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Sharp Estimates of Generalized Zalcman Functional of Early Coefficients for Ma-Minda Type Functions

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Abstract. Let φ be an analytic function in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ which has the form $\varphi(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots$ with $p_1 > 0$, $p_2, p_3 \in \mathbb{R}$. For given such φ, let $\mathcal{S}^*(\varphi)$, $\mathcal{K}(\varphi)$ and $\mathcal{R}(\varphi)$ denote the classes of standardly normalized analytic functions f in $\mathbb{D}$ which satisfy

$$\frac{zf'(z)}{f(z)} \prec \varphi(z), \quad 1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z) \quad f'(z) \prec \varphi(z), \quad z \in \mathbb{D},$$

respectively, where $\prec$ means the usual subordination. In this paper, we find the sharp bounds of $|a_2 a_3 - a_4|$, where $a_n := f^{(n)}(0)/n!$, $n \in \mathbb{N}$, over classes $\mathcal{S}^*(\varphi)$, $\mathcal{K}(\varphi)$ and $\mathcal{R}(\varphi)$.

1. Introduction

Let $\mathcal{H}$ be the class of analytic functions in $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and let $\mathcal{A}$ be its subclass of f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D}. \tag{1}$$

The subclass of $\mathcal{A}$ consisting of univalent functions is denoted by $\mathcal{S}$.

For analytic functions f and g we say that f is subordinate to g and write $f \prec g$, if there is an analytic function $\omega : \mathbb{D} \to \mathbb{D}$ with $\omega(0) = 0$ such that $f = g \circ \omega$ in $\mathbb{D}$. If g is univalent, then $f \prec g$ is equivalent to $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$.

Given $\varphi \in \mathcal{H}$ of the form

$$\varphi(z) = 1 + \sum_{n=1}^{\infty} p_n z^n, \quad z \in \mathbb{D}, \tag{2}$$

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let $\mathcal{S}^*(\varphi)$, $\mathcal{K}(\varphi)$ and $\mathcal{R}(\varphi)$ denote the classes of functions $f \in \mathcal{A}$ which satisfy

$$\frac{zf'(z)}{f(z)} \prec \varphi(z), \quad 1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z), \quad f'(z) \prec \varphi(z), \quad z \in \mathbb{D}, \tag{3}$$

respectively. Let $\mathcal{P}$ be the class of functions $\varphi \in \mathcal{H}$ of the form (2) having a positive real part in $\mathbb{D}$, i.e., the Carathéodory class of functions. When $\varphi \in \mathcal{P}$, then functions in the classes $\mathcal{S}^*(\varphi)$ and $\mathcal{K}(\varphi)$ are called Ma-Minda starlike functions and Ma-Minda convex functions, respectively [12]. Therefore functions in $\mathcal{R}(\varphi)$ can be called of bounded turning of Ma-Minda type. For $\varphi \in \mathcal{P}$ the inclusions $\mathcal{S}^*(\varphi) \subset \mathcal{S}$, $\mathcal{K}(\varphi) \subset \mathcal{S}$ and $\mathcal{R}(\varphi) \subset \mathcal{S}$ hold evidently. Let us emphasize, that in our consideration functions φ is not restricted to the class $\mathcal{P}$, however throughout the whole paper we will assume that $p_1 > 0$, $p_2, p_3 \in \mathbb{R}$ in its power series (2).

Given $0 \le \alpha < 1$ and $0 < \beta \le 1$, define

$$\varphi_\alpha(z) := \frac{1 + (1 - 2\alpha)z}{1 - z} = 1 + 2(1 - \alpha)\sum_{k=1}^{\infty} z^k, \quad z \in \mathbb{D}, \tag{4}$$

and

$$\varphi^*_\beta(z) := \left(\frac{1 + z}{1 - z}\right)^\beta = 1 + 2\beta z + 2\beta^2 z^2 + \frac{2}{3}\beta(1 + 2\beta^2)z^3 + \cdots, \quad z \in \mathbb{D}. \tag{5}$$

Let

$$\varphi_P(z) := 1 + \frac{2}{\pi^2}\left(\log\frac{1 + \sqrt{z}}{1 - \sqrt{z}}\right)^2 = 1 + \frac{8}{\pi^2}z + \frac{16}{3\pi^2}z^2 + \frac{184}{45\pi^2}z^3 + \cdots, \quad z \in \mathbb{D}. \tag{6}$$

Substituting $\varphi = \varphi_\alpha$, $\varphi = \varphi^*_\beta$ and $\varphi = \varphi_P$ into (3) we obtain several classes that some of these will be examined subsequently:

- $\mathcal{S}^*(\alpha) := \mathcal{S}^*(\varphi_\alpha)$ – the class of starlike functions of order α;
- $\mathcal{SS}^*_\beta := \mathcal{S}^*(\varphi^*_\beta)$ – the class of strongly starlike functions of order β;
- $\mathcal{S}^*_P := \mathcal{S}^*(\varphi_P)$ – the class of parabolic starlike functions;
- $\mathcal{K}(\alpha) := \mathcal{K}(\varphi_\alpha)$ – the class of convex functions of order α;
- $\mathcal{SK}_\beta := \mathcal{K}(\varphi^*_\beta)$ – the class of strongly convex functions of order β;
- $\mathcal{UCV} := \mathcal{K}(\varphi_P)$ – the class of uniformly convex functions;
- $\mathcal{R}(\alpha) := \mathcal{R}(\varphi_\alpha)$ – the class of functions of bounded turning of order α.

In this paper, we computed the sharp upper bound of the functional $J_{2,3}(f) := a_2a_3 - a_4$ over the classes $\mathcal{S}^*(\varphi)$, $\mathcal{K}(\varphi)$ and $\mathcal{R}(\varphi)$, respectively. The functional $J_{2,3}$ is a specific case of the generalized Zalcman functional $J_{n,m}(f) := a_na_m - a_{n+m-1}$, $n, m \in \mathbb{N} \setminus \{1\}$, which was investigated by Ma [11] for $f \in \mathcal{S}$ (see also [14] for relevant results on this functional). On the other hand, many authors (*cf.* [1–6, 8, 15]) computed the upper bound for the functional $J_{2,3}$ over various subclasses of $\mathcal{A}$ to obtain a bound for Hankel determinant

$$H_{3,1}(f) := \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}, \quad f \in \mathcal{A},$$

of third order using the inequality

$$|H_3(f)| \le |a_3||a_2a_4 - a_3^2| + |a_4||a_2a_3 - a_4| + |a_5||a_3 - a_2^2|, \quad f \in \mathcal{A}.$$

Refer to [9] for the study of the functional $H_{2,2}(f) := a_2a_4 - a_3^2$, i.e., the Hankel determinant of the second order over the classes $\mathcal{S}^*(\varphi)$ and $\mathcal{K}(\varphi)$.

In Section 2 we introduce some lemmas which will be used for proofs main results. Sharp bounds for the functional $J_{2,3}$ over the classes $\mathcal{S}^*(\varphi)$, $\mathcal{K}(\varphi)$ and $\mathcal{R}(\varphi)$ are computed in Sections 3, 4 and 5, respectively. Some specific functions are examined in each section also.

2. Preliminary results

Let $\mathcal{B}_0$ be a subclass of $\mathcal{H}$ of functions ω of the form

$$\omega(z) = \sum_{n=1}^{\infty} c_n z^n, \quad z \in \mathbb{D}, \tag{7}$$

such that $\omega(0) = 0$ which map $\mathbb{D}$ into itself, and called Schwarz functions. Clearly, $\omega \in \mathcal{B}_0$ if and only if $\varphi := (1+\omega)/(1-\omega) \in \mathcal{P}$.

