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# The Global Behavior of a Quadratic Difference Equation

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**Abstract.** In this paper we will present the Julia set and the global behavior of a quadratic second order difference equation of type

$$x_{n+1} = ax_nx_{n-1} + ax_{n-1}^2 + bx_{n-1}$$

where $a > 0$ and $0 \leq b < 1$ with non-negative initial conditions.

## 1. Introduction

In general, polynomial difference equations and polynomial maps in the plane have been studied in both the real and complex domains (see [3, 4]). First results on quadratic polynomial difference equation have been obtained in [1, 2] but these results gave us only a part of the basins of attraction of equilibrium points and period-two solutions. In [6], the general second order difference equation is completely investigated and described the regions of initial conditions in the first quadrant for which all solutions tend to equilibrium points, period-two solutions, or the point at infinity, except for the case of infinitely many period-two solutions. Also, the new method of proof for the non-hyperbolic case is presented in [6]. Our results are based on the theorems which hold for monotone difference equations. Our principal tool is the theory of monotone maps, and in particular cooperative maps, which guarantee the existence and uniqueness of the stable and unstable invariant manifolds for the fixed points and periodic points (see [5]). Consider the difference equation

$$x_{n+1} = f(x_n, x_{n-1}), \quad n = 0, 1, \ldots \tag{1}$$

where $f$ is a continuous and increasing function in both variables. The following result has been obtained in [1]:

**Theorem 1.1.** *Let $I \subseteq R$ and let $f \in C[I \times I, I]$ be a function which increases in both variables. Then for every solution of Eq.*(1) *the subsequences $\{x_{2n}\}_{n=0}^{\infty}$ and $\{x_{2n+1}\}_{n=-1}^{\infty}$ of even and odd terms of the solution do exactly one of the following:*

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(*i*) *Eventually they are both monotonically increasing.*

(*ii*) *Eventually they are both monotonically decreasing.*

(*iii*) *One of them is monotonically increasing and the other is monotonically decreasing.*

As a consequence of Theorem 1.1 every bounded solution of Eq.(1) approaches either an equilibrium solution or period-two solution and every unbounded solution is asymptotic to the point at infinity in a monotonic way. Thus the major problem in dynamics of Eq.(1) is the problem of determinig the basins of attraction of three different types of attractors: the equilibrium solutions, period-two solution(s) and the point(s) at infinity. The following result can be proved by using the techniques of proof of Theorem 11 in [5].

**Theorem 1.2.** *Consider Eq.*(1) *where $f$ is increasing function in its arguments and assume that there is no minimal period-two solution. Assume that $E_1(x_1, y_1)$ and $E_2(x_2, y_2)$ are two consecutive equilibrium points in North-East ordering that satisfy*

$$(x_1, y_1) \preceq_{ne} (x_2, y_2)$$

*and that $E_1$ is a local attractor and $E_2$ is a saddle point or a non-hyperbolic point with second characteristic root in interval $(-1, 1)$, with the neighborhoods where $f$ is strictly increasing. Then the basin of attraction $\mathcal{B}(E_1)$ of $E_1$ is the region below the global stable manifold $\mathcal{W}^s(E_2)$. More precisely*

$$\mathcal{B}(E_1) = \{(x, y) : \exists y_u : \ y < y_u, \quad (x, y_u) \in \mathcal{W}^s(E_2)\}.$$

*The basin of attraction $\mathcal{B}(E_2) = \mathcal{W}^s(E_2)$ is exactly the global stable manifold of $E_2$. The global stable manifold extend to the boundary of the domain of Eq.*(1)*. If there exists a period-two solution, then the end points of the global stable manifold are exactly the period two solution.*

