for each $j \in \{1, 2, \ldots\} \setminus \{j_1, \ldots, j_l\}$.

If $C = \emptyset$, then put $c = 0$ and go to Step 5. Otherwise, since each $\{x_{i_1}, x_{i_2}, \ldots\}$, $\{i_1, i_2, \ldots\} \in C$ satisfies our agreement, use Proposition 4.2 to print

$$c = \max\{\dim(\{x_{i_1}, x_{i_2}, \ldots\}) : \{i_1, i_2, \ldots\} \in C\}$$

and go to Step 5.

Step 5: Print $\dim_q(X) = \max\{d, f, c\}$.

Example 4.4. We consider the Alexandroff topological space X of Example 3.13. We will compute the quasi covering dimension of X by performing the following steps.

Step 1: As we have seen in Example 3.13 the incidence matrix T of X is:

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 1 & 1 & 1 & 1 & \dots \\ 0 & 0 & 1 & 1 & 1 & 1 & \dots \\ 0 & 0 & 1 & 1 & 1 & 1 & \dots \\ 0 & 0 & 1 & 1 & 1 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

Step 2: Since

$$c_1 + c_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \\ 1 \\ 1 \\ \vdots \end{pmatrix} \geqslant \mathbf{1},$$

by Proposition 4.2 we have $d = \dim(X) = \max(c_1 + c_3) - 1 = 2 - 1 = 1$. Therefore, by Proposition 4.3 we have $\dim_q(X) \geqslant 1$.

Step 3: We have $\mathcal{F} = \{\{2\}, \{1, 2\}\}$ with $f = \max\{\dim(\{x_2\}), \dim(\{x_1, x_2\})\} = 0$.

Step 4: We have $C = \{\{2, 3, 4, \dots\}\}$ with $c = \dim(\{x_2, x_3, x_4, \dots\}) = 0$.

Step 5: Therefore, $\dim_q(X) = \max\{d, f, c\} = 1$.

Example 4.5. Let the following sets $X = \{x_1, x_2, x_3, \dots\}$, $W_1 = \{x_1, x_4, x_5, x_6, \dots\}$, $W_2 = \{x_2, x_4, x_5, x_6, \dots\}$ and $W_i = \{x_i, x_{i+1}, x_{i+2}, x_{i+3}, \dots\}$, for $i = 3, 4, 5, \dots$ be given. We consider the Alexandroff countable space (X, τ), where τ is the topology which is generated by the family $\{\emptyset, W_1, W_2, W_3, \dots, X\}$. The quasi covering dimension of X can be found via the following steps.

Step 1: The incidence matrix T of X is the following matrix:

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & 0 & \dots \\ 1 & 1 & 1 & 1 & 0 & 0 & \dots \\ 1 & 1 & 1 & 1 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

since $W_i = U_i$, for each $i = 1, 2, 3, \dots$.

Step 2: For the space X we have:

$$c_1 + c_2 + c_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 3 \\ 3 \\ 3 \\ \vdots \end{pmatrix} \geqslant \mathbf{1},$$

$c_r \neq \mathbf{1}$, for every $r \in \{1, 2, \ldots\}$ and $c_{r_1} + c_{r_2} \ngeqslant \mathbf{1}$, for every $r_1, r_2 \in \{1, 2, \ldots\}$ with $r_1 \neq r_2$. Thus, $d = \dim(X) = 2$ and by Proposition 4.3 we have $\dim_q(X) \geqslant 2$.

Step 3: We have $\mathcal{F} = \emptyset$. Thus, $f = 0$.

Step 4: The set C consists of the countable sets $\{1, 4, 5, 6, \ldots\}$, $\{2, 4, 5, 6, \ldots\}$, $\{1, 2, 4, 5, 6, \ldots\}$, $\{1, 3, 4, 5, 6, \ldots\}$, $\{2, 3, 4, 5, 6, \ldots\}$, and $\{i, i+1, i+2, \ldots\}$, where $i = 3, 4, 5, \ldots$. For the covering dimension $\dim(\{x_1, x_4, x_5, x_6, \ldots\})$ we observe that the corresponding incidence matrix of W_1 is the following matrix:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 1 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 1 & 1 & 0 & 0 & \ldots \\ 1 & 1 & 1 & 1 & 1 & 0 & \ldots \\ 1 & 1 & 1 & 1 & 1 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

Since the first column of the above matrix is equal to $\mathbf{1}$, we have $\dim(W_1) = 0$. Similarly, $\dim(W_i) = 0$, for every $i = 2, 3, 4, \ldots$. Following the same thinking, we observe that $\dim(\{x_1, x_2, x_4, x_5, x_6, \ldots\}) = 1$, $\dim(\{x_1, x_3, x_4, x_5, x_6, \ldots\}) = 1$ and $\dim(\{x_2, x_3, x_4, x_5, x_6, \ldots\}) = 1$. Therefore, $c = 1$.

Step 5: Thus, $\dim_q(X) = \max\{d, f, c\} = 2$.

Acknowledgements

The authors would like to thank both referees for the careful reading of the paper and the useful comments.

References

[1] P. Alexandroff, Diskrete Räume, Mat. Sb. 2 (1937) 501–518.
[2] D. Boyadzhiev, D.N. Georgiou, A.C. Megaritis, F. Sereti, A study of a covering dimension of finite lattices, Appl. Math. Comput. 333 (2018) 276–285.
[3] R. Engelking, Theory of Dimensions, Finite and Infinite, Heldermann Verlag, Berlin, 1995.
[4] A.V. Evako, R. Kopperman, Y.V. Mukhin, Dimensional properties of graphs and digital spaces, J. Math. Imaging Vision 6(2-3) (1996) 109–119.
[5] D.N. Georgiou, S.-E. Han, A.C. Megaritis, Dimensions of the type dim and Alexandroff spaces, J. Egyptian Math. Soc. 21 (2013) 311–317.
[6] D.N. Georgiou, A.C. Megaritis, Covering dimension and finite spaces, Appl. Math. Comput. 218 (2014) 3122–3130.
[7] D.N. Georgiou, A.C. Megaritis, An algorithm of polynomial order for computing the covering dimension of a finite space, Appl. Math. Comput. 231 (2014) 276–283.
[8] D.N. Georgiou, A.C. Megaritis, S. Moshokoa, A computing procedure for the small inductive dimension of a finite T_0-space, Comput. Appl. Math. 34 (2015) 401–415.
[9] D.N. Georgiou, A.C. Megaritis, F. Sereti, A topological dimension greater than or equal to the classical covering dimension, Houston J. Math. 43 (2017) 283–298.
[10] D.N. Georgiou, A.C. Megaritis, F. Sereti, A study of the quasi covering dimension of finite spaces using matrix theory, Hacet. J. Math. Statist. 46 (2017) 124–138.
[11] E. Khalimsky, R. Kopperman, P.R. Meyer, Computer graphics and connected topologies on finite ordered sets, Topology Appl. 36 (1990) 1–17.
[12] E.H. Kronheimer, The topology of digital images, Special issue on digital topology, Topology Appl. 46 (1992) 279–303.
[13] A.R. Pears, Dimension theory of general spaces, Cambridge University Press, Cambridge, 1975.
[14] A. Rosenfeld, Digital topology, Amer. Math. Monthly 86(8) (1979) 621–630.
[15] P. Wiederhold, R.G. Wilson, Dimension for Alexandroff spaces, In: Vision Geometry, R.A. Melter and A.Y. Wu (Eds.), Proceedings of The Society of Photo-Optical Instrumentation Engineers (SPIE) 1832 (1993) 13–22.
[16] P. Wiederhold, R.G. Wilson, The Krull dimension of Alexandroff T_0-spaces, Papers on General Topology and Applications, Ann. New York Acad. Sci. 806 (1996) 444–453.