In [13], Prokhorov and Szynal investigated the sharp upper bound for the functional Ψ over the class $\mathcal{B}_0$, where

$$\Psi(\mu, \nu) := |c_3 + \mu c_1 c_2 + \nu c_1^3|, \quad (\mu, \nu) \in \mathbb{R}^2 \tag{8}$$

and c_i $(i = 1, 2, 3)$ are the coefficients of functions in $\mathcal{B}_0$ with the form given by (7). Moreover the extremal functions for each cases $(\mu, \nu) \in D_i$ $(i = 1, 2, \cdots, 12)$ were given in [13, p. 135]. Here, D_i $(i = 1, 2, \cdots, 12)$ are the set defined as in [13, p. 127] such that $\cup_{i=1}^{12} D_i = \mathbb{R}^2$. Recall that the extremal functions are given by

I. $\omega(z) = z^3$, when $(\mu, \nu) \in D_1 \cup D_2 \cup \{(2, 1)\}$;

II. $\omega(z) = z$, when $(\mu, \nu) \in \bigcup_{k=3}^{7} D_k$.

However the explicit form of the extremal functions for the cases $(\mu, \nu) \in D_8 \cup D_9$, $(\mu, \nu) \in D_{10} \cup D_{11} \setminus \{(2, 1)\}$ and $(\mu, \nu) \in D_{12}$ have not been dealt with at all until now. In this section we will obtain the extremal functions $\omega \in \mathcal{B}_0$ with the explicit form for the cases above.

To do it, the following result shown by Kwon *et al.* [7] is required. We remark here that a special case of the proposition below matches to [10, Lemma 2.3]. Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$.

Proposition 2.1 ([7]). *Let $\varphi \in \mathcal{P}$ be of the form* (2) *with $p_1 \in [0, 2)$ and for $\zeta \in \mathbb{T}$,*

$$2p_2 = p_1^2 + \zeta(4 - p_1^2). \tag{9}$$

Then φ must be of the form

$$\varphi(z) = \frac{1 + \rho(1+\zeta)z + \zeta z^2}{1 - \rho(1-\zeta)z - \zeta z^2}, \quad z \in \mathbb{D}, \tag{10}$$

where $\rho \in [0, 1)$.

Let $\omega \in \mathcal{B}_0$ be of the form (7) and $c_2 = (1 - c_1^2)\zeta$ holds for some $\zeta \in \mathbb{T}$. Then $\varphi := (1+\omega)/(1-\omega) \in \mathcal{P}$ is of the form (2) and therefore

$$p_1 = 2c_1, \quad p_2 = 2(c_1^2 + c_2), \quad p_3 = 2(c_1^3 + 2c_1c_2 + c_3).$$

Hence and from equality $c_2 = (1 - c_1^2)\zeta$, it follows that (9) holds. By Proposition 2.1 the function φ is of the form (10). Since $\omega = (\varphi - 1)/(\varphi + 1)$, we get the following lemma.

Lemma 2.2. *Let $\omega \in \mathcal{B}_0$ be of the form (7) with $c_1 \in [0, 1)$ and $c_2 = (1 - c_1^2)\zeta$ for some $\zeta \in \mathbb{T}$. Then ω must be of the form*

$$\omega(z) = \frac{z(\rho + \zeta z)}{1 + \rho\zeta z}, \quad z \in \mathbb{D}, \tag{11}$$

where $\rho \in [0, 1)$.

From Lemma 2.2, the statements III, IV and V in [13, p. 135] can be replaced by III′, IV′ and V′ below, respectively, i.e., the extremal function ω has the form (11) with

III′. $\rho = \sqrt{(\mu + 1)/(3(\mu + 1 + \nu))}$ and $\zeta = -1$, when $(\mu, \nu) \in D_8 \cup D_9$;

IV′. $\rho = \sqrt{(3\mu^2 - 2(\mu^2 + 2)\nu)/(3(\nu - 1)(4\nu - \mu^2))}$ and $\zeta = e^{i\theta_0}$, where θ_0 is defined by

$$\theta_0 = \pm \arccos\left(\frac{\mu[2(\mu^2 + 2) - (\mu^2 + 8)\nu]}{2[3\mu^2 - 2(\mu^2 + 2)\nu]}\right),$$

when $(\mu, \nu) \in D_{10} \cup D_{11} \setminus \{(2, 1)\}$;

V′. $\rho = \sqrt{(\mu - 1)/(3(\mu - 1 - \nu))}$ and $\zeta = 1$, when $(\mu, \nu) \in D_{12}$.

With the aid of [13, Lemma 2] and the extremal functions given in I, II, III′, IV′, V′, from here, we will obtain the sharp bounds of $|a_2a_3 - a_4|$ over the classes $\mathcal{S}^*(\varphi)$, $\mathcal{K}(\varphi)$ and $\mathcal{R}(\varphi)$.

3. The class $\mathcal{S}^*(\varphi)$

In this section, we deal with the class $\mathcal{S}^*(\varphi)$. Given $\varphi \in \mathcal{H}$ of the form (2) with $p_1 > 0$, p_2 and $p_3 \in \mathbb{R}$, let $f \in \mathcal{S}^*(\varphi)$ be of the form (1). Then there exists $\omega \in \mathcal{B}_0$ of the form (7) such that

$$\frac{zf'(z)}{f(z)} = \varphi(\omega(z)), \quad z \in \mathbb{D}. \tag{12}$$

Substituting the series (1), (2) and (7) into (12) by equating the coefficient we get

$$a_2 = p_1c_1, \quad a_3 = \frac{1}{2}[p_1c_2 + (p_1^2 + p_2)c_1^2] \quad \text{and} \quad a_4 = \frac{1}{6}[2p_1c_3 + (3p_1^2 + 4p_2)c_1c_2 + (p_1^3 + 3p_1p_2 + 2p_3)c_1^3]. \tag{13}$$

Hence

$$|a_2a_3 - a_4| = \frac{1}{3}p_1\Psi(\hat{\mu}, \hat{\nu}), \tag{14}$$

where Ψ is defined by (8),

$$\hat{\mu} = \frac{2p_2}{p_1}, \quad \hat{\nu} = \frac{p_3 - p_1^3}{p_1}.$$

Thus by applying the result in [13, Lemma 2], the sharp bound of (14) is one of the following values:

$$A_1 := \frac{1}{3}p_1, \quad A_2 := \frac{1}{3}|p_3 - p_1^3|, \quad A_3 := \frac{2\sqrt{3}(p_1 + 2|p_2|)^{3/2}}{27\sqrt{p_1 - p_1^3 + 2|p_2| + p_3}},$$

$$A_4 := \frac{2\sqrt{3}(p_3 - p_1^3)(p_2^2 - p_1^2)^{3/2}}{27(p_2^2 + p_1^4 - p_1p_3)\sqrt{p_1(p_3 - p_1 - p_1^3)}} \quad \text{and} \quad A_5 := \frac{2\sqrt{3}(2|p_2| - p_1)^{3/2}}{27\sqrt{2|p_2| - p_1 + p_1^3 - p_3}}.$$

Now, for each $i = 1, \dots, 5$ consider the functions whose coefficients satisfy equality $|a_2a_3 - a_4| = A_i$. To do this, define

$$f(z) = z \exp\left[\int_0^z \frac{\varphi(\omega(\xi)) - 1}{\xi} \mathrm{d}\xi\right], \quad z \in \mathbb{D}. \tag{15}$$

Taking $\mu = \hat{\mu}$ and $\nu = \hat{\nu}$ in [13, Lemma 2], we get the following functions which are extremal ones for each case.