Now, the theorems applied in [6] provided the two continuous curves $\mathcal{W}^s(E_2)$ (stable manifold) and $\mathcal{W}^u(E_2)$ (unstable manifold), both passing through the point $E_2(x_2, y_2)$ according to Theorem 1.2, such that $\mathcal{W}^s(E_2)$ is a graph of a decreasing function and $\mathcal{W}^u(E_2)$ is a graph of an increasing function. The curve $\mathcal{W}^s(E_2)$ splits the first quadrant of initial conditions into two disjoint regions, but we do not know the explicit form of the curve $\mathcal{W}^u(E_2)$. In this paper we investigate the following difference equation

$$x_{n+1} = ax_n x_{n-1} + ax_{n-1}^2 + bx_{n-1} \tag{2}$$

where $a > 0$ and $b \in [0, 1)$, that has infinitely many period-two solutions and we expose the explicit form of the curve that separates the first quadrant into two basins of attraction of a locally stable equilibrium point and of the point at infinity. One of the major problems in the dynamics of polynomial maps is determining the basin of attraction of the point at infinity and in particular the boundary of the basin known as the Julia set, what we managed to do for Eq.(2). We also obtain the global dynamics in the interior of the Julia set, which includes all the points for which solutions are not asymptotic to the point at infinity. It turns out that the Julia set for Eq.(2) is the union of the stable manifolds of some saddle equilibrium points, nonhyperbolic equilibrium points or period-two points. We first list some results needed for the proofs of our theorems. The main result for studying local stability of equilibria is linearized stability Theorem 1.1 in [7].

**Theorem 1.3.** *(linerized stability): Consider the difference equation*

$$x_{n+1} = f(x_n, x_{n-1}) \tag{3}$$

*and let $\bar{x}$ be an equilibrium point of difference equation* (3)*. Let $p = \frac{\partial f(\bar{x},\bar{x})}{\partial u}$ and $q = \frac{\partial f(\bar{x},\bar{x})}{\partial v}$ denote the partial derivatives of $f(u, v)$ evaluated at the equilibrium $\bar{x}$. Let $\lambda_1$ and $\lambda_2$ roots of the quadratic equation $\lambda^2 - p\lambda - q = 0$.*

*a) If $|\lambda_1| < 1$ and $|\lambda_2| < 1$, then the equilibrium $\bar{x}$ is locally asymptotically stable (sink).*

b) *If* $|\lambda_1| > 1$ *or* $|\lambda_2| > 1$, *then the equilibrium* $\bar{x}$ *is unstable.*

c) $|\lambda_1| < 1$ *and* $|\lambda_2| < 1 \Leftrightarrow |p| < 1 - q < 2$. *Equilibrium* $\bar{x}$ *is a sink.*

d) $|\lambda_1| > 1$ *and* $|\lambda_2| > 1 \Leftrightarrow |q| > 1$ *and* $|p| < |1 - q|$. *Equilibrium* $\bar{x}$ *is a repeller.*

e) $|\lambda_1| > 1$ *and* $|\lambda_2| < 1 \Leftrightarrow |p| > |1 - q|$. *Equilibrium* $\bar{x}$ *is a saddle point.*

f) $|\lambda_1| = 1$ *or* $|\lambda_2| = 1 \Leftrightarrow |p| = |1 - q|$ *or* $q = -1$ *and* $|p| \leq 2$. *Equilibrium* $\bar{x}$ *is called a non-hyperbolic point.*

The next theorem (Theorem 1.4.1. in [8]) is a very useful tool in establishing bounds for the solutions of nonlinear equations in terms of the solutions of equations with known behaviour.

**Theorem 1.4.** *Let* $I$ *be an interval of real numbers, let* $k$ *be a positive integer, and let* $F : I^{k+1} \to I$ *be a function which is increasing in all its arguments. Assume that* $\{x_n\}_{n=-k}^{\infty}, \{y_n\}_{n=-k}^{\infty}$ *and* $\{z_n\}_{n=-k}^{\infty}$ *are sequences of real numbers such that*

$$x_{n+1} \leq F(x_n, \ldots, x_{n-k}), \quad n = 0, 1, \ldots$$

$$y_{n+1} = F(y_n, \ldots, y_{n-k}), \quad n = 0, 1, \ldots$$

$$z_{n+1} \geq F(z_n, \ldots, z_{n-k}), \quad n = 0, 1, \ldots$$

*and*

$$x_n \leq y_n \leq z_n, \quad for \quad all \quad -k \leq n \leq 0.$$

*Then*

$$x_n \leq y_n \leq z_n, \quad for \quad all \quad n > 0.$$

## 2. Main results

By applying the Theorem 1.3 we obtain the following results on local stability of the zero equilibrium of Eq.(2):

**Proposition 2.1.** *The zero equilibrium of Eq.(2) is one of the following:*

a) *locally asymptotically stable if* $b < 1$,

b) *non-hyperbolic and locally stable if* $b = 1$,

c) *unstable if* $b > 1$.

The linearized equation at the positive equilibrium $\bar{x}$ is

$$z_{n+1} = pz_n + qz_{n-1} = a\bar{x}z_n + (3a\bar{x} + b)z_{n-1}.$$

Now, in view of Theorem 1.3 we obtain the following results on local stability of the positive equilibrium of Eq.(2):

**Proposition 2.2.** *The positive equilibrium of Eq.(2) is one of the following:*

a) *locally asymptotically stable if* $p + q < 1$,

b) *non-hyperbolic and locally stable if* $p + q = 1$, *c) unstable if* $p + q > 1$,

*d) saddle point if* $p > |q - 1|$,

*e) repeller if* $1 - q < p < q - 1$.

**Theorem 2.3.** *Consider the difference equation (2) in the first quadrant of initial conditions, where* $a > 0$ *and* $b \in [0, 1)$. *Then Eq.(2) has a zero equilibrium and a unique positive equilibrium* $\bar{x} = \frac{1-b}{2a}$. *The line* $a(x + y) = 1 - b$ *is the Julia set and separates the first quadrant into two regions: the region below the line is the basin of attraction of point* $E_0(0, 0)$, *the region above the line is the basin of attraction of the point at infinity and every point on the line except* $E_+(\bar{x}, \bar{x})$ *is a period-two solution of Eq.(2)* .

*Proof.* The equilibrium points of Eq. (2) are the solutions of quadratic equation

$$x = ax^2 + ax^2 + bx,$$

which implies two equilibria: zero equilibrium and positive equilibrium $\bar{x} = \frac{1-b}{2a}$. Since $a > 0$ and $b \in [0, 1)$ then by applying Proposition (2.1) the zero equilibrium is locally asymptotically stable. Denote by $f(x, y) = axy + ay^2 + by$ and let $p$ and $q$ denote the partial derivatives of function $f$ at point $E_+$. By straightforward calculation we obtain that the following hold:

$$p + q = 4a\bar{x} + b = 2 - b > 1,$$

$$q - p = 2a\bar{x} + b = 1.$$

Hence, according to Proposition (2.2) the positive equilibrium is an unstable non-hyperbolic point. Period-two solution $u, v$ satisfies the system

$$u = (av + au + b)u$$

$$v = (au + av + b)v.$$

Since $v > 0$ this implies $a(u + v) + b = 1$. Therefore every point of the set $\{(x, y) : a(x + y) + b = 1\}$ is a period-two solution of Eq.(2) except point $E_+$. Let $\{x_n\}$ be a solution of Eq.(2) for initial condition $(x_{-1}, x_0)$ which lies below the line $a(x + y) + b = 1$. Then

$$a(x_{-1} + x_0) + b < 1$$

and

$$\begin{aligned} x_1 &= (a(x_{-1} + x_0) + b)x_{-1} < x_{-1}, \\ x_2 &= (a(x_0 + x_1) + b)x_0 < (a(x_0 + x_{-1}) + b)x_0 < x_0. \end{aligned}$$

Thus $(x_1, x_2)$ and $(x_{-1}, x_0)$ are two points in North-East ordering $(x_1, x_2) \leq_{ne} (x_{-1}, x_0)$ which means that the point $(x_1, x_2)$ is also below the line $a(x + y) + b = 1$ and so

$$a(x_1 + x_2) + b < 1.$$

Similarly we find

$$\begin{aligned} x_3 &= (a(x_2 + x_1) + b)x_1 < x_1, \\ x_4 &= (a(x_3 + x_2) + b)x_2 < (a(x_1 + x_2) + b)x_2 < x_2. \end{aligned}$$

Continuing in this way we get

$$(0, 0) \leq_{ne} \ldots \leq_{ne} (x_3, x_4) \leq_{ne} (x_1, x_2) \leq_{ne} (x_{-1}, x_0)$$

which implies that both subsequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ are monotonically decreasing and bounded below by 0. Since below the line $a(x+y)+b = 1$ there are no period-two solutions it must be $x_{2n} \to 0$ and $x_{2n+1} \to 0$. On the other hand, if we consider solution $\{x_n\}$ of Eq.(2) for initial condition $(x_{-1}, x_0)$ which lies above the line $a(x + y) + b = 1$ then

$$a(x_{-1} + x_0) + b > 1$$

and by applying the method shown above we obtain the following condition:

$$(x_{-1}, x_0) \leq_{ne} (x_1, x_2) \leq_{ne} (x_3, x_4) \leq_{ne} \dots$$

Therefore both subsequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ are monotonically increasing, hence $x_{2n} \to \infty$ and $x_{2n+1} \to \infty$ as $n \to \infty$. □

In view of Theorem 1.4 which implies results on difference inequalities we get the following:

**Proposition 2.4.** *Consider the difference equation of type*

$$x_{n+1} = ax_n x_{n-1} + bx_{n-1}^2 + cx_{n-1}, \tag{4}$$

*where parameters $a, b, c$ satisfy conditions $a, b > 0$ and $0 \leq c < 1$. Then the global stable manifold of the positive equilibrium is between two lines*

$$p_1 : min\{a, b\}(x + y) + c = 1 \tag{5}$$

*and*

$$p_2 : max\{a, b\}(x + y) + c = 1. \tag{6}$$

*Proof.* It is easy to show that Eq. (4) has two equilibria: since $c < 1$ the zero equilibrium is always locally asymptotically stable thus the positive equilibrium must be unstable equilibrium point. More precisely, if the positive equilibrium is a saddle point or a non-hyperbolic point then by Theorem 4 in [6] there exists a global stable manifold which contains point $E_+(\bar{x}, \bar{x})$, where $\bar{x}$ is the positive equilibrium. In this case global behavior of Eq. (4) is described by Theorem 9 in [6]. If the positive equilibrium is a repeller then there exists a period-two solution. By using Theorem 8 in [6] we obtain that the period-two solution is a saddle point and there are two global stable manifolds which contain points $P_1(u, v)$ and $P_2(v, u)$ wehre $(u, v)$ is unique period-two solution of Eq.(4). In this case the global behavior of Eq.(4) is described by Theorem 10 in [6]. Furthermore

$$x_{n+1} \geq min\{a, b\}(x_n x_{n-1} + x_{n-1}^2) + cx_{n-1}$$

and

$$x_{n+1} \leq max\{a, b\}(x_n x_{n-1} + x_{n-1}^2) + cx_{n-1}$$

for all $n$, by applying Theorem (1.4) for solution $\{x_n\}$ of Eq.(4) the following inequality holds

$$y_n \leq x_n \leq z_n,$$

for all $n$, where $\{y_n\}$ is a solution of the difference equation

$$x_{n+1} = min\{a, b\}(x_n x_{n-1} + x_{n-1}^2) + cx_{n-1}, \tag{7}$$

and $\{z_n\}$ is a solution of the difference equation

$$x_{n+1} = max\{a, b\}(x_n x_{n-1} + x_{n-1}^2) + cx_{n-1}. \tag{8}$$

Since Eq. (7) and Eq. (8) satisfy all conditions of Theorem (2.3) this implies that the statement of Proposition (2.4) holds. □

Let

$$A = min\{a, b\} \quad and \quad B = max\{a, b\}.$$

One can show that the distance between lines $p_1$ and $p_2$ is given by

$$d = d(p_1, p_2) = \frac{1 - c}{AB\sqrt{2}}(B - A).$$

If we set $C = \frac{A+B}{2}$ then the line

$$p_3 : C(x + y) + c = 1 \tag{9}$$

is between the lines $p_1$ and $p_2$. Clearly, smