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Classes of Analytic Functions Related to Blaschke Products

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Abstract. Basing on the well known Riesz Theorem on the canonical factorization of bounded analytic functions, we distinguished the subfamilies of the Carathéodory class of functions. The basic properties of introduced classes have been proved. Related classes of analytic functions are discussed also.

1. Introduction

Let $\mathcal{H}$ be the class of all analytic functions in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. Subclasses of $\mathcal{H}$, particularly subclasses of univalent functions are the basic subject to study in the geometric function theory. There are several classic methods of distinguishing such subfamilies. One of them is based on geometrical view. In this way e.g., Study [15] in 1913 introduced convex functions, Alexander [1] in 1915 starlike functions and Robertson [14] in 1936 functions convex in one direction. References for these and other classes see e.g., [5]. Many of such families of functions have an analytical description expressed in term of Carathéodory class of functions, i.e., the family $\mathcal{P}$ of functions $p \in \mathcal{H}$ normalized by $p(0) := 1$ with a positive real part. Let us say that $\mathcal{F}$ in $\mathcal{H}$ is such a class. Therefore each subclass, say $\widetilde{\mathcal{P}}$ of $\mathcal{P}$, creates a corresponding subclass $\widetilde{\mathcal{F}}$ in $\mathcal{F}$. One can restrict the range of $p(\mathbb{D})$ to a given domain e.g., to a sector, a disk, a conic domain etc. lying in the right halfplane to define the subclass $\widetilde{\mathcal{P}}$ and the corresponding subclass $\widetilde{\mathcal{F}}$. One can modify the power series of $p \in \mathcal{P}$, e.g., by removing a polynomial part. Given $m \in \mathbb{N}$, consider a subfamily $\mathcal{P}_m$ of $\mathcal{P}$ of all p of the form

$$p(z) = 1 + \sum_{k=m}^{\infty} c_k z^k, \quad z \in \mathbb{D}. \tag{1}$$

One can restrict the coefficients of the above series to real, negative etc. The concepts of α-convexity and γ-starlikeness were at the base of the subclass construction technique illustrate the technique by using of real or complex parameters. This method has been intensively developed over the last 50 years.

In this paper, we propose a method of distinguishing subclasses in $\mathcal{P}$ based on famous Riesz Theorem on the factorization of functions in the Hardy classes, so in particular of bounded analytic functions. Since there is a one to one relationship between the class $\mathcal{P}$ and the class $\mathcal{B}_0$ of Schwarz functions, i.e., of self

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analytic mapping of $\mathbb{D}$ with a fixed point at the origin and in general between the class $\mathcal{P}_m$ and the class $\mathcal{B}_{0,m}$ of Schwarz functions having zero at the origin of order of at least m, the factorization of the class $\mathcal{B}_0$ can be transferred to the class $\mathcal{P}$, so in the next step to the corresponding class $\mathcal{F}$. Thus classes such as starlike, convex and other can be factorized through the suitable representation of $\mathcal{P}$. What is important, they are defined by zeros of Schwarz functions. In this way, the geometry of zeros both of their module and of oscillation plays a fundamental role for the corresponding families in $\mathcal{H}$.

In addition to the definition of new classes, the main purpose of the work is to prove the basic theorems, i.e., growth and distortion theorems (Theorems 3.3 and 3.7) and Theorem 3.5 in the newly introduced families of Carathéodory functions that are tools for examining other classes. Next we apply them to discuss the radii of convexity. In the last section we show that the upper bounds of the classical well known coefficient functionals on the whole class $\mathcal{P}$ can be expressed in therm of zeros when instead of $\mathcal{P}$ we take its subclass related to the same zeros and that the new results are more detailed than the classical.

Given $r \in (0,1)$, let $\mathbb{T}_r := \{z \in \mathbb{C} : |z| = r\}$ and let $\mathbb{T} := \mathbb{T}_1$. Let $\mathcal{B}$ be the class of all $\omega \in \mathcal{H}$ such that $|\omega(z)| \le 1$ for $z \in \mathbb{D}$, and $\mathcal{B}^0$ be its subclass of non-vanishing functions in $\mathbb{D}$. Let $\mathcal{B}_0$ be the class of Schwarz functions, i.e., of all $\omega \in \mathcal{B}$ keeping the origin fixed.

Let $\mathbb{D}^0 := \mathbb{D} \setminus \{0\}$. Given $\alpha \in \mathbb{D}$, let

$$\varphi_\alpha(z) := \frac{z-\alpha}{1-\overline{\alpha}z}, \quad z \in \mathbb{D},$$

denote the Blaschke factor. A sequence of points $\mathit{\Lambda} = (\alpha_k) \in (\mathbb{D}^0)^\infty$ is said to satisfy the Blaschke condition if

$$\sum_{k=1}^{\infty}(1-|\alpha_k|) < \infty,$$

which guaranties convergence of the product

$$B_{\mathit{\Lambda}}(z) := \prod_{k\in\mathbb{N}} \frac{-|\alpha_k|}{\alpha_k}\varphi_{\alpha_k}(z), \quad z \in \mathbb{D}.$$

A function $B(z) := z^m B_{\mathit{\Lambda}}(z),\ z \in \mathbb{D}$, with $m \in \mathbb{N} \cup \{0\}$, is called the Blaschke product. When $\mathit{\Lambda}(\mathbb{N}) = \emptyset$, set $B_{\mathit{\Lambda}}(z) := 1,\ z \in \mathbb{D}$, and then

$$B_{\mathit{\Lambda}}(z) = z^m, \quad z \in \mathbb{D}.$$

Given $k \in \mathbb{N}$, let $\Lambda_k := (\mathbb{D}^0)^k$ and let $\Lambda_0 := \{0\}^1$. Let Λ_∞ be the subset of $(\mathbb{D}^0)^\infty$ of all sequences which satisfy the Blaschke condition. Let

$$\Lambda := \bigcup_{k\in\mathbb{N}\cup\{0,\infty\}} \Lambda_k.$$

2. Subclasses of analytic functions

For $f \in \mathcal{H}$ let $Z(f)$ be the set of all zeros of f in $\mathbb{D}^0$ counting with their multiplicities. Since $Z(f)$ is countable, it can be considered as an element of Λ. As it is known, the sequence $Z(\omega)$ of each bounded analytic function ω, so in particular, of each Schwarz function, satisfies the Blaschke condition. Moreover, by Riesz Theorem (e.g., [6, p. 283], [2, p. 20]) each $\omega \in \mathcal{B}_0$ has a unique canonical factorization

$$\omega(z) = z^m B_{Z(\omega)}\varphi(z), \quad z \in \mathbb{D}, \tag{2}$$

where $m \in \mathbb{N}$ and $\varphi \in \mathcal{B}^0$. Thus $B(z) = z^m B_{Z(\omega)}(z)$ for $z \in \mathbb{D}$, is the Blaschke product with the same zeros as the function ω. In additional, the function φ can be uniquely represented as a product of some inner and some outer function. Vice versa, each function

$$\omega(z) := z^m B_{\mathit{\Lambda}}(z)\varphi(z), \quad z \in \mathbb{D}, \tag{3}$$

with $m \in \mathbb{N}$, $\mathit{\Lambda} \in \Lambda$ and $\varphi \in \mathcal{B}^0$, is a Schwarz function. This is a starting point for further considerations.

Definition 2.1. *Given $m \in \mathbb{N}$ and $\Lambda \in \Lambda$, let $\mathcal{B}(m, \Lambda)$ denote the class of functions of the form* (3)*, where $\varphi \in \mathcal{B}$.*
Let $\mathcal{B}^0(m, \Lambda)$ be the class of functions of the form (3)*, where $\varphi \in \mathcal{B}^0$.*
When $B_\Lambda \equiv 1$, i.e., when $\Lambda(\mathbb{N}) = \emptyset$, we will write $\mathcal{B}(m)$ and $\mathcal{B}^0(m)$ instead of $\mathcal{B}(m, \Lambda)$ and $\mathcal{B}^0(m, \Lambda)$, respectively.

Closely related to the class $\mathcal{B}_0$ is the class $\mathcal{P}$. Namely, if $\omega \in \mathcal{B}_0$, then

$$p := \frac{1+\omega}{1-\omega} \in \mathcal{P}. \tag{4}$$

Vice versa, if $p \in \mathcal{P}$, then

$$\omega := \frac{1-p}{1+p} \in \mathcal{B}_0. \tag{5}$$

Definition 2.2. *Given $m \in \mathbb{N}$ and $\Lambda \in \Lambda$, let $\mathcal{P}(m, \Lambda)$ denote the class of functions $p \in \mathcal{H}$ of the form*

$$p(z) = \frac{1 + z^m B_\Lambda(z)\varphi(z)}{1 - z^m B_\Lambda(z)\varphi(z)}, \quad z \in \mathbb{D},\ \varphi \in \mathcal{B}. \tag{6}$$

Let $\mathcal{P}^0(m, \Lambda)$ be the class of functions of the form (6)*, where $\varphi \in \mathcal{B}^0$.*
When $B_\Lambda \equiv 1$, we will write $\mathcal{P}(m)$ and $\mathcal{P}^0(m)$ instead of $\mathcal{P}(m, \Lambda)$ and $\mathcal{P}^0(m, \Lambda)$, respectively.

Clearly, $\mathcal{P}(m, \Lambda) \subset \mathcal{P}_m$. When $p \in \mathcal{P}$, then ω given by (5) is in $\mathcal{B}_0$, so has a unique factorization (2). Hence and by (4) a function p is uniquely representation as

$$p(z) = \frac{1 + z^m B(z)\varphi(z)}{1 - z^m B(z)\varphi(z)}, \quad z \in \mathbb{D},$$

with $\varphi \in \mathcal{B}^0$.