(1) $|a_2a_3 - a_4| = A_1$ holds for $\hat{f_1} := f$, where f is the function defined by (15) with $\omega(z) = z^3$, $z \in \mathbb{D}$;

(2) $|a_2a_3 - a_4| = A_2$ holds for $\hat{f_2} := f$, where f is the function defined by (15) with $\omega(z) = z$, $z \in \mathbb{D}$;

(3) $|a_2a_3 - a_4| = A_3$ holds for $\hat{f_3} := f$, where f is the function defined by (15) with ω defined by (11), where

$$\rho = \sqrt{\frac{2p_2 + p_1}{3(2p_2 + p_1 - p_1^3 + p_3)}} \quad \text{and} \quad \zeta = -1;$$

(4) $|a_2a_3 - a_4| = A_4$ holds for $\hat{f_4} := f$, where f is the function defined by (15) with ω defined by (11), where

$$\rho = \sqrt{\frac{p_1^5 + 3p_1p_2^2 + 2p_1^3p_2^2 - p_1^2p_3 - 2p_2^2p_3}{3(p_1 + p_1^3 - p_3)(p_1^4 + p_2^2 - p_1p_3)}}, \quad \zeta = \mathrm{e}^{\mathrm{i}\theta_0}$$

and

$$\theta_0 = \pm \arccos\left(\frac{p_2(2p_1^5 + 2p_1p_2^2 + p_1^3(1 + p_2^2) - 2p_1^2p_3 - p_2^2p_3)}{p_1(p_1^5 + 3p_1p_2^2 + 2p_1^3p_2^2 - p_1^2p_3 - 2p_2^2p_3)}\right);$$

(5) $|a_2a_3 - a_4| = A_5$ holds for $\hat{f_5} := f$, where f is the function defined by (15) with ω defined by (11), where

$$\rho = \sqrt{\frac{2p_2 - p_1}{3(2p_2 - p_1 + p_1^3 - p_3)}} \quad \text{and} \quad \zeta = 1.$$

From the above consideration it follows the following sharp upper bound of the functional $J_{2,3}$ over the class $\mathcal{S}^*(\varphi)$.

Theorem 3.1. *Let $\varphi \in \mathcal{H}$ be of the form given by* (2) *with $p_1 > 0$, $p_2, p_3 \in \mathbb{R}$ and let*

$$\hat{\sigma}_1 := p_1^3 - p_1 - 2|p_2| + \frac{4(p_1 + 2|p_2|)^3}{27p_1^2}, \quad \hat{\sigma}_2 := p_1^3 + \frac{p_1|p_2|(p_1 + 2|p_2|)}{p_1^2 + p_1|p_2| + p_2^2},$$

$$\hat{\sigma}_3 := p_1^3 + \frac{p_1|p_2|(2|p_2| - p_1)}{p_2^2 - p_1|p_2| + p_1^2} \quad \text{and} \quad \hat{\sigma}_4 := p_1^3 + \frac{p_2^2 + 2p_1^2}{3p_1}.$$

Let $f \in \mathcal{S}^*(\varphi)$ be of the form given by (1)*. Then the following sharp inequalities hold:*

A. When $|p_2| \le p_1/4$:

(a) If $p_1^3 - p_1 \le p_3 \le p_1^3 + p_1$, then $|a_2a_3 - a_4| \le A_1$ and the extremal function is $\hat{f_1}$;

(b) If $p_3 \le p_1^3 - p_1$ or $p_3 \ge p_1^3 + p_1$, then $|a_2a_3 - a_4| \le A_2$ and the extremal function is $\hat{f_2}$.

B. When $p_1/4 \le |p_2| \le p_1$:

(a) *If* $\hat{\sigma}_1 \le p_3 \le p_1^3 + p_1$, *then* $|a_2a_3 - a_4| \le A_1$ *and the extremal function is* $\hat{f}_1$;

(b) *If* $p_3 \le (3p_1^3 - 2p_1 - 4|p_2|)/3$ *or* $p_3 \ge p_1^3 + p_1$, *then* $|a_2a_3 - a_4| \le A_2$ *and the extremal function is* $\hat{f}_2$;

(c) *If* $(3p_1^3 - 2p_1 - 4|p_2|)/3 \le p_3 \le \hat{\sigma}_1$, *then* $|a_2a_3 - a_4| \le A_3$ *and the extremal function is* $\hat{f}_3$.

C. *When* $p_1 < |p_2| \le 2p_1$:

(a) *If* $p_3 \le (3p_1^3 - 2p_1 - 4|p_2|)/3$ *or* $p_3 \ge \hat{\sigma}_4$, *then* $|a_2a_3 - a_4| \le A_2$ *and the extremal function is* $\hat{f}_2$;

(b) *If* $(3p_1^3 - 2p_1 - 4|p_2|)/3 \le p_3 \le \hat{\sigma}_2$, *then* $|a_2a_3 - a_4| \le A_3$ *and the extremal function is* $\hat{f}_3$;

(c) *If* $\hat{\sigma}_2 \le p_3 \le \hat{\sigma}_4$, *then* $|a_2a_3 - a_4| \le A_4$ *and the extremal function is* $\hat{f}_4$.

D. *When* $|p_2| \ge 2p_1$:

(a) *If* $p_3 \le (3p_1^3 - 2p_1 - 4|p_2|)/3$ *or* $p_3 \ge (3p_1^3 - 2p_1 + 4|p_2|)/3$, *then* $|a_2a_3 - a_4| \le A_2$ *and the extremal function is* $\hat{f}_2$;

(b) *If* $(3p_1^3 - 2p_1 - 4|p_2|)/3 \le p_3 \le \hat{\sigma}_2$, *then* $|a_2a_3 - a_4| \le A_3$ *and the extremal function is* $\hat{f}_3$;

(c) *If* $\hat{\sigma}_2 \le p_3 \le \hat{\sigma}_3$, *then* $|a_2a_3 - a_4| \le A_4$ *and the extremal function is* $\hat{f}_4$;

(d) *If* $\hat{\sigma}_3 \le p_3 \le (3p_1^3 - 2p_1 + 4|p_2|)/3$, *then* $|a_2a_3 - a_4| \le A_5$ *and the extremal function is* $\hat{f}_5$.

Example 3.2. *(see [4, Theorem 2.1]) Let* $\alpha \in [0,1)$ *and let* $f \in \mathcal{S}^*(\alpha) = \mathcal{S}^*(\varphi_\alpha)$, *where* φ_α *is defined by* (4). *Since* $p_1 = p_2 = p_3 = 2(1-\alpha)$, *we see that* $p_1/4 \le |p_2| \le p_1$ *for all* $\alpha \in [0,1)$. *Note that* $\hat{\sigma}_1 > p_3$ *for all* $\alpha \in [0,1)$, *since* $\hat{\sigma}_1 - p_3 = 8(1-\alpha)^3$. *Note also that* $p_3 - (3p_1^3 - 2p_1 - 4|p_2|)/3 = -2(1-\alpha)(1-8\alpha+4\alpha^2)$. *Therefore, for* $\alpha \in [0,(2-\sqrt{3})/2]$ *the inequality* $p_3 \le (3p_1^3 - 2p_1 - 4|p_2|)/3$ *holds. Hence by Theorem 3.1.B.(b) we have*

$$|a_2a_3 - a_4| \le A_2 = \frac{2}{3}(3 - 11\alpha + 12\alpha^2 - 4\alpha^3)$$

when $\alpha \in [0,(2-\sqrt{3})/2]$. *The equality holds for the function*

$$\hat{f}_2(z) = z\exp\left(\int_0^z \frac{\varphi_\alpha(\xi)-1}{\xi}\mathrm{d}\xi\right) = \frac{z}{(1-z)^{2(1-\alpha)}}, \quad z \in \mathbb{D},$$

which is in $\mathcal{S}^*(\alpha)$. *On the other hand, for* $\alpha \in [(2-\sqrt{3})/2,1)$, *the inequality* $p_3 \ge (3p_1^3 - 2p_1 - 4|p_2|)/3$ *holds and this fact with Theorem 3.1.B.(c) yield the sharp inequality*

$$|a_2a_3 - a_4| \le A_3 = \frac{2(1-\alpha)}{3\sqrt{\alpha(2-\alpha)}}.$$

The equality holds for the function

$$\hat{f}_3(z) = z\exp\left(\int_0^z \frac{\varphi_\alpha(\xi^3)-1}{\xi}\mathrm{d}\xi\right) = z\exp\left(\int_0^z \frac{2(1-\alpha)(\rho-\xi)}{1-2\rho\xi+\xi^2}\mathrm{d}\xi\right), \quad z \in \mathbb{D},$$

with $\rho = 1/(2\sqrt{\alpha(2-\alpha)})$, *which is in* $\mathcal{S}^*(\alpha)$.

Example 3.3. *(see [3, Theorem 2.1]) Let* $\beta \in (0,1]$ *and let* $f \in \mathcal{SS}^*_\beta = \mathcal{S}^*(\varphi^*_\beta)$, *where* φ^*_β *is defined by* (5). *We have* $p_1 = 2\beta$, $p_2 = 2\beta^2$ *and* $p_3 = 2\beta(1+2\beta^2)/3$. *Firstly, let* $\beta \in (0,1/4]$. *Then* $p_2 \le p_1/4$ *and* $p_1^3 - p_1 \le p_3 \le p_1^3 + p_1$. *Hence by Theorem 3.1.A.(a), we get the sharp inequality*

$$|a_2a_3 - a_4| \le \frac{2}{3}\beta. \tag{16}$$

The equality holds for the function

$$\hat{f_1}(z) = z\exp\left[\int_0^z \frac{1}{\xi}\left(\left(\frac{1+\xi^3}{1-\xi^3}\right)^\beta - 1\right)\mathrm{d}\xi\right], \quad z \in \mathbb{D},$$

which is in $\mathcal{SS}^*_\beta$. *Now, let fix* $\beta \in [1/4, 1]$. *Then* $p_1/4 \le |p_2| \le p_1$ *and* $p_3 \le p_1^3 + p_1$. *Note also that* $(3p_1^3 - 2p_1 - 4p_2)/3 \le p_3$ *when* $\beta \in [1/4, (2+\sqrt{34})/10]$ *and* $(3p_1^3 - 2p_1 - 4p_2)/3 \ge p_3$ *when* $\beta \in [(2+\sqrt{34})/10, 1]$. *We have*

$$\hat{\sigma}_1 - p_3 = \frac{4}{27}\beta(-16 - 15\beta + 69\beta^2 + 16\beta^3).$$

Hence $\hat{\sigma}_1 \le p_3$ *for* $\beta \in [1/4, \beta_1]$ *and* $\hat{\sigma}_1 \ge p_3$ *for* $\beta \in [\beta_1, 1]$, *where* $\beta_1 \approx 0.559$ *is the zero of the equation* $-16 - 15x + 69x^2 + 16x^3 = 0$. *Consequently, for* $\beta \in [1/4, \beta_1]$, *by Theorem 3.1.B.(a) the sharp inequality* (16) *holds. The equality holds for* $\hat{f_1}$ *defined above. For* $\beta \in [\beta_1, (2+\sqrt{34})/10]$, *by Theorem 3.1.B.(c) we get the sharp inequality*

$$|a_2a_3 - a_4| \le A_3 = \frac{2\sqrt{2}\beta(1+2\beta)^{3/2}}{9\sqrt{2+3\beta-5\beta^2}}.$$

The equality holds for the function

$$\hat{f_3}(z) = z\exp\left[\int_0^z \frac{1}{\xi}\left(\left(\frac{1-\xi^2}{1-2\rho\xi+\xi^2}\right)^\beta - 1\right)\mathrm{d}\xi\right], \quad z \in \mathbb{D},$$

where $\rho = \sqrt{(1+2\beta)/(3(1+3\beta-4\beta^2))}$, *which is* $\mathcal{SS}^*_\beta$. *When* $\beta \in [(2+\sqrt{34})/10, 1]$, *by applying Theorem 3.1.B.(b) we get the sharp inequality*

$$|a_2a_3 - a_4| \le A_2 = \frac{2}{9}\beta(10\beta^2 - 1).$$

The equality holds for the function

$$\hat{f_2}(z) = z\exp\left[\int_0^z \frac{1}{\xi}\left(\left(\frac{1+\xi}{1-\xi}\right)^\beta - 1\right)\mathrm{d}\xi\right], \quad z \in \mathbb{D}.$$

which is in $\mathcal{SS}^*_\beta$.

4. The class $\mathcal{K}(\varphi)$

Given φ be of the form (2) with $p_1 > 0$, $p_2, p_3 \in \mathbb{R}$, let $f \in \mathcal{K}(\varphi)$ be of the form (1). Since $zf'(z) \in \mathcal{S}^*(\varphi)$, from (13) we obtain

$$a_2 = \frac{1}{2}p_1c_1, \quad a_3 = \frac{1}{6}[p_1c_2 + (p_1^2+p_2)c_1^2] \quad \text{and} \quad a_4 = \frac{1}{24}[2p_1c_3 + (3p_1^2+4p_2)c_1c_2 + (p_1^3+3p_1p_2+2p_3)c_1^3].$$

Hence

$$|a_2a_3 - a_4| = \frac{1}{12}p_1\Psi(\tilde{\mu}, \tilde{\nu}),$$

where Ψ is defined by (8),

$$\tilde{\mu} = \frac{p_1^2 + 4p_2}{2p_1}, \quad \tilde{\nu} = \frac{2p_3 + p_1p_2 - p_1^3}{2p_1}.$$

Therefore, by applying the result in [13, Lemma 2] the sharp bound of the functional $J_{2,3}$ over the class $\mathcal{K}(\varphi)$ is among the following values:

$$B_1 := \frac{1}{12}p_1, \quad B_2 := \frac{1}{24}|p_1^3 - p_1p_2 - 2p_3|, \quad B_3 := \frac{\sqrt{3}(|p_1^2 + 4p_2| + 2p_1)^{3/2}}{108\sqrt{|p_1^2 + 4p_2| + 2p_1 - p_1^3 + p_1p_2 + 2p_3}},$$

$$B_4 := \frac{\sqrt{6}(-p_1^3 + p_1p_2 + 2p_3)(p_1^4 + 8p_1^2p_2 + 16p_2^2 - 16p_1^2)^{3/2}}{432(9p_1^4 + 16p_2^2 - 16p_1p_3)\sqrt{p_1(-p_1^3 + p_1p_2 + 2p_3 - 2p_1)}}$$

and

$$B_5 := \frac{\sqrt{3}(|p_1^2 + 4p_2| - 2p_1)^{3/2}}{108\sqrt{|p_1^2 + 4p_2| - 2p_1 + p_1^3 - p_1p_2 - 2p_3}}.$$

Now, for each $i = 1, \ldots, 5$ consider the functions whose coefficients satisfy equality $|a_2a_3 - a_4| = B_i$. To do this, define

$$f(z) = \int_0^z \left(\exp\left[\int_0^\zeta \frac{\varphi(\omega(\xi)) - 1}{\xi}\mathrm{d}\xi\right]\right)\mathrm{d}\zeta, \quad z \in \mathbb{D}, \tag{17}$$

where $\omega \in \mathcal{B}_0$. Taking $\mu = \tilde{\mu}$ and $\nu = \tilde{\nu}$ in [13, Lemma 2], we get the following functions which are extremal ones for each case.

(1) $|a_2a_3 - a_4| = B_1$ holds for $\tilde{f}_1 := f$, where f is the function defined by (17) with $\omega(z) = z^3$, $z \in \mathbb{D}$;

(2) $|a_2a_3 - a_4| = B_2$ holds for $\tilde{f}_2 := f$, where f is the function defined by (17) with $\omega(z) = z$, $z \in \mathbb{D}$;

(3) $|a_2a_3 - a_4| = B_3$ holds for $\tilde{f}_3 := f$, where f is the function defined by (17) with ω defined by (11), where

$$\rho = \sqrt{\frac{p_1^2 + 4p_2 + 2p_1}{3(p_1^2 + 4p_2 + 2p_1 + 2p_3 + p_1p_2 - p_1^3)}} \quad \text{and} \quad \zeta = -1;$$

(4) $|a_2a_3 - a_4| = B_4$ holds for $\tilde{f}_4 := f$, where f is the function defined by (17) with ω defined by (11), where $\rho = \sqrt{(2\kappa_1)/(3\kappa_2)}$, $\zeta = \mathrm{e}^{\mathrm{i}\theta_0}$ and $\theta_0 = \pm\arccos(\kappa_3/\kappa_4)$, and where

$$\kappa_1 := p_1^7 + 16p_1(3 - p_2)p_2^2 + 8p_1^3p_2(2 + p_2) + p_1^5(11 + 7p_2) - 2p_1^4p_3 - 32p_2^2p_3 - 16p_1^2(1 + p_2)p_3,$$

$$\kappa_2 := (p_1^3 + p_1(2 - p_2) - 2p_3)(9p_1^4 + 16p_2^2 - 16p_1p_3),$$

$$\kappa_3 := (p_1^2 + 4p_2)[4p_1(8p_1^2 + (p_1^2 + 4p_2)^2) + (p_1^4 + 16p_2^2 + 8p_1^2(4 + p_2))(p_1^3 - p_1p_2 - 2p_3)]$$

and

$$\kappa_4 := 24p_1^2(p_1^2 + 4p_2)^2 + 8p_1(8p_1^2 + (p_1^2 + 4p_2)^2)(p_1^3 - p_1p_2 - 2p_3);$$

(5) $|a_2a_3 - a_4| = B_5$ holds for $\tilde{f}_5 := f$, where f is the function defined by (17) with ω defined by (11), where

$$\rho = \sqrt{\frac{p_1^2 + 4p_2 - 2p_1}{3(p_1^2 + 4p_2 - 2p_1 - 2p_3 - p_1p_2 + p_1^3)}} \quad \text{and} \quad \zeta = 1.$$

From the above consideration it follows the following sharp upper bound of the functional $J_{2,3}$ over the class $\mathcal{K}(\varphi)$.

Theorem 4.1. *Let $\varphi \in \mathcal{H}$ be of the form* (2) *with $p_1 > 0$, $p_2, p_3 \in \mathbb{R}$ and let*

$$\tilde{\sigma}_1 := \frac{1}{2}p_1^3 - \frac{1}{2}p_1p_2 - p_1 - \frac{1}{2}|p_1^2 + 4p_2| + \frac{(|p_1^2 + 4p_2| + 2p_1)^3}{54p_1^2},$$

$$\tilde{\sigma}_2 := \frac{1}{2}p_1^3 - \frac{1}{2}p_1p_2 + \frac{p_1(2p_1^4 + 16p_1^2p_2 + 32p_2^2 + 4p_1|p_1^2 + 4p_2|)}{p_1^4 + 8p_1^2p_2 + 16p_2^2 + 16p_1^2 + 4p_1|p_1^2 + 4p_2|},$$

$$\tilde{\sigma}_3 := \frac{1}{2}p_1^3 - \frac{1}{2}p_1p_2 + \frac{p_1(2p_1^4 + 16p_1^2p_2 + 32p_2^2 - 4p_1|p_1^2 + 4p_2|)}{p_1^4 + 8p_1^2p_2 + 16p_2^2 + 16p_1^2 - 4p_1|p_1^2 + 4p_2|}$$

and

$$\tilde{\sigma}_4 := \frac{1}{2}p_1^3 - \frac{1}{2}p_1p_2 + \frac{p_1^4 + 8p_1^2p_2 + 16p_2^2 + 32p_1^2}{48p_1}.$$

Let $f \in \mathcal{K}(\varphi)$ be of the form given by (1)*. Then the following sharp inequalities hold:*

A. *When $|p_1^2 + 4p_2| \le p_1$:*
   - *(a) If $(p_1^3 - 2p_1 - p_1p_2)/2 \le p_3 \le (p_1^3 + 2p_1 - p_1p_2)/2$, then $|a_2a_3 - a_4| \le B_1$ and the extremal function is $\tilde{f}_1$;*
   - *(b) If $p_3 \le (p_1^3 - 2p_1 - p_1p_2)/2$ or $p_3 \ge (p_1^3 + 2p_1 - p_1p_2)/2$, then $|a_2a_3 - a_4| \le B_2$ and the extremal function is $\tilde{f}_2$.*

B. *When $p_1 \le |p_1^2 + 4p_2| \le 4p_1$:*
   - *(a) If $\tilde{\sigma}_1 \le p_3 \le (p_1^3 - p_1p_2 + 2p_1)/2$, then $|a_2a_3 - a_4| \le B_1$ and the extremal function is $\tilde{f}_1$;*
   - *(b) If $p_3 \le (3p_1^3 - 3p_1p_2 - 4p_1 - 2|p_1^2 + 4p_2|)/6$ or $p_3 \ge (p_1^3 - p_1p_2 + 2p_1)/2$, then $|a_2a_3 - a_4| \le B_2$ and the extremal function is $\tilde{f}_2$;*
   - *(c) If $(3p_1^3 - 3p_1p_2 - 4p_1 - 2|p_1^2 + 4p_2|)/6 \le p_3 \le \tilde{\sigma}_1$, then $|a_2a_3 - a_4| \le B_3$ and the extremal function is $\tilde{f}_3$.*

C. *When $4p_1 < |p_1^2 + 4p_2| \le 8p_1$:*
   - *(a) If $p_3 \le (3p_1^3 - 3p_1p_2 - 4p_1 - 2|p_1^2 + 4p_2|)/6$ or $p_3 \ge \tilde{\sigma}_4$, then $|a_2a_3 - a_4| \le B_2$ and the extremal function is $\tilde{f}_2$;*
   - *(b) If $(3p_1^3 - 3p_1p_2 - 4p_1 - 2|p_1^2 + 4p_2|)/6 \le p_3 \le \tilde{\sigma}_2$, then $|a_2a_3 - a_4| \le B_3$ and the extremal function is $\tilde{f}_3$;*
   - *(c) If $\tilde{\sigma}_2 \le p_3 \le \tilde{\sigma}_4$, then $|a_2a_3 - a_4| \le B_4$ and the extremal function is $\tilde{f}_4$.*

D. *When $|p_1^2 + 4p_2| \ge 8p_1$:*
   - *(a) If $p_3 \le (3p_1^3 - 3p_1p_2 - 4p_1 - 2|p_1^2 + 4p_2|)/6$ or $p_3 \ge (3p_1^3 - 3p_1p_2 - 4p_1 + 2|p_1^2 + 4p_2|)/6$, then $|a_2a_3 - a_4| \le B_2$ and the extremal function is $\tilde{f}_2$;*
   - *(b) If $(3p_1^3 - 3p_1p_2 - 4p_1 - 2|p_1^2 + 4p_2|)/6 \le p_3 \le \tilde{\sigma}_2$, then $|a_2a_3 - a_4| \le B_3$ and the extremal function is $\tilde{f}_3$;*
   - *(c) If $\tilde{\sigma}_2 \le p_3 \le \tilde{\sigma}_3$, then $|a_2a_3 - a_4| \le B_4$ and the extremal function is $\tilde{f}_4$;*
   - *(d) If $\tilde{\sigma}_3 \le p_3 \le (3p_1^3 - 3p_1p_2 - 4p_1 + 2|p_1^2 + 4p_2|)/6$, then $|a_2a_3 - a_4| \le B_5$ and the extremal function is $\tilde{f}_5$.*

Example 4.2. *Let $\alpha \in [0, 1)$ and let $f \in \mathcal{K}(\alpha) = \mathcal{K}(\varphi_\alpha)$, where φ_α is defined by* (4)*. Since $p_1 = p_2 = p_3 = 2(1 - \alpha)$, it follows that $4p_1 < p_1^2 + 4p_2 < 8p_1$ for all $\alpha \in [0, 1)$. Note also that $p_3 > (3p_1^3 - 3p_1p_2 - 4p_1 - 2|p_1^2 + 4p_2|)/6$ for all $\alpha \in [0, 1)$. We have*

$$p_3 - \tilde{\sigma}_4 = \frac{1}{6}(1 - \alpha)^2(-17 + 25\alpha).$$

Thus $p_3 \geq \tilde{\sigma}_4$ *for* $\alpha \in [17/25, 1)$. *Therefore, by Theorem 4.1.C.(a) we get the sharp inequality*

$$|a_2a_3 - a_4| \leq B_2 = \frac{1}{6}\alpha(3 - 5\alpha + 2\alpha^2).$$

The equality holds for the function

$$\tilde{f_2}(z) = \frac{1}{1-2\alpha}\left((1-z)^{2\alpha-1} - 1\right), \quad z \in \mathbb{D},$$

which is in $\mathcal{K}(\alpha)$. *When* $\alpha \in [0, 17/25]$, *then* $p_3 \leq \tilde{\sigma}_4$. *We have*

$$\tilde{\sigma}_2 - p_3 = \frac{2(1-\alpha)^2(24 - 47\alpha + 17\alpha^2 - 2\alpha^3)}{19 - 8\alpha + \alpha^2}.$$

If $\alpha \in [0, \alpha_1]$, *where* $\alpha_1 \approx 0.653$ *is the zero of the equation* $24 - 47x + 17x^2 - 2x^3 = 0$, *then* $\tilde{\sigma}_2 \geq p_3$. *Therefore by Theorem 4.1.C.(b) we get the sharp inequality*

$$|a_2a_3 - a_4| \leq B_3 = \frac{\sqrt{6}(1-\alpha)(4-\alpha)^{3/2}}{54\sqrt{2 + \alpha - \alpha^2}}, \quad \alpha \in [0, \alpha_1].$$

The equality holds for the function

$$\tilde{f_3}(z) = \int_0^z \left[\exp\left(2(1-\alpha)\int_0^\zeta \frac{(\rho - \xi)}{1 - 2\rho\xi + \xi^2}\mathrm{d}\xi\right)\right]\mathrm{d}\zeta, \quad z \in \mathbb{D},$$

with $\rho = \sqrt{(4-\alpha)/(6(\alpha^2 - \alpha - 2))}$, *which is in* $\mathcal{K}(\alpha)$. *If* $\alpha \in [\alpha_1, 17/25]$, *then* $\tilde{\sigma}_2 \leq p_3$. *Therefore by Theorem 4.1.C.(c) we get the sharp inequality*

$$|a_2a_3 - a_4| \leq B_4 = \frac{\sqrt{3}\alpha(3-2\alpha)(5-\alpha)^{3/2}}{486\sqrt{2\alpha - 1}}, \quad \alpha \in [\alpha_1, 17/25].$$

The equality holds for the function

$$\tilde{f_4}(z) = \int_0^z \left[\exp\left(2(1-\alpha)\int_0^\zeta \frac{(\rho + \zeta\xi)}{1 + \rho(\zeta - 1)\xi - \zeta\xi^2}\mathrm{d}\xi\right)\right]\mathrm{d}\zeta, \quad z \in \mathbb{D},$$

where

$$\rho = \sqrt{\frac{-27 + 57\alpha - 26\alpha^2 + 4\alpha^3}{27(1-\alpha)^2(-1 + 2\alpha)}}$$

and $\zeta = \mathrm{e}^{\mathrm{i}\theta_0}$ *with*

$$\theta_0 = \arccos\left(\frac{(-3+\alpha)(-22 + 41\alpha - 13\alpha^2 + 2\alpha^3)}{54 - 114\alpha + 52\alpha^2 - 8\alpha^3}\right),$$

which is in $\tilde{f_4} \in \mathcal{K}(\alpha)$.

Example 4.3. *Let* $\beta \in (0, 1]$ *and consider the function* $f \in \mathcal{SK}_\beta = \mathcal{K}(\varphi^*_\beta)$, *where* φ^*_β *is defined by* (5). *Then* $p_1 = 2\beta$, $p_2 = 2\beta^2$ *and* $p_3 = 2\beta(1 + 2\beta^2)/3$. *Note that* $(p_1^3 - 2p_1 - p_1p_2)/2 < p_3 < (p_1^3 + 2p_1 - p_1p_2)/2$ *for all* $\beta \in (0, 1]$. *Firstly, let* $\beta \in (0, 1/6]$. *Then* $p_1^2 + 4p_2 \leq p_1$. *Thus from Theorem 4.1.A.(a) we get the sharp inequality*

$$|a_2a_3 - a_4| \leq \frac{1}{6}\beta. \tag{18}$$

The equality holds for the function

$$\tilde{f_1}(z) = \int_0^z \left(\exp\left[\int_0^\zeta \frac{1}{\xi} \left(\left(\frac{1+\xi^3}{1-\xi^3} \right)^\beta - 1 \right) \mathrm{d}\xi \right] \right) \mathrm{d}\zeta, \quad z \in \mathbb{D}, \tag{19}$$

which is in $\mathcal{SK}_\beta$. *Let now* $\beta \in [1/6, 2/3]$. *Since*

$$\tilde{\sigma}_1 - p_3 = -\frac{2}{27}\beta(32 + 45\beta - 117\beta^2 - 108\beta^3),$$

we see that $\tilde{\sigma}_1 \le p_3$ *when* $\beta \in [1/6, \beta_2]$ *and* $\tilde{\sigma}_1 \ge p_3$ *when* $\beta \in [\beta_2, 2/3]$, *where* $\beta_2 \approx 0.568$ *is the zero of the equation* $32 + 45x - 117x^2 + 108x^3 = 0$. *Therefore, if* $\beta \in [1/6, \beta_2]$ *by Theorem 4.1.B.(a) the sharp inequality* (18) *holds with* $\tilde{f_1}$ *defined by* (19) *as the extremal function. If* $\beta \in [\beta_2, 2/3]$, *then taking into account that*

$$\frac{1}{6}(3p_1^3 - 3p_1p_2 - 4p_1 - 2|p_1^2 + 2p_2|) - p_3 = -\frac{2}{3}\beta(3 + 6\beta - \beta^2) < 0, \quad \beta \in (0, 1], \tag{20}$$

by Theorem 4.1.B.(c) we get the sharp inequality

$$|a_2a_3 - a_4| \le \frac{\beta(1+3\beta)^{3/2}}{9\sqrt{4 + 9\beta - \beta^2}}. \tag{21}$$

The equality holds for the function

$$\tilde{f_3}(z) = \int_0^z \left(\exp\left[\int_0^\zeta \frac{1}{\xi} \left(\left(\frac{1-\xi^2}{1 - 2\rho\xi + \xi^2} \right)^\beta - 1 \right) \mathrm{d}\xi \right] \right) \mathrm{d}\zeta, \quad z \in \mathbb{D}, \tag{22}$$

where $\rho = \sqrt{(1+3\beta)/(4+9\beta-\beta^2)}$, *which is in* $\mathcal{SK}_\beta$. *Let now* $\beta \in [2/3, 1]$. *Since*

$$p_3 - \tilde{\sigma}_2 = \frac{2\beta(4 - 12\beta - 49\beta^2 - 6\beta^3 - 9\beta^4)}{3(4 + 6\beta + 9\beta^2)} < 0, \quad \beta \in [2/3, 1],$$

by (20) *we have* $(3p_1^3 - 3p_1p_2 - 4p_1 - 2|p_1^2 + 4p_2|)/6 < p_3$. *Thus from Theorem 4.1.C.(b) it follows that the sharp inequality* (21) *holds with the* $\tilde{f_3}$ *defined by* (22) *as the extremal function. Summarizing, we get the following sharp result. Let* $\beta \in (0, 1]$ *and* $f \in \mathcal{SK}_\beta$ *be of the form* (1). *Then*

$$|a_2a_3 - a_4| \le \begin{cases} \frac{\beta}{6}, & \beta \in (0, \beta_2], \\ \frac{\beta(1+3\beta)^{3/2}}{9\sqrt{4+9\beta-\beta^2}}, & \beta \in [\beta_2, 1]. \end{cases}$$

Example 4.4. *Let* $f \in \mathcal{UCV} = \mathcal{K}(\varphi_P)$, *where* φ_P *is defined by* (6). *Since* $p_1 = 8/\pi^2$, $p_2 = 16/(3\pi^2)$ *and* $p_3 = 184/(45\pi^2)$, *we can easily check that* $\tilde{\sigma}_1 < p_3 < (p_1^3 - p_1p_2 + 2p_1)/2$. *Therefore by Theorem 4.1.B.(a) we get the sharp inequality*

$$|a_2a_3 - a_4| \le \frac{2}{3\pi^2}.$$

The equality holds for the function

$$\tilde{f_1}(z) = \int_0^z \left(\exp\left[\int_0^\zeta \frac{1}{\xi} \left(\varphi_P(\xi^3) - 1 \right) \mathrm{d}\xi \right] \right) \mathrm{d}\zeta, \quad z \in \mathbb{D},$$

which is in $\mathcal{UCV}$.

5. The class $\mathcal{R}(\varphi)$

Given $\varphi \in \mathcal{H}$ of the form (2) with $p_1 > 0$, $p_2, p_3 \in \mathbb{R}$, let $f \in \mathcal{R}(\varphi)$ be of the form (1). Then there exists $\omega \in \mathcal{B}_0$ of the form (7) such that

$$f'(z) = \varphi(\omega(z)), \quad z \in \mathbb{D}. \tag{23}$$

Substituting the series (1), (2) and (7) into (23) by equating the coefficient we get

$$a_2 = \frac{1}{2}c_1p_1, \quad a_3 = \frac{1}{3}(c_2p_1 + c_1^2p_2) \quad \text{and} \quad a_4 = \frac{1}{4}(c_3p_1 + 2c_1c_2p_2 + c_1^3p_3).$$

Hence

$$|a_2a_3 - a_4| = \frac{1}{4}p_1\Psi(\mathring{\mu}, \mathring{\nu}),$$

where Ψ is defined by (8),

$$\mathring{\mu} = \frac{2(3p_2 - p_1^2)}{3p_1}, \quad \mathring{\nu} = \frac{3p_3 - 2p_1p_2}{3p_1}.$$

Therefore, by applying the result in [13, Lemma 2], the sharp bound of the functional $J_{2,3}$ over the class $\mathcal{R}(\varphi)$ is among the following values:

$$C_1 := \frac{1}{4}p_1, \quad C_2 := \frac{1}{12}|3p_3 - 2p_1p_2|, \quad C_3 := \frac{\sqrt{3}(2|3p_2 - p_1^2| + 3p_1)^{3/2}}{54\sqrt{2|3p_2 - p_1^2| + 3p_1 - 2p_1p_2 + 3p_3}},$$

$$C_4 := \frac{(3p_3 - 2p_1p_2)(9p_2^2 - 6p_1^2p_2 + p_1^4 - 9p_1^2)^{3/2}}{54[(p_1^2 - 3p_2)^2 + 3p_1(2p_1p_2 - 3p_3)]\sqrt{p_1(3p_3 - 2p_1p_2 - 3p_1)}}$$

and

$$C_5 := \frac{\sqrt{3}(2|3p_2 - p_1^2| - 3p_1)^{3/2}}{54\sqrt{2|3p_2 - p_1^2| - 3p_1 + 2p_1p_2 - 3p_3}}.$$

Define

$$f(z) = \int_0^z \varphi(w(\xi))\mathrm{d}\xi, \quad z \in \mathbb{D}, \tag{24}$$

and by applying the analogue methods in Section 3 and 4, let us define the functions $\mathring{f}_i$ $(i = 1, \ldots, 5)$ as follows:

(1) $\mathring{f}_1 = f$, where f is the function defined by (24) with $\omega(z) = z^3$, $z \in \mathbb{D}$;

(2) $\mathring{f}_2 = f$, where f is the function defined by (24) with $\omega(z) = z$, $z \in \mathbb{D}$;

(3) $\mathring{f}_3 = f$, where f is the function defined by (24) with ω defined by (11), where

$$\rho = \sqrt{\frac{3p_1 - 2p_1^2 + 6p_2}{3(3p_1 - 2p_1^2 + 6p_2 - 2p_1p_2 + 3p_3)}}, \quad \zeta = -1;$$

(4) $\mathring{f}_4 = f$, where f is the function defined by (24) with ω defined by (11), where $\rho = \sqrt{\kappa_1/(3\kappa_2)}$, $\zeta = e^{i\theta_0}$ and $\theta_0 = \pm\arccos(\kappa_3/\kappa_4)$, and where

$$\kappa_1 := -12p_1^3p_2(3+2p_2) + p_1^5(9+4p_2) + 9p_1p_2^2(9+4p_2) - 6p_1^4p_3 - 54p_2^2p_3 + 9p_1^2(-3+4p_2)p_3,$$

$$\kappa_2 := (p_1(3+2p_2) - 3p_3)(p_1^4 + 9p_2^2 - 9p_1p_3),$$

$$\kappa_3 := (p_1^2 - 3p_2)[-2p_1^5(3+p_2) - 18p_1p_2^2(3+p_2) + 3p_1^3(-9+4p_2^2) + 3p_1^4p_3 - 18p_1^2(-3+p_2)p_3 + 27p_2^2p_3]$$

and

$$\kappa_4 := 3p_1[-12p_1^3p_2(3+2p_2) + p_1^5(9+4p_2) + 9p_1p_2^2(9+4p_2) - 6p_1^4p_3 - 54p_2^2p_3 + 9p_1^2(-3+4p_2)p_3];$$

(5) $\mathring{f}_5 = f$, where f is the function defined by (24) with ω defined by (11), where

$$\rho = \sqrt{\frac{3p_1 + 2p_1^2 - 6p_2}{3(3p_1 + 2p_1^2 - 6p_2 - 2p_1p_2 + 3p_3)}}, \quad \zeta = 1.$$

The following sharp upper bound of the functional $J_{2,3}$ over the class $\mathcal{R}(\varphi)$ holds.

Theorem 5.1. *Let $\varphi \in \mathcal{H}$ be of the form* (2) *with $p_1 > 0$, $p_2, p_3 \in \mathbb{R}$ and let*

$$\mathring{\sigma}_1 := \frac{2}{3}p_1p_2 - p_1 - \frac{2}{3}|3p_2 - p_1^2| + \frac{4(2|3p_2 - p_1^2| + 3p_1)^3}{729p_1^2}, \quad \mathring{\sigma}_2 := \frac{2}{3}p_1p_2 + \frac{p_1[2(3p_2 - p_1^2)^2 + 3p_1|3p_2 - p_1^2|]}{(p_1^2 - 3p_2)^2 + 9p_1^2 + 3p_1|p_1^2 - 3p_2|},$$

$$\mathring{\sigma}_3 := \frac{2}{3}p_1p_2 + \frac{p_1[2(3p_2 - p_1^2)^2 - 3p_1|3p_2 - p_1^2|]}{(p_1^2 - 3p_2)^2 + 9p_1^2 - 3p_1|p_1^2 - 3p_2|} \quad \text{and} \quad \mathring{\sigma}_4 := \frac{2}{3}p_1p_2 + \frac{9p_2^2 - 6p_1^2p_2 + p_1^4 + 18p_1^2}{27p_1}.$$

Let $f \in \mathcal{R}(\varphi)$ be of the form (1). *Then the following sharp inequalities hold:*

A. When $4|3p_2 - p_1^2| \le 3p_1$:

- *(a) If $p_1(2p_2 - 3)/3 \le p_3 \le p_1(2p_2 + 3)/3$, then $|a_2a_3 - a_4| \le C_1$ and the extremal function is $\mathring{f}_1$;*
- *(b) If $p_3 \le p_1(2p_2 - 3)/3$ or $p_3 \ge p_1(2p_2 + 3)/3$, then $|a_2a_3 - a_4| \le C_2$ and the extremal function is $\mathring{f}_2$.*

B. When $3p_1 \le 4|3p_2 - p_1^2| \le 12p_1$:

- *(a) If $\mathring{\sigma}_1 \le p_3 \le p_1(2p_2 + 3)/3$, then $|a_2a_3 - a_4| \le C_1$ and the extremal function is $\mathring{f}_1$;*
- *(b) If $p_3 \le (6p_1p_2 - 4|p_1^2 - 3p_2| - 6p_1)/9$ or $p_3 \ge p_1(2p_2 + 3)/3$, then $|a_2a_3 - a_4| \le C_2$ and the extremal function is $\mathring{f}_2$;*
- *(c) If $(6p_1p_2 - 4|p_1^2 - 3p_2| - 6p_1)/9 \le p_3 \le \mathring{\sigma}_1$, then $|a_2a_3 - a_4| \le C_3$ and the extremal function is $\mathring{f}_3$.*

C. When $3p_1 < |3p_2 - p_1^2| \le 6p_1$:

- *(a) If $p_3 \le (6p_1p_2 - 4|p_1^2 - 3p_2| - 6p_1)/9$ or $p_3 \ge \mathring{\sigma}_4$, then $|a_2a_3 - a_4| \le C_2$ and the extremal function is $\mathring{f}_2$;*
- *(b) If $(6p_1p_2 - 4|p_1^2 - 3p_2| - 6p_1)/9 \le p_3 \le \mathring{\sigma}_2$, then $|a_2a_3 - a_4| \le C_3$ and the extremal function is $\mathring{f}_3$;*
- *(c) If $\mathring{\sigma}_2 \le p_3 \le \mathring{\sigma}_4$, then $|a_2a_3 - a_4| \le C_4$ and the extremal function is $\mathring{f}_4$.*

D. When $|3p_2 - p_1^2| \ge 6p_1$:

- *(a) If $p_3 \le (6p_1p_2 - 4|p_1^2 - 3p_2| - 6p_1)/9$ or $p_3 \ge (6p_1p_2 + 4|p_1^2 - 3p_2| - 6p_1)/9$, then $|a_2a_3 - a_4| \le C_2$ and the extremal function is $\mathring{f}_2$;*

(b) *If* $(6p_1p_2 - 4|p_1^2 - 3p_2| - 6p_1)/9 \le p_3 \le \mathring{\sigma}_2$, *then* $|a_2a_3 - a_4| \le C_3$ *and the extremal function is* $\mathring{f}_3$;

(c) *If* $\mathring{\sigma}_2 \le p_3 \le \mathring{\sigma}_3$, *then* $|a_2a_3 - a_4| \le C_4$ *and the extremal function is* $\mathring{f}_4$;

(d) *If* $\mathring{\sigma}_3 \le p_3 \le (6p_1p_2 + 4|p_1^2 - 3p_2| - 6p_1)/9$, *then* $|a_2a_3 - a_4| \le C_5$ *and the extremal function is* $\mathring{f}_5$.

Example 5.2. *(see [6, Theorem 2.1]) Let* $\alpha \in [0, 1)$ *and* $f \in \mathcal{R}(\alpha) = \mathcal{R}(\varphi_\alpha)$, *where* φ_α *is defined by* (4). *Since* $p_1 = p_2 = p_3 = 2(1-\alpha)$, *we see that* $3p_1 \le 4|3p_2 - p_1^2| \le 12p_1$ *for all* $\alpha \in [0, 1)$. *We have*

$$\mathring{\sigma}_1 - p_3 = -\frac{16}{729}(1-\alpha)^2(59 + 152\alpha + 32\alpha^2) < 0, \quad \alpha \in [0, 1)$$

and

$$p_3 - \frac{1}{3}p_1(2p_2 + 3) = -\frac{8}{3}(1-\alpha)^2 < 0, \quad \alpha \in [0, 1).$$

Thus $\mathring{\sigma}_1 \le p_3 \le p_1(2p_2 + 3)/3$ *and by Theorem 5.1.B.(a) we get the sharp inequality*

$$|a_2a_3 - a_4| \le \frac{1}{2}(1-\alpha).$$

The equality holds the function

$$\mathring{f}_1(z) = \int_0^z \frac{1 + (1-2\alpha)\xi^3}{1 - \xi^3} d\xi, \quad z \in \mathbb{D},$$

which is in $\mathcal{R}(\alpha)$.

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