The Carathéodory class is the basic tool for analytic description of the well known classes of analytic functions. Let us recall some of them. Given $m \in \mathbb{N}$, let $\mathcal{A}_m$ of be the subset of $\mathcal{H}$ of all f of the form

$$f(z) = z + \sum_{k=m}^{\infty} a_{k+1} z^{k+1}, \quad z \in \mathbb{D}.$$

Let $\mathcal{A} := \mathcal{A}_1$ and let $\mathcal{S}$ be the class of all univalent functions in $\mathcal{A}$.

Let $\mathcal{S}^*$ be the class of starlike functions, i.e., of all $f \in \mathcal{A}$ such that

$$zf'(z) = f(z)p(z), \quad z \in \mathbb{D},$$

for some $p \in \mathcal{P}$ (see e.g., [3, p. 41]). The class $\mathcal{S}^*$ contains all univalent functions in $\mathcal{A}$ which map a disk $\mathbb{D}$ onto starlike domains with respect to the origin.

Let $\mathcal{S}^c$ be the class convex functions, i.e., of all $f \in \mathcal{A}$ such that

$$1 + \frac{zf''(z)}{f'(z)} = p(z), \quad z \in \mathbb{D},$$

for some $p \in \mathcal{P}$ (see e.g., [3, p. 42]). The class $\mathcal{S}^c$ contains all univalent functions in $\mathcal{A}$ which map a disk $\mathbb{D}$ onto convex domains.

Let $\mathcal{T}$ be the class of all functions $f \in \mathcal{A}$ such that

$$f(z) = zp(z), \quad z \in \mathbb{D},$$

for some $p \in \mathcal{P}$. The class $\mathcal{T}$ was considered by many authors (see e.g., [9], [10]).

Let $\mathcal{P}'$ be the class of functions of bounded rotation, i.e., of all $f \in \mathcal{A}$ such that

$$f'(z) = p(z), \quad z \in \mathbb{D}, \tag{7}$$

for some $p \in \mathcal{P}$ (see e.g., [5, Vol. I, p. 101]). The condition (7) is the well known criterium of univalence due to Noshiro [13] and Warschawski [16] (see also [5, p. 88]).

Given $m \in \mathbb{N}$, let

$$\mathcal{S}^*_m := \mathcal{S}^* \cap \mathcal{A}_m, \quad \mathcal{S}^c_m := \mathcal{S}^c \cap \mathcal{A}_m, \quad \mathcal{T}_m := \mathcal{T} \cap \mathcal{A}_m, \quad \mathcal{P}'_m := \mathcal{P}' \cap \mathcal{A}_m.$$

Having classes $\mathcal{P}(m, \Lambda)$ we now define some classes of analytic functions. Many others can be defined in a similar way.

Definition 2.3. *Given $m \in \mathbb{N}$ and $\Lambda \in \Lambda$, let*

1. *$\mathcal{T}(m, \Lambda)$ denote the class of functions $f \in \mathcal{A}$ such that*

$$f(z) = z\frac{1 + z^m B_\Lambda(z)\varphi(z)}{1 - z^m B_\Lambda(z)\varphi(z)}, \quad z \in \mathbb{D}, \ \varphi \in \mathcal{B}; \tag{8}$$

2. *$\mathcal{P}'(m, \Lambda)$ denote the class of functions $f \in \mathcal{A}$ such that*

$$f'(z) = \frac{1 + z^m B_\Lambda(z)\varphi(z)}{1 - z^m B_\Lambda(z)\varphi(z)}, \quad z \in \mathbb{D}, \ \varphi \in \mathcal{B}; \tag{9}$$

3. *$\mathcal{S}^*(m, \Lambda)$ denote the class of functions $f \in \mathcal{A}$ such that*

$$zf'(z) = f(z)\frac{1 + z^m B_\Lambda(z)\varphi(z)}{1 - z^m B_\Lambda(z)\varphi(z)}, \quad z \in \mathbb{D}, \ \varphi \in \mathcal{B}; \tag{10}$$

4. *$\mathcal{S}^c(m, \Lambda)$ denote the class of functions $f \in \mathcal{A}$ such that*

$$1 + \frac{zf''(z)}{f'(z)} = \frac{1 + z^m B_\Lambda(z)\varphi(z)}{1 - z^m B_\Lambda(z)\varphi(z)}, \quad z \in \mathbb{D}, \ \varphi \in \mathcal{B}. \tag{11}$$

5. *Let $\mathcal{T}^0(m, \Lambda)$, $\mathcal{P}'^{,0}(m, \Lambda)$, $\mathcal{S}^{*,0}(m, \Lambda)$ and $\mathcal{S}^{c,0}(m, \Lambda)$ be classes of functions satisfying* (8)-(11)*, respectively, where $\varphi \in \mathcal{B}^0$.*

When $\Lambda(\mathbb{N}) = \emptyset$, for short we will write $\mathcal{T}(m)$, $\mathcal{P}'(m)$, $\mathcal{S}^*(m)$ and $\mathcal{S}^c(m)$ for the corresponding classes.

Clearly, $\mathcal{T}(m, \Lambda) \subset \mathcal{T}_m$, $\mathcal{P}'(m, \Lambda) \subset \mathcal{P}'_m$, $\mathcal{S}^*(m, \Lambda) \subset \mathcal{S}^*_m$ and $\mathcal{S}^c(m, \Lambda) \subset \mathcal{S}^c_m$.

3. Growth and Distortion Theorems

Given $r \in (0, 1)$ and $f \in \mathcal{H}$, let

$$M_r(f) := \max_{z \in \mathbb{T}_r} |f(z)| = \max_{z \in \overline{\mathbb{D}}_r} |f(z)|.$$

In particular, let $M_r(\Lambda) := M_r(B_\Lambda)$.

We will now prove the growths theorems for the class $\mathcal{P}(m, \Lambda)$.

Theorem 3.1. *Let $m \in \mathbb{N}$ and $\Lambda \in \Lambda$. If $p \in \mathcal{P}(m, \Lambda)$ has the form* (6)*, then for $z \in \mathbb{D}$,*

$$|p(z)| \le \frac{1 + |\varphi(0)||z| + |z|^m(|\varphi(0)| + |z|)|B_\Lambda(z)|}{1 + |\varphi(0)||z| - |z|^m(|\varphi(0)| + |z|)|B_\Lambda(z)|} \tag{12}$$

and

$$\operatorname{Re} p(z) \ge \frac{1 + |\varphi(0)||z| - |z|^m(|\varphi(0)| + |z|)|B_\Lambda(z)|}{1 + |\varphi(0)||z| + |z|^m(|\varphi(0)| + |z|)|B_\Lambda(z)|}. \tag{13}$$

Proof. Let

$$\omega(z) := z^m B_\Lambda(z)\varphi(z), \quad z \in \mathbb{D}.$$

Since (see e.g., [4, Corollary 1.3, p. 4])

$$|\varphi(z)| \leq \frac{|\varphi(0)| + |z|}{1 + |\varphi(0)||z|} \leq 1, \quad z \in \mathbb{D},$$

we have

$$|\omega(z)| \leq |z|^m |B_\Lambda(z)| \frac{|\varphi(0)| + |z|}{1 + |\varphi(0)||z|} < 1, \quad z \in \mathbb{D}. \tag{14}$$

Hence and by the fact that a function $[0, 1) \ni r \to \gamma(r) := (1 + r)/(1 - r)$ is increasing, from (14) we have

$$\begin{aligned} |p(z)| = \left|\frac{1 + \omega(z)}{1 - \omega(z)}\right| \leq \frac{1 + |\omega(z)|}{1 - |\omega(z)|} = \gamma(|\omega(z)|) \qquad (15)\\ \leq \gamma\left(|z|^m |B_\Lambda(z)| \frac{|\varphi(0)| + |z|}{1 + |\varphi(0)||z|}\right) \\ = \frac{1 + |\varphi(0)||z| + |z|^m(|\varphi(0)| + |z|)|B_\Lambda(z)|}{1 + |\varphi(0)||z| - |z|^m(|\varphi(0)| + |z|)|B_\Lambda(z)|}, \quad z \in \mathbb{D}, \end{aligned}$$

which shows (12).

Using the second inequality in (15) we have

$$\begin{aligned} \operatorname{Re} p(z) = \operatorname{Re} \frac{1 + \omega(z)}{1 - \omega(z)} = \frac{1 - |\omega(z)|^2}{|1 - \omega(z)|^2} \\ \geq \frac{1 - |\omega(z)|}{1 + |\omega(z)|} = \frac{1}{\frac{1 + |\omega(z)|}{1 - |\omega(z)|}} = \frac{1}{\gamma(|\omega(z)|)} \\ \geq \frac{1 + |\varphi(0)||z| - |z|^m(|\varphi(0)| + |z|)|B_\Lambda(z)|}{1 + |\varphi(0)||z| + |z|^m(|\varphi(0)| + |z|)|B_\Lambda(z)|}, \quad z \in \mathbb{D}, \end{aligned}$$

which shows (13). □

Theorem 3.2. *Let $m \in \mathbb{N}$ and $\Lambda \in \Lambda$. If $p \in \mathcal{P}(m, \Lambda)$, then for $z \in \mathbb{D}$,*

$$|p(z)| \leq \frac{1 + |z|^m |B_\Lambda(z)|}{1 - |z|^m |B_\Lambda(z)|} \tag{16}$$

and

$$\operatorname{Re} p(z) \geq \frac{1 - |z|^m |B_\Lambda(z)|}{1 + |z|^m |B_\Lambda(z)|}. \tag{17}$$

Proof. Let $z \in \mathbb{D}$. Set $x := |\varphi(0)|$, $r := |z|$ and $a := |B_\Lambda(z)|$. Then $x \in [0, 1]$, $r \in [0, 1)$, $a \in [0, 1]$ and the inequality (12) takes the form

$$|p(z)| \leq \frac{1 + ar^{m+1} + (r + ar^m)x}{1 - ar^{m+1} + (r - ar^m)x} =: \gamma(x), \quad x \in [0, 1].$$

Since the function γ is increasing, we have

$$|p(z)| \leq \gamma(x) \leq \gamma(1) = \frac{1 + |z|^m |B_\Lambda(z)|}{1 - |z|^m |B_\Lambda(z)|}, \quad z \in \mathbb{D},$$

i.e., the inequality (16).

Because the function $1/\gamma$ is decreasing, the inequality (17) follows from the inequality (13). □

Theorem 3.3. *Let $m \in \mathbb{N}$ and $\Lambda \in \Lambda$. If $p \in \mathcal{P}(m, \Lambda)$ and $r \in (0, 1)$, then for $z \in \overline{\mathbb{D}}_r$,*

$$|p(z)| \leq \frac{1 + r^m M_r(\Lambda)}{1 - r^m M_r(\Lambda)} \tag{18}$$

and

$$\operatorname{Re} p(z) \geq \frac{1 - r^m M_r(\Lambda)}{1 + r^m M_r(\Lambda)}. \tag{19}$$

Both inequalities are sharp for $\Lambda = (\alpha_k)$ either with $\alpha_k \in (0, 1)$ or with $\alpha_k \in (-1, 0)$.

Proof. Let $r \in (0, 1)$ and $z \in \mathbb{T}_r$. Set $x := |B_\Lambda(z)|$. Then $0 \leq x \leq 1$. Since a function

$$\gamma(x) := \frac{1 + r^m x}{1 - r^m x}, \quad x \in [0, 1],$$

is increasing, in view of (16) we have

$$|p(z)| \leq \gamma(x) \leq \gamma(M_r(\Lambda)) = \frac{1 + r^m M_r(\Lambda)}{1 - r^m M_r(\Lambda)},$$

which shows (18).

Because the function $1/\gamma$ is decreasing, the inequality (19) follows from the inequality (17).

Let now $\Lambda = (\alpha_k) \in \Lambda$ with $\alpha_k \in (0, 1)$. Since

$$\max_{z \in \mathbb{T}_r} |\varphi_{\alpha_k}(z)| = \frac{r + \alpha_k}{1 + r\alpha_k}, \quad k \in \mathbb{N},$$

and

$$B_\Lambda(-r) = \prod_{k \in \mathbb{N}} (-\varphi_{\alpha_k}(-r)) = \prod_{k \in \mathbb{N}} \frac{r + \alpha_k}{1 + r\alpha_k} = \prod_{k \in \mathbb{N}} \max_{z \in \mathbb{T}_r} |\varphi_{\alpha_k}(z)|,$$

we see that $M_r(\Lambda) = B_\Lambda(-r)$. Thus for the function p given by (6) with $\varphi \equiv 1$ when m is even and with $\varphi \equiv -1$ when m is odd we have

$$p(-r) = \frac{1 + r^m B_\Lambda(-r)}{1 - r^m B_\Lambda(-r)} = \frac{1 + r^m M_r}{1 - r^m M_r},$$

which makes the equality in (18).

To get equality in (19), take a function p given by (6) with $\varphi \equiv -1$ when m is even, and with $\varphi \equiv 1$ when m is odd. Then we have

$$p(-r) = \frac{1 - r^m B_\Lambda(-r)}{1 + r^m B_\Lambda(-r)} = \frac{1 - r^m M_r}{1 + r^m M_r}.$$

The case $\alpha_k \in (-1, 0)$ follows analogously. □

Recall that the inequality

$$|\varphi'(z)| \leq \frac{1 - |\varphi(z)|^2}{1 - |z|^2}, \quad z \in \mathbb{D}. \tag{20}$$

is a part of Schwarz-Pick Lemma (e.g., [4, p. 2]). In the class $\mathcal{B}$ the equality in (20) holds only for conformal automorphisms of $\mathbb{D}$ or for constant functions of modulus 1.

Theorem 3.4. *Let $m \in \mathbb{N}$ and $\Lambda \in \Lambda$. If $p \in \mathcal{P}(m, \Lambda)$, then for $z \in \mathbb{D}$,*

$$\left|\frac{p'(z)}{p(z)}\right| \leq \frac{2|z|^{m-1}}{1 - |z|^2} \cdot \frac{m(1 - |z|^2)|B_\Lambda(z)| + |z|(1 - |B_\Lambda(z)|^2)}{1 - |z|^{2m}|B_\Lambda(z)|^2}. \tag{21}$$

Proof. Since

$$p'(z) = 2z^{m-1}\frac{z\varphi'(z)B_\Lambda(z) + m\varphi(z)B_\Lambda(z) + z\varphi(z)B'_\Lambda(z)}{(1 - z^m\varphi(z)B_\Lambda(z))^2}, \quad z \in \mathbb{D},$$

so

$$\frac{p'(z)}{p(z)} = 2z^{m-1}\frac{(z\varphi'(z) + m\varphi(z))\,B_\Lambda(z) + z\varphi(z)B'_\Lambda(z)}{1 - z^{2m}\varphi^2(z)B^2_\Lambda(z)}, \quad z \in \mathbb{D}.$$

Using twice the inequality (20) for the function φ and for $\varphi := B_\Lambda$, we have

$$\begin{aligned}\left|\frac{p'(z)}{p(z)}\right| &= 2|z|^{m-1}\frac{\left|(z\varphi'(z) + m\varphi(z))\,B_\Lambda(z) + z\varphi(z)B'_\Lambda(z)\right|}{\left|1 - z^{2m}\varphi^2(z)B^2_\Lambda(z)\right|}\\
&\le 2|z|^{m-1}\frac{(|z||\varphi'(z)| + m|\varphi(z)|)\,|B_\Lambda(z)| + |z||\varphi(z)||B'_\Lambda(z)|}{1 - |z|^{2m}|\varphi(z)|^2|B_\Lambda(z)|^2}\\
&\le 2|z|^{m-1}\frac{\left(|z|\dfrac{1-|\varphi(z)|^2}{1-|z|^2} + m|\varphi(z)|\right)|B_\Lambda(z)| + |z||\varphi(z)|\dfrac{1-|B_\Lambda(z)|^2}{1-|z|^2}}{1 - |z|^{2m}|\varphi(z)|^2|B_\Lambda(z)|^2}\\
&= 2\frac{|z|^{m-1}}{1-|z|^2}\\
&\times\frac{\left(|z|(1-|\varphi(z)|^2) + m(1-|z|^2)|\varphi(z)|\right)|B_\Lambda(z)| + |z||\varphi(z)|\left(1-|B_\Lambda(z)|^2\right)}{1 - |z|^{2m}|\varphi(z)|^2|B_\Lambda(z)|^2}\end{aligned}$$

for $z \in \mathbb{D}$. Set $r := |z| \in [0, 1)$, $x := |\varphi(z)| \in [0, 1]$ and $a := |B_\Lambda(z)| \in [0, 1]$. Let

$$\gamma(x) := \frac{-arx^2 + \left(m(1-r^2)a + r(1-a^2)\right)x + ar}{1 - r^{2m}a^2x^2}, \quad x \in [0, 1].$$

We will now show that

$$\gamma(x) \le \gamma(1), \quad x \in [0, 1], \tag{22}$$

i.e., that

$$\begin{aligned}&\frac{-arx^2 + \left(m(1-r^2)a + r(1-a^2)\right)x + ar}{1 - r^{2m}a^2x^2}\\
&\le \frac{m(1-r^2)a + r(1-a^2)}{1 - r^{2m}a^2}, \quad x \in [0, 1].\end{aligned}$$

which can be equivalently written as

$$\begin{aligned}&\left(m(1-r^2)a + r\left(1-a^2\right)\right)(1-x)\\
&+\left(m\left(1-r^2\right)a + r\left(1-a^2\right)\right)a^2r^{2m}x(1-x) - ar\left(1-x^2\right)\\
&+a^3r^{2m+1}\left(1-x^2\right) \ge 0, \quad x \in [0, 1].\end{aligned}$$

The above inequality has the form

$$(1-x)F_m(x) \ge 0, \quad x \in [0, 1],$$

where

$$F_m(x) := A_m - B_mx,$$

with

$$A_m := m(1-r^2)a + r(1-a^2) - ar(1-a^2r^{2m})$$

and

$$B_m := ar(1-a^2r^{2m}) - \left(m\left(1-r^2\right)a + r\left(1-a^2\right)\right)a^2r^{2m}.$$

Thus to prove the inequality (22) we need to show that

$$F_m(x) \geq 0, \quad x \in [0,1]. \tag{23}$$

First we shall prove that

$$B_m > 0. \tag{24}$$

Let

$$\varrho(a) := r^{2m}a^3 - \left(r^{2m} + m(1-r^2)r^{2m-1}\right)a^2 - r^{2m}a + 1, \quad a \in [0,1].$$

Since

$$B_m = ar\varrho(a), \quad a \in [0,1],$$

it is enough to show that

$$\varrho(a) > 0, \quad a \in [0,1]. \tag{25}$$

The above inequality obviously holds for $r = 0$. Thus let $r \in (0,1)$. Note that

$$\varrho(0) > 0 \tag{26}$$

and

$$\begin{aligned}\varrho(1) &= 1 - r^{2m} - m(1-r^2)r^{2m-1} \\ &= (1-r^2)\left(1 + r^2 + \cdots + r^{2m-2}\right) - m(1-r^2)r^{2m-1} \\ &= (1-r^2)\left(1 + r^2 + \cdots + r^{2m-2} - mr^{2m-1}\right) \\ &= (1-r^2)\left((1-r^{2m-1}) + (r^2 - r^{2m-1}) + \cdots + (r^{2m-2} - r^{2m-1})\right) > 0.\end{aligned} \tag{27}$$

Since

$$\varrho'(a) = 3r^{2m}a^2 - 2\left(r^{2m} + m(1-r^2)r^{2m-1}\right)a - r^{2m}, \quad a \in [0,1],$$

so $\varrho'(0) = -r^{2m} < 0$, $\varrho'(1) = -2m(1-r^2)r^{2m-1} < 0$ and the coefficient $3r^{2m}$ is positive. Thus $\varrho'(a) < 0$ for $a \in [0,1]$. Because ϱ is the decreasing function, the inequality (25), so (24) follows from (26) and (27).

By (24) we have

$$F_m(x) \geq F_m(1), \quad x \in [0,1]. \tag{28}$$

Observe now that

$$F_1(1) > 0. \tag{29}$$

Indeed,

$$\begin{aligned}F_1(1) &= A_1 - B_1 \\ &= (1-r^2)a + r(1-a^2) - ar(1-a^2r^2) - ar(1-a^2r^2) \\ &\quad + \left((1-r^2)a + r(1-a^2)\right)a^2r^2 \\ &= \left((1-r^2)a + r(1-a^2)\right)(1+a^2r^2) - 2ar(1-a^2r^2)\end{aligned}$$

$$= (a+r)(1-ar)(1+a^2r^2) - 2ar(1-a^2r^2)$$
$$= (1-ar)\Big((a+r)(1+a^2r^2) - 2ar(1+ar)\Big)$$
$$= (1-ar)\left(a + a^3r^2 + r + a^2r^3 - 2ar - 2a^2r^2\right)$$
$$= (1-ar)\left(a - ar + r - ar + a^3r^2 - a^2r^2 + a^2r^3 - a^2r^2\right)$$
$$= (1-ar)\Big(a(1-r) + r(1-a) - a^2r^2(1-a) - a^2r^2(1-r)\Big)$$
$$= (1-ar)\Big((1-r)(a-a^2r^2) + (1-a)(r-a^2r^2)\Big)$$
$$= (1-ar)\Big(a(1-r)(1-ar^2) + r(1-a)(1-a^2r)\Big) > 0.$$

We will now show that

$$F_{m+1}(1) \geq F_m(1). \tag{30}$$

We have

$$\begin{aligned} &F_{m+1}(1) - F_m(1)\\ &= A_{m+1} - A_m - (B_{m+1} - B_m)\\ &= (m+1)(1-r^2)a + r(1-a^2) - ar(1-a^2r^{2m+2})\\ &-m(1-r^2)a - r(1-a^2) + ar(1-a^2r^{2m})\\ &-ar\left(1-a^2r^{2m+2}\right) + \Big((m+1)(1-r^2)a + r(1-a^2)\Big)a^2r^{2m+2}\\ &+ar(1-a^2r^{2m}) - \Big(m\left(1-r^2\right)a + r\left(1-a^2\right)\Big)a^2r^{2m}\\ &= (1-r^2)a + 2a^3r^{2m+3} - 2a^3r^{2m+1}\\ &+\Big(-\Big(m(1-r^2)a + r(1-a^2)\Big)(1-r^2) + ar^2(1-r^2)\Big)a^2r^{2m}\\ &= a(1-r^2)\Big(1 - 2a^2r^{2m+1} - \Big(m(1-r^2)a + r(1-a^2)\Big)ar^{2m} + a^2r^{2m+2}\Big)\\ &= a(1-r^2)\sigma(a), \end{aligned}$$

where

$$\sigma(a) := r^{2m+1}a^3 - \left(2r^{2m+1} + m(1-r^2)r^{2m} - r^{2m+2}\right)a^2 - r^{2m+1}a + 1, \quad a \in [0,1].$$

Thus to confirm (30), we need to show that

$$\sigma(a) > 0, \quad a \in [0,1]. \tag{31}$$

Note that

$$\sigma(0) > 0 \tag{32}$$

and

$$\begin{aligned} \sigma(1) &= 1 - r^{2m+1} - r^{2m+1} + r^{2m+2} - mr^{2m} + mr^{2m+2} \\ &= (1-r)\left(1 + r + r^2 + \cdots + r^{2m}\right) - r^{2m+1}(1-r) - mr^{2m}(1-r^2)\\ &= (1-r)\Big(1 + r + r^2 + \cdots + r^{2m} - r^{2m+1} - mr^{2m}(1+r)\Big)\\ &= (1-r)\Big(1 + r + r^2 + \cdots + r^{2m} - r^{2m+1} - mr^{2m} - mr^{2m+1}\Big) \end{aligned} \tag{33}$$

$$= (1-r)\Big((1-r^{2m+1}) + (r-r^{2m}) + (r^2-r^{2m}) + \ldots + (r^m - r^{2m})$$

$$+(r^{m+1}-r^{2m+1}) + (r^{m+2}-r^{2m+1}) + \cdots + (r^{2m}-r^{2m+1})\Big)$$

$$= (1-r)\Big((1-r^{2m+1}) + r(1-r^{2m-1}) + r^2(1-r^{2m-2})$$

$$+ \cdots + r^m(1-r^m) + r^{m+1}(1-r^m) + r^{m+2}(1-r^{m-1}) + \cdots + r^{2m}(1-r)\Big) > 0.$$

Since

$$\sigma'(a) = 3r^{2m+1}a^2 - 2\left(2r^{2m+1} + m(1-r^2)r^{2m} - r^{2m+2}\right)a - r^{2m+1}, \quad a \in [0,1],$$

so

$$\sigma'(0) = -r^{2m+1} < 0,$$

$$\sigma'(1) = 3r^{2m+1} - 2\left(2r^{2m+1} + m(1-r^2)r^{2m} - r^{2m+2}\right) - r^{2m+1}$$

$$= -2r^{2m}(1-r)(r+m(1+r)) < 0,$$

and the coefficient $3r^{2m+1}$ is positive. Thus $\sigma'(a) < 0$ for $a \in [0,1]$. Because the function σ is decreasing, the inequality (31), so (30) follows from (32) and (33).

The inequalities (28)-(30) complete the proof of the inequality (23). □

Theorem 3.5. *Let $m \in \mathbb{N}$ and $\Lambda \in$. If $p \in \mathcal{P}(m,\Lambda)$ and $r \in (0,1)$, then for $z \in \overline{\mathbb{D}}_r$,*

$$\left|\frac{p'(z)}{p(z)}\right| \le \frac{2r^{m-1}(m(1-r^2)M_r(\Lambda) + r(1-M_r^2(\Lambda)))}{(1-r^2)(1-r^{2m}M_r^2(\Lambda))} \tag{34}$$

and

$$\operatorname{Re}\frac{zp'(z)}{p(z)} \ge -\frac{2r^m(m(1-r^2)M_r(\Lambda) + r(1-M_r^2(\Lambda)))}{(1-r^2)(1-r^{2m}M_r^2(\Lambda))}. \tag{35}$$

Both inequalities are sharp when $\Lambda = (\alpha)$ with $\alpha \in (-1,1)\setminus\{0\}$.

Proof. Fix $r \in (0,1)$ and $z \in \mathbb{T}_r$. Then the inequality (21) has the form

$$\left|\frac{p'(z)}{p(z)}\right| \le \frac{2r^{m-1}}{1-r^2}\cdot\frac{m(1-r^2)|B_\Lambda(z)| + r(1-|B_\Lambda(z)|^2)}{1-r^{2m}|B_\Lambda(z)|^2}. \tag{36}$$

Define

$$\gamma(x) := \frac{-rx^2 + m(1-r^2)x + r}{1-r^{2m}x^2}, \quad x \in [0,1].$$

We shall prove that the function γ is increasing. We have

$$\gamma'(x) = \frac{(-2rx + m(1-r^2))(1-r^{2m}x^2) + 2r^{2m}x(-rx^2 + m(1-r^2)x + r)}{(1-r^{2m}x^2)^2} \tag{37}$$

$$= \frac{\varrho(x)}{(1-r^{2m}x^2)^2}, \quad x \in [0,1],$$

where

$$\varrho(x) := mr^{2m}(1-r^2)x^2 - 2r(1-r^{2m})x + m(1-r^2), \quad x \in [0,1].$$

Note that

$$\varrho(0) > 0 \tag{38}$$

and

$$
\begin{aligned}
\varrho(1) &= mr^{2m}(1-r^2) - 2r(1-r^{2m}) + m(1-r^2) \qquad (39)\\
&= (1-r^2)\left(mr^{2m} - 2r(1+r^2+\cdots+r^{2m-2}) + m\right)\\
&= (1-r^2)\left(mr^{2m} - (r+r^3+\cdots+r^{2m-1}) + m - (r+r^3+\ldots+r^{2m-1})\right)\\
&= (1-r^2)\left((r^{2m}-r) + (r^{2m}-r^3) + \ldots + (r^{2m}-r^{2m-1})\right.\\
&\qquad \left. +(1-r) + (1-r^3) + \cdots + (1-r^{2m-1})\right)\\
&= (1-r^2)\left(-r(1-r^{2m-1}) - r^3(1-r^{2m-3}) - \cdots - r^{2m-1}(1-r)\right.\\
&\qquad \left. +(1-r) + (1-r^3) + \cdots + (1-r^{2m-1})\right)\\
&= (1-r^2)\\
&\quad \times\left((1-r)(1-r^{2m-1}) + (1-r^3)(1-r^{2m-3}) + \cdots + (1-r^{2m-1})(1-r)\right) > 0.
\end{aligned}
$$

Moreover $\varrho'(x) = 0$ iff

$$x = \frac{r(1-r^{2m})}{mr^{2m}(1-r^2)} =: x_0.$$

Observe first that

$$x_0 = \frac{r(1-r^{2m})}{mr^{2m}(1-r^2)} > 1, \tag{40}$$

i.e., that

$$1 - r^{2m} > mr^{2m-1}(1-r^2). \tag{41}$$

Indeed,

$$
\begin{aligned}
&(1-r^{2m}) - mr^{2m-1}(1-r^2)\\
&= (1-r^2)\left(1+r^2+\cdots+r^{2m-2}\right) - mr^{2m-1}(1-r^2)\\
&= (1-r^2)\left(1+r^2+\cdots+r^{2m-2} - mr^{2m-1}\right)\\
&= (1-r^2)\left((1-r^{2m-1}) + (r^2-r^{2m-1}) + \cdots + (r^{2m-2}-r^{2m-1})\right)\\
&= (1-r^2)\left((1-r^{2m-1}) + r^2(1-r^{2m-3}) + \cdots + r^{2m-2}(1-r)\right) > 0,
\end{aligned}
$$

which confirms the inequality (41), so (40). Hence, by (38) and (39) it follows that $\varrho(x) > 0$ for $x \in [0,1]$. Consequently, due to (37) we see that $\gamma'(x) > 0$ for $x \in [0,1]$. Thus the function γ is increasing and therefore

$$\gamma\left(|B_\Lambda(z)|\right) \le \gamma(M_r).$$

Hence and from (36) we have

$$
\begin{aligned}
\left|\frac{p'(z)}{p(z)}\right| &\le \frac{2r^{m-1}}{1-r^2}\cdot\gamma\left(|B_\Lambda(z)|\right) \le \frac{2r^{m-1}}{1-r^2}\cdot\gamma(M_r(\Lambda))\\
&= \frac{2r^{m-1}(m(1-r^2)M_r(\Lambda) + r(1-M_r^2(\Lambda)))}{(1-r^2)(1-r^{2m}M_r^2(\Lambda))},
\end{aligned}
$$

which shows (34).

The inequality (35) follows directly from (34).

Let $\mathit{\Lambda} = (\alpha)$ with $\alpha \in (-1,1) \setminus \{0\}$. Then $B_{\mathit{\Lambda}} = \varphi_\alpha$ and for a function p of the form (6) with $\varphi \equiv 1$ we have

$$\frac{p'(z)}{p(z)} = 2z^{m-1}\frac{m\varphi_\alpha(z) + z\varphi'_\alpha(z)}{1 - z^{2m}\varphi_\alpha^2(z)} \tag{42}$$

$$= 2z^{m-1}\frac{m(z-\alpha)(1-\alpha z) + z(1-|\alpha|^2)}{(1-\alpha z)^2 - z^{2m}(z-\alpha)^2}, \quad z \in \mathbb{D}.$$

Since

$$M_r(\mathit{\Lambda}) = \frac{r + |\alpha|}{1 + |\alpha| r}, \quad r \in (0,1), \tag{43}$$

from (42) it follows that the equality in (34) holds for $\alpha \in (0,1)$ at $z := -r$ and for $\alpha \in (-1,0)$ at $z := r$. $\square$

The following two statements can be called distortion theorems for $\mathcal{P}(m;\mathit{\Lambda})$.

Theorem 3.6. *Let $m \in \mathbb{N}$ and $\mathit{\Lambda} \in \Lambda$. If $p \in \mathcal{P}(m,\mathit{\Lambda})$, then for $z \in \mathbb{D}$,*

$$|p'(z)| \le \frac{2|z|^{m-1}}{1-|z|^2} \cdot \frac{m(1-|z|^2)|B_{\mathit{\Lambda}}(z)| + |z|(1-|B_{\mathit{\Lambda}}(z)|^2)}{(1-|z|^m|B_{\mathit{\Lambda}}(z)|)^2} \tag{44}$$

and

$$\mathrm{Re}(zp'(z)) \ge -\frac{2|z|^m}{1-|z|^2} \cdot \frac{m(1-|z|^2)|B_{\mathit{\Lambda}}(z)| + |z|(1-|B_{\mathit{\Lambda}}(z)|^2)}{(1-|z|^m|B_{\mathit{\Lambda}}(z)|)^2}. \tag{45}$$

Proof. In view of (21) and (16) we have

$$|p'(z)| \le \frac{2|z|^{m-1}}{1-|z|^2} \cdot \frac{m(1-|z|^2)|B_{\mathit{\Lambda}}(z)| + |z|(1-|B_{\mathit{\Lambda}}(z)|^2)}{1-|z|^{2m}|B_{\mathit{\Lambda}}(z)|^2}|p(z)|$$

$$\le \frac{2|z|^{m-1}}{1-|z|^2} \cdot \frac{m(1-|z|^2)|B_{\mathit{\Lambda}}(z)| + |z|(1-|B_{\mathit{\Lambda}}(z)|^2)}{1-|z|^{2m}|B_{\mathit{\Lambda}}(z)|^2} \cdot \frac{1+|z|^m|B_{\mathit{\Lambda}}(z)|}{1-|z|^m|B_{\mathit{\Lambda}}(z)|}$$

$$= \frac{2|z|^{m-1}}{1-|z|^2} \cdot \frac{m(1-|z|^2)|B_{\mathit{\Lambda}}(z)| + |z|(1-|B_{\mathit{\Lambda}}(z)|^2)}{(1-|z|^m|B_{\mathit{\Lambda}}(z)|)^2}, \quad z \in \mathbb{D},$$

which proves (44).

The inequality (45) follows directly from the inequality (44). $\square$

Using the inequalities (34) and (18) we get

Theorem 3.7. *Let $m \in \mathbb{N}$ and $\mathit{\Lambda} \in \Lambda$. If $p \in \mathcal{P}(m,\mathit{\Lambda})$ and $r \in (0,1)$, then for $z \in \overline{\mathbb{D}}_r$,*

$$|p'(z)| \le \frac{2r^{m-1}(m(1-r^2)M_r(\mathit{\Lambda}) + r(1-M_r^2(\mathit{\Lambda})))}{(1-r^2)(1-r^m M_r(\mathit{\Lambda}))^2}$$

and

$$\mathrm{Re}(zp'(z)) \ge -\frac{2r^m(m(1-r^2)M_r(\mathit{\Lambda}) + r(1-M_r^2(\mathit{\Lambda})))}{(1-r^2)(1-r^m M_r(\mathit{\Lambda}))^2}.$$

Both inequalities are sharp when $\mathit{\Lambda} = (\alpha)$ with $\alpha \in (-1,1) \setminus \{0\}$.

When $\mathit{\Lambda}(\mathbb{N}) = \emptyset$, the results of Theorems 3.3, 3.5 and 3.7 for the class $\mathcal{P}(m)$ are the same as for the class $\mathcal{P}_m$. Recall that the inequalities (46) was proved by MacGregor [10, Lemma 1].

Theorem 3.8. *Let $m \in \mathbb{N}$. If $p \in \mathcal{P}(m)$ and $r \in (0,1)$, then for $z \in \overline{\mathbb{D}}_r$,*

$$|p(z)| \le \frac{1+r^m}{1-r^m}, \quad \operatorname{Re} p(z) \ge \frac{1-r^m}{1+r^m},$$

$$\left|\frac{p'(z)}{p(z)}\right| \le \frac{2mr^{m-1}}{1-r^{2m}}, \quad \operatorname{Re}\frac{zp'(z)}{p(z)} \ge -\frac{2mr^m}{1-r^{2m}}, \tag{46}$$

$$|p'(z)| \le \frac{2mr^{m-1}}{(1-r^m)^2}, \quad \operatorname{Re}(zp'(z)) \ge -\frac{2mr^m}{(1-r^m)^2}.$$

The inequalities are sharp with

$$p(z) := \frac{1+z^m}{1-z^m}, \quad z \in \mathbb{D},$$

as the extremal function.

When $\Lambda := (\alpha)$, where $\alpha \in \mathbb{D}^0$, then $B_\Lambda = \varphi_\alpha$ and M_r is as (43). In this case results of Theorems 3.3, 3.5 and 3.7 can be rewritten as follows.

Theorem 3.9. *Let $m \in \mathbb{N}$ and $\alpha \in \mathbb{D}^0$. If $p \in \mathcal{P}(m;(\alpha))$ and $r \in (0,1)$, then for $z \in \overline{\mathbb{D}}_r$,*

$$|p(z)| \le \frac{r^m(r+|\alpha|)+|\alpha|r+1}{-r^m(r+|\alpha|)+|\alpha|r+1},$$

$$\operatorname{Re} p(z) \ge \frac{-r^m(r+|\alpha|)+|\alpha|r+1}{r^m(r+|\alpha|)+|\alpha|r+1},$$

$$\left|\frac{p'(z)}{p(z)}\right| \le \frac{r^{m-1}\left[2m|\alpha|r^2+2\left(m+1+(m-1)|\alpha|^2\right)r+2m|\alpha|\right]}{-r^{2m}(r+|\alpha|)^2+(|\alpha|r+1)^2},$$

$$\operatorname{Re}\frac{zp'(z)}{p(z)} \ge -\frac{r^{m}\left[2m|\alpha|r^2+2\left(m+1+(m-1)|\alpha|^2\right)r+2m|\alpha|\right]}{-r^{2m}(r+|\alpha|)^2+(|\alpha|r+1)^2},$$

$$|p'(z)| \le \frac{r^{m-1}\left[2m|\alpha|r^2+2\left(m+1+(m-1)|\alpha|^2\right)r+2m|\alpha|\right]}{r^{2m}(1+|\alpha|^2)-2r^m\left(|\alpha|r^2+(1+|\alpha|^2)r+|\alpha|\right)+(|\alpha|r+1)^2},$$

$$\operatorname{Re}(zp'(z)) \ge \frac{r^{m}\left[2m|\alpha|r^2+2\left(m+1+(m-1)|\alpha|^2\right)r+2m|\alpha|\right]}{r^{2m}(1+|\alpha|^2)-2r^m\left(|\alpha|r^2+(1+|\alpha|^2)r+|\alpha|\right)+(|\alpha|r+1)^2}.$$

4. Radii of convexity

We apply now the results of the previous section to discuss radii of convexity.

Definition 4.1. *For $\mathcal{F} \subset \mathcal{A}$ set*

$$R_{\mathcal{S}^c}(\mathcal{F}) := \sup\left\{r \in (0,1] : \operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} > 0,\ f \in \mathcal{F},\ z \in \mathbb{D}_r\right\}.$$

The number $R_{\mathcal{S}^c}(\mathcal{F})$ is called the radius of convexity for $\mathcal{F}$.

I. The case $\mathcal{F} := \mathcal{P}'(m;\Lambda)$.

Theorem 4.2. *Let $m \in \mathbb{N}$ and $\Lambda \in \Lambda$. Then*

$$R_{\mathcal{S}^c}(\mathcal{P}'(m;\Lambda)) \geq r_0, \tag{47}$$

where $r := r_0$ is the smallest root in $(0,1)$ of the equation

$$(1-r^2)(1-r^{2m}M_r^2(\Lambda)) - 2r^m(m(1-r^2)M_r(\Lambda) + r(1-M_r^2(\Lambda))) = 0. \tag{48}$$

For $\alpha \in (-1,1)\setminus\{0\}$ equality in (47) holds.

Proof. Let $f \in \mathcal{P}'(m;\Lambda)$. Then $f' = p$ for some $p \in \mathcal{P}(m;\Lambda)$. Hence and from (35), for each $r \in (0,1)$ we have

$$\begin{aligned} \operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} &= 1 + \operatorname{Re}\frac{zp'(z)}{p(z)} \\ &\geq 1 - \frac{2r^m(m(1-r^2)M_r(\Lambda) + r(1-M_r^2(\Lambda)))}{(1-r^2)(1-r^{2m}M_r^2(\Lambda))}, \quad z \in \mathbb{D}_r, \end{aligned} \tag{49}$$

which leads to (47).

When $\alpha \in (-1,1)\setminus\{0\}$ the inequality (35) is sharp, so is (49) which yields equality in (50). □

Substituting $M_r = 1$ for the equation (48) we see that r_0 is the unique root in $(0,1)$ of the equation $r^{2m} + 2mr^m - 1 = 0$, i.e.,

$$r_0 = \left(\sqrt{m^2+1} - m\right)^{1/m}.$$

Then $r_0 = R_{\mathcal{S}^c}(\mathcal{P}'(m))$ with the extremal function

$$f(z) := \int_0^z \frac{1+\zeta^m}{1-\zeta^m} d\zeta, \quad z \in \mathbb{D}.$$

Recalling the radius of convexity $R_{\mathcal{S}^c}(\mathcal{P}'_m)$ for the class $\mathcal{P}'_m$ which was found by MacGregor [10, Theorem 2] we have

Theorem 4.3. *Let $m \in \mathbb{N}$. Then*

$$R_{\mathcal{S}^c}(\mathcal{P}'_m) = R_{\mathcal{S}^c}(\mathcal{P}'(m)) = \left(\sqrt{m^2+1} - m\right)^{1/m}.$$

The case $m = 1$, i.e., the radius of convexity $R_{\mathcal{S}^c}(\mathcal{P}')$ was computed also by MacGregor in the former paper [9, Theorem 2].

For $\Lambda := (\alpha)$, $\alpha \in \mathbb{D}^0$, M_r is given by (43) and then the equation (48) reduces to (51).

Corollary 4.4. *Let $m \in \mathbb{N}$ and $\alpha \in \mathbb{D}^0$. Then*

$$R_{\mathcal{S}^c}(\mathcal{P}'(m;(\alpha)) \geq r_0, \tag{50}$$

where $r := r_0$ is the smallest root in $(0,1)$ of the equation

$$r^{2m}(r+|\alpha|)^2 + 2r^m\left(m|\alpha|(r^2+1) + (m+1+(m-1)|\alpha|^2)r\right) - (|\alpha|r+1)^2 = 0. \tag{51}$$

For $\alpha \in (-1,1)\setminus\{0\}$ equality in (50) holds.

Since the function

$$(0,1) \ni r \mapsto r^4 + 4|\alpha|r^3 + 4r^2 - 1,$$

is strictly increasing, the case $m = 1$ yields

Corollary 4.5. *Let $\alpha \in \mathbb{D}^0$. Then*

$$R_{\mathcal{S}^c}(\mathcal{P}'(1;(\alpha)) \geq r_0, \tag{52}$$

where $r := r_0$ is the unique root in $(0,1)$ of the equation

$$r^4 + 4|\alpha|r^3 + 4r^2 - 1 = 0.$$

For $\alpha \in (-1,1)\setminus\{0\}$ equality in (52) holds.

II. The case $\mathcal{F} := \mathcal{S}^*(m;\Lambda)$.

Theorem 4.6. *Let $m \in \mathbb{N}$ and $\Lambda \in \Lambda$. Then*

$$R_{\mathcal{S}^c}(\mathcal{S}^*(m;\Lambda)) \geq r_0 \tag{53}$$

where $r := r_0$ is the smallest root in $(0,1]$ of the equation

$$\begin{aligned}(1-r^m M_r(\Lambda))(1-r^2)(1-r^{2m}M_r^2(\Lambda))& \\ -2r^m(1+r^m M_r(\Lambda))(m(1-r^2)M_r(\Lambda)+r(1-M_r^2(\Lambda)))&=0.\end{aligned} \tag{54}$$

For $\Lambda = (\alpha)$, where $\alpha \in (-1,1)\setminus\{0\}$, the equality in (53) holds.

Proof. Let $f \in \mathcal{S}^*(m;\Lambda)$. Then

$$\frac{zf'(z)}{f(z)} = p(z), \quad z \in \mathbb{D},$$

for some $p \in \mathcal{P}(m;\Lambda)$. Hence, from (19) and (35), for each $r \in (0,1)$ we have

$$\begin{aligned}\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} &= \operatorname{Re} p(z) + \operatorname{Re}\frac{zp'(z)}{p(z)} \\ &\geq \frac{1-r^m M_r(\Lambda)}{1+r^m M_r(\Lambda)} - \frac{2r^m(m(1-r^2)M_r(\Lambda)+r(1-M_r^2(\Lambda)))}{(1-r^2)(1-r^{2m}M_r^2(\Lambda))} \\ &= \frac{(1-r^m M_r(\Lambda))(1-r^2)(1-r^{2m}M_r^2(\Lambda))}{(1+r^m M_r(\Lambda))(1-r^2)(1-r^{2m}M_r^2(\Lambda))} \\ &\quad - \frac{2r^m(1+r^m M_r(\Lambda))(m(1-r^2)M_r(\Lambda)+r(1-M_r^2(\Lambda)))}{(1+r^m M_r(\Lambda))(1-r^2)(1-r^{2m}M_r^2(\Lambda))}, \quad z \in \mathbb{D}_r,\end{aligned} \tag{55}$$

which leads to (53).

When $\alpha \in (-1,1)\setminus\{0\}$ the inequalities (19) and (35) are sharp, so is (55) which yields equality in (55). □

Substituting $M_r = 1$ for the equation (54) we see that r_0 is the unique root in $(0,1)$ of the equation

$$(r^m+1)\left(r^{2m}-2(1+m)r^m+1\right)=0,$$

i.e.,

$$r_0 = \left(1+m-\sqrt{m^2+2m}\right)^{1/m}.$$

Then $r_0 = R_{\mathcal{S}^c}(\mathcal{S}^*(m))$ with the extremal function $f \in \mathcal{A}$ satisfying the differential equation

$$\frac{zf'(z)}{f(z)} = \frac{1+z^m}{1-z^m}, \quad z \in \mathbb{D}.$$

Moreover we can state that $R_{\mathcal{S}^c}(\mathcal{S}^*_m) = r_0$. Unfortunately, we were not able to find to whom belongs this result although we are sure that the result is well known for specialists. Thus we have

Theorem 4.7. *Let $m \in \mathbb{N}$. Then*

$$R_{\mathcal{S}^c}(\mathcal{S}^*_m) = R_{\mathcal{S}^c}(\mathcal{S}^*(m)) = \left(1+m-\sqrt{m^2+2m}\right)^{1/m}.$$

Let us recall that the radius of convexity for class $\mathcal{S}^*$ was found by Nevanlinna [12] and it is the same as for the class $\mathcal{S}$. Thus we have

Theorem 4.8.

$$R_{\mathcal{S}^c}(\mathcal{S}) = R_{\mathcal{S}^c}(\mathcal{S}^*) = R_{\mathcal{S}^c}(\mathcal{S}^*(1)) = 2 - \sqrt{3}.$$

When $\Lambda := (\alpha)$, where $\alpha \in \mathbb{D}^0$, then substituting M_r given by (43) for the equation (54) we obtain

Corollary 4.9. *Let* $\alpha \in \mathbb{D}^0$. *Then*

$$R_{\mathcal{S}^c}(\mathcal{S}^*(1;(\alpha)) \geq r_0 \tag{56}$$

where

$$r_0 = \frac{1}{2}\left(u_0 - \sqrt{u_0 - 4}\right), \tag{57}$$

with $u := u_0$ *being the unique root in* $(2, +\infty)$ *of the equation*

$$u^3 - 4(|\alpha|^2 + 2)u - 16|\alpha| = 0. \tag{58}$$

For $\alpha \in (-1, 1) \setminus \{0\}$ *the equality in* (56) *holds.*

Proof. For $m = 1$ and $M_r(\Lambda)$ is given by (43) the equation (54) takes the form

$$r^8 - (4|\alpha|^2 + 6)r^6 - 16|\alpha|r^5 + 16|\alpha|r^3 + (4|\alpha|^2 + 6)r^2 - 1 = 0$$

equivalently written as

$$(r^2 - 1)(r^6 - (4|\alpha| + 5)r^4 - 16|\alpha|r^3 - (4|\alpha|^2 + 5)r^2 + 1) = 0.$$

Since the second factor is a symmetrical polynomial, by setting $u := r + r^{-1}$ it takes the form $u^3 - 4(|\alpha|^2 + 2)u - 16|\alpha|$, where $u \in (2, +\infty)$. It is easily to see that the function

$$(2, +\infty) \ni u \mapsto u^3 - 4(|\alpha|^2 + 2)u - 16|\alpha|$$

is strictly increasing, so the equation (58) has a unique root u_0 in $(2, +\infty)$. Thus r_0 given by (57) is the unique solution in $(0, 1)$ of the equation $u_0 = r_0 + r_0^{-1}$, which leads to (56). □

5. Coefficient functionals

In this section we discuss some basic coefficients problems for the class $\mathcal{P}(m, (\alpha))$, where $m \in \mathbb{N}$ and $\alpha \in \mathbb{D}^0$. Let $p \in \mathcal{P}(m, (\alpha))$. Then

$$p(z)\,(1 - z^m\varphi(z)\varphi_\alpha(z)) = 1 + z^m\varphi(z)\varphi_\alpha(z), \quad z \in \mathbb{D}, \tag{59}$$

for some $\varphi \in \mathcal{B}^0$, i.e., equivalently

$$p(z)\,(1 - \overline{\alpha}z - z^m\varphi(z)(z - \alpha)) = 1 - \overline{\alpha}z + z^m\varphi(z)(z - \alpha), \quad z \in \mathbb{D}.$$

Putting into the above equation the series (1) and the series

$$\varphi(z) = \sum_{n=0}^{\infty} b_n z^n, \quad z \in \mathbb{D}, \tag{60}$$

by comparing corresponding coefficients we get

$$c_m = -2\alpha b_0. \tag{61}$$

Moreover, when $m = 1$, then

$$c_2 = -2\alpha b_1 + 2\left(1 - |\alpha|^2\right) b_0 + 2\alpha^2 b_0^2 \tag{62}$$

and when $m > 1$, then

$$c_{m+1} = -2\alpha b_1 + 2\left(1 - |\alpha|^2\right) b_0. \tag{63}$$

Since $|b_0| = |\varphi(0)| \leq 1$, from (61) we have

Theorem 5.1. *Let $m \in \mathbb{N}$ and $\alpha \in \mathbb{D}^0$. If $p \in \mathcal{P}(m,(\alpha))$ is of the form* (1), *then*

$$|c_m| \leq 2|\alpha|. \tag{64}$$

The result is sharp. Equality in (64) *holds for the function p given by* (59) *with $\varphi \equiv -e^{-i\theta}$, where $\theta := \operatorname{Arg}\alpha \in [0, 2\pi)$.*

Given $m \in \mathbb{N}$ and $\lambda \in \mathbb{R}$, consider the functional

$$\Phi_{m,\lambda}(p) := |c_{m+1} - \lambda c_m^2|$$

over the class $\mathcal{P}$ of functions p of the form (1). In particular, the functional $\Phi_\lambda := \Phi_{1,\lambda}$ plays a fundamental role in many extremal coefficient problems. Ma and Minda [8, Lemma 1] proved the following sharp result for the whole class $\mathcal{P}$:

$$|c_2 - \lambda c_1^2| \leq \begin{cases} 2|1-2\lambda|, & \lambda \in (-\infty, 0] \cup [1, +\infty), \\ 2, & \lambda \in (0,1). \end{cases} \tag{65}$$

We compute now the upper bound of Φ_λ in the class $\mathcal{P}(1,(\alpha))$. It should be expected that the result is more detailed then the bounds in (65) and so is.

Theorem 5.2. *Let $\alpha \in \mathbb{D}^0$ and $p \in \mathcal{P}(1,(\alpha))$ be of the form* (1). *If $|\alpha| \in (0, \sqrt{2}-1]$, then*

$$\left|c_2 - \lambda c_1^2\right| \leq 2\left(|\alpha|^2\left(|1-2\lambda|-1\right)+1\right), \quad \lambda \in \mathbb{R}. \tag{66}$$

If $|\alpha| \in (\sqrt{2}-1, 1)$, then

$$\left|c_2 - \lambda c_1^2\right| \tag{67}$$

$$\leq \begin{cases} 2\left(|\alpha|^2\left(|1-2\lambda|-1\right)+1\right), & \lambda \leq \dfrac{|\alpha|^2-2|\alpha|+1}{4|\alpha|^2} \vee \lambda \geq \dfrac{3|\alpha|^2+2|\alpha|-1}{4|\alpha|^2}, \\ \dfrac{|\alpha|^4-4|1-2\lambda||\alpha|^3+2|\alpha|^2+1}{2|\alpha|-2|\alpha|^2|1-2\lambda|}, & \dfrac{|\alpha|^2-2|\alpha|+1}{4|\alpha|^2} < \lambda < \dfrac{3|\alpha|^2+2|\alpha|-1}{4|\alpha|^2}. \end{cases}$$

In particular,

$$\left|c_2 - \frac{1}{2}c_1^2\right| \leq \begin{cases} 2(1-|\alpha|^2), & |\alpha| \in (0, \sqrt{2}-1], \\ \dfrac{(1+|\alpha|^2)^2}{2|\alpha|}, & |\alpha| \in (\sqrt{2}-1, 1). \end{cases} \tag{68}$$

and

$$|c_2| \leq 2. \tag{69}$$

The result is sharp. Let $\alpha := |\alpha|e^{i\theta}$, $\theta \in [0, 2\pi)$. The equality in (66) *and in the first inequality in* (67) *holds for the function p given by* (59) *with $\varphi \equiv \pm e^{-2i\theta}$. The equality in the second inequality in* (67) *holds for the function p given by* (59) *with*

$$\varphi(z) := \pm e^{-2i\theta}\frac{e^{-i\theta}z - x_0}{1 - e^{-i\theta}x_0 z}, \quad z \in \mathbb{D}, \tag{70}$$

where

$$x_0 := \frac{1-|\alpha|^2}{2|\alpha|(1-|\alpha||1-2\lambda|)}. \tag{71}$$

Proof. Since (see e.g., [5, Vol. II, p. 78])

$$|b_1| \leq 1 - |b_0|^2, \tag{72}$$

from (61) for $m = 1$ and (62) we have

$$\begin{aligned} |c_2 - \lambda c_1^2| &= 2\left|-\alpha b_1 + \left(1-|\alpha|^2\right) b_0 + \alpha^2(1-2\lambda)b_0^2\right| \\ &\leq 2\left[|\alpha|\left(1-|b_0|^2\right) + \left(1-|\alpha|^2\right)|b_0| + |\alpha|^2|1-2\lambda|\,|b_0|^2\right] \\ &= 2\left[|\alpha|\left(|\alpha||1-2\lambda|-1\right)|b_0|^2 + \left(1-|\alpha|^2\right)|b_0| + |\alpha|\right] =: \gamma(|b_0|), \end{aligned} \tag{73}$$

where

$$\gamma(x) = 2\left[|\alpha|\left(|\alpha||1-2\lambda|-1\right)x^2 + \left(1-|\alpha|^2\right)x + |\alpha|\right], \quad x \in [0,1].$$

(a) When $|\alpha||1-2\lambda| - 1 \geq 0$, i.e., when

$$\lambda \in \left(-\infty, \frac{|\alpha|-1}{2|\alpha|}\right] \cup \left[\frac{|\alpha|+1}{2|\alpha|}, +\infty\right),$$

we have $\gamma'(x) \geq 0$ for $x \in [0,1]$, and consequently

$$\gamma(x) \leq \gamma(1) = 2\left(|\alpha|^2(|1-2\lambda|-1)+1\right), \quad x \in [0,1]. \tag{74}$$

(b) Assume now that $|\alpha||1-2\lambda| - 1 < 0$, i.e., that

$$\lambda \in \left(\frac{|\alpha|-1}{2|\alpha|}, \frac{|\alpha|+1}{2|\alpha|}\right). \tag{75}$$

We have $\gamma'(x) = 0$ only for $x := x_0$, where x_0 is given by (71). Thus $x_0 \geq 1$ iff

$$|1-2\lambda| \geq \frac{|\alpha|^2 + 2|\alpha| - 1}{2|\alpha|^2}.$$

Taking into account (75) we see that the above inequality holds: when $|\alpha| \in (0, \sqrt{2}-1]$ for λ as in (75), and when $|\alpha| \in (\sqrt{2}-1, 1)$ for

$$\lambda \in \left(\frac{|\alpha|-1}{2|\alpha|}, \frac{|\alpha|^2-2|\alpha|+1}{4|\alpha|^2}\right] \cup \left[\frac{3|\alpha|^2+2|\alpha|-1}{4|\alpha|^2}, \frac{|\alpha|+1}{2|\alpha|}\right). \tag{76}$$

Hence and by the case (a) it follows that $\gamma'(x) \geq 0$ for $x \in [0,1]$ so the inequality (74) holds when $|\alpha| \in (0, \sqrt{2}-1]$ for all $\lambda \in \mathbb{R}$, and when $|\alpha| \in (\sqrt{2}-1, 1)$ for λ as in (76). This and (73) proves the inequality (66) and the first inequality in (67). The second inequality in (67) is a consequence of the inequality

$$\gamma(x) \leq \gamma(x_0) = \frac{|\alpha|^4 - 4|1-2\lambda||\alpha|^3 + 2|\alpha|^2 + 1}{2|\alpha| - 2|\alpha|^2|1-2\lambda|}$$

which holds for

$$\lambda \in \left(\frac{|\alpha|^2-2|\alpha|+1}{4|\alpha|^2}, \frac{3|\alpha|^2+2|\alpha|-1}{4|\alpha|^2}\right).$$

Since

$$\frac{|\alpha|^2-2|\alpha|+1}{4|\alpha|^2} < \frac{1}{2} < \frac{3|\alpha|^2+2|\alpha|-1}{4|\alpha|^2},$$

for $|\alpha| \in (\sqrt{2}-1, 1)$, so the inequalities (66) and (67) with $\lambda := 1/2$ reduce to the inequality (68).

For $\lambda := 0$ the inequalities (66) and (67) reduce to (69).

It remains to discuss the sharpness. Let $\alpha := |\alpha|e^{i\theta}$, $\theta \in [0, 2\pi)$. It is obvious that the equality in (66) and in the first inequality in (67) holds for the function p given by (59) either with $\varphi \equiv e^{-2i\theta}$ or with $\varphi \equiv -e^{-2i\theta}$. Let $\lambda \le 1/2$. For

$$\varphi(z) = -e^{-2i\theta} \frac{e^{-i\theta}z - x_0}{1 - e^{-i\theta}x_0 z} = e^{-2i\theta}x_0 - (1 - x_0^2)e^{-3i\theta}z + \ldots, \quad z \in \mathbb{D},$$

we have

$$\begin{aligned} &\left|-\alpha b_1 + \left(1 - |\alpha|^2\right) b_0 + \alpha^2(1 - 2\lambda)b_0^2\right| \\ &= \left||\alpha|(1 - x_0^2)e^{-2i\theta} + \left(1 - |\alpha|^2\right) x_0 e^{-2i\theta} + (1 - 2\lambda)|\alpha|^2 x_0^2 e^{-2i\theta}\right| \\ &= |\alpha|(1 - x_0^2) + \left(1 - |\alpha|^2\right) x_0 + (1 - 2\lambda)|\alpha|^2 x_0^2, \end{aligned}$$

which yields the equality in the second inequality in (67). The case $\lambda > 1/2$ follows in a similar way. □

Remark 5.3. One can check that the upper bounds in (66) and (67) do not exceed of the upper bounds in (65). Setting $|\alpha| = 1$ the inequalities (66) and (67) reduce to the inequality (65).

We consider now the case $m > 1$.

Theorem 5.4. *Let $\alpha \in \mathbb{D}^0$, $m > 1$ and $p \in \mathcal{P}(m, (\alpha))$ be of the form* (1). *If $|\alpha| \in (0, \sqrt{2} - 1]$, then*

$$\left|c_{m+1} - \lambda c_m^2\right| \le 2\left(|\alpha|^2 (2|\lambda| - 1) + 1\right), \quad \lambda \in \mathbb{R}. \tag{77}$$

If $|\alpha| \in (\sqrt{2} - 1, 1)$, then

$$\left|c_{m+1} - \lambda c_m^2\right| \tag{78}$$

$$\le \begin{cases} 2\left(|\alpha|^2 (2|\lambda| - 1) + 1\right), & \lambda \le \dfrac{-|\alpha|^2 - 2|\alpha| + 1}{4|\alpha|^2} \vee \lambda \ge \dfrac{|\alpha|^2 + 2|\alpha| - 1}{4|\alpha|^2}, \\ \dfrac{|\alpha|^4 - 8|\lambda||\alpha|^3 + 2|\alpha|^2 + 1}{2|\alpha| - 4|\alpha|^2|\lambda|}, & \dfrac{-|\alpha|^2 - 2|\alpha| + 1}{4|\alpha|^2} < \lambda < \dfrac{|\alpha|^2 + 2|\alpha| - 1}{4|\alpha|^2}. \end{cases}$$

In particular,

$$\left|c_{m+1} - \frac{1}{2}c_m^2\right| \le 2 \tag{79}$$

and

$$|c_{m+1}| \le \begin{cases} 2(1 - |\alpha|^2), & \alpha \in (0, \sqrt{2} - 1], \\ \dfrac{(1 + |\alpha|^2)^2}{2|\alpha|}, & \alpha \in (\sqrt{2} - 1, 1). \end{cases} \tag{80}$$

The result is sharp. The equality in (77) *and in the first inequality in* (78) *holds for the function p given by* (59) *with $\varphi \equiv \pm e^{-2i\theta}$, where $\theta := \operatorname{Arg}\alpha \in [0, 2\pi)$. The equality in the second inequality in* (78) *holds for the function p given by* (59) *with φ given by* (70) *where*

$$x_0 = \frac{1 - |\alpha|^2}{2|\alpha|(1 - 2|\alpha||\lambda|)}. \tag{81}$$

Proof. From (61), (63) and (72) it follows that

$$\left|c_{m+1} - \lambda c_m^2\right| = 2\left|-\alpha b_1 + \left(1 - |\alpha|^2\right) b_0 - 2\lambda\alpha^2 b_0^2\right| \tag{82}$$

$$\leq 2\left[|\alpha|(1 - |b_0|^2) + (1 - |\alpha|^2)|b_0| + 2|\lambda||\alpha|^2|b_0|^2\right]$$

$$= 2\left[|\alpha|\,(2|\alpha||\lambda| - 1|)\,|b_0|^2 + (1 - |\alpha|^2)|b_0| + |\alpha|\right] =: \gamma(|b_0|),$$

where

$$\gamma(x) = 2\left[|\alpha|(2|\alpha||\lambda| - 1|)x^2 + (1 - |\alpha|^2)x + |\alpha|\right], \quad x \in [0, 1].$$

(a) When $2|\alpha||\lambda| - 1 \geq 0$, i.e., when

$$\lambda \in \left(-\infty, \frac{-1}{2|\alpha|}\right] \cup \left[\frac{1}{2|\alpha|}, +\infty\right),$$

we have $\gamma'(x) \geq 0$ for $x \in [0, 1]$, and consequently

$$\gamma(x) \leq \gamma(1) = 2\left(|\alpha|^2\,(2|\lambda| - 1) + 1\right)), \quad x \in [0, 1]. \tag{83}$$

(b) Assume now that $2|\alpha||\lambda| - 1 < 0$, i.e., that

$$\lambda \in \left(\frac{-1}{2|\alpha|}, \frac{1}{2|\alpha|}\right). \tag{84}$$

We have $\gamma'(x) = 0$ only for $x := x_0$, where x_0 is given by (81). Thus $x_0 \geq 1$ iff

$$|\lambda| > \frac{|\alpha|^2 + 2|\alpha| - 1}{4|\alpha|^2}.$$

Taking into account (84) we see that the above inequality holds: when $|\alpha| \in (0, \sqrt{2} - 1]$ for λ as in (84), and when $|\alpha| \in (\sqrt{2} - 1, 1)$ for

$$\lambda \in \left(\frac{-1}{2|\alpha|}, \frac{-|\alpha|^2 - 2|\alpha| + 1}{4|\alpha|^2}\right) \cup \left(\frac{|\alpha|^2 + 2|\alpha| - 1}{4|\alpha|^2}, \frac{1}{2|\alpha|}\right). \tag{85}$$

Hence and by the case (a) it follows that $\gamma'(x) \geq 0$ for $x \in [0, 1]$ so the inequality (83) holds when $|\alpha| \in (0, \sqrt{2} - 1]$ for all $\lambda \in \mathbb{R}$, and when $|\alpha| \in (\sqrt{2} - 1, 1)$ for λ as in (85). This and (82) proves the inequality (77) and the first inequality in (78). The second inequality in (78) is a consequence of the inequality

$$\gamma(x) \leq \gamma(x_0) = \frac{|\alpha|^4 - 8|\lambda||\alpha|^3 + 2|\alpha|^2 + 1}{2|\alpha| - 4|\alpha|^2|\lambda|}$$

which holds for

$$\lambda \in \left(\frac{-|\alpha|^2 - 2|\alpha| + 1}{4|\alpha|^2}, \frac{|\alpha|^2 + 2|\alpha| - 1}{4|\alpha|^2}\right).$$

Since

$$\frac{1}{2} > \frac{|\alpha|^2 + 2|\alpha| - 1}{4|\alpha|^2},$$

for $|\alpha| \in (\sqrt{2} - 1, 1)$, so the inequalities (77) and (78) with $\lambda := 1/2$ reduce to the inequality (79).

For $\lambda := 0$ the inequalities (77) and (78) reduce to (80).

The sharpness follows analogously as in Theorem 5.2. □

6. Additional remarks

In this paper we were mainly interested in the class $\mathcal{P}(m,\Lambda)$. Examples of radii of convexity show that some computational problems can be considered in the corresponding classes of analytic functions. The class $\mathcal{P}^0(m,\Lambda)$ and related families of analytic functions were defined also however they were not examined here. Although the results for the class $\mathcal{P}(m,\Lambda)$ are valid for the class $\mathcal{P}^0(m,\Lambda)$ they can not be sharp in $\mathcal{P}^0(m,\Lambda)$ in general. One of the extremal function p for the inequalities (67) is defined with using the function φ given by (70) having zero in $\mathbb{D}$, so $\varphi \notin \mathcal{B}^0$ and therefore $p \notin \mathcal{P}^0(m,\Lambda)$. The computational techniques for $\mathcal{P}^0(m,\Lambda)$, i.e., when $\varphi \in \mathcal{B}^0$, have to be more sophisticated based on knowledge on the class on bounded non-vanishing analytic functions. Let us mention that in 1968 Krzyż [7] conjectured that

$$|b_n| \leq \frac{2}{\mathrm{e}}, \quad n \in \mathbb{N},$$

for $\varphi \in \mathcal{B}^0$ of the form (60) with equality only for the function

$$\varphi_n(z) := \exp\left(\frac{z^n - 1}{z^n + 1}\right) = \frac{1}{\mathrm{e}} + \frac{2}{\mathrm{e}} z + \ldots, z \in \mathbb{D},$$

and its rotations. A review of the results of this conjecture can be found in [11].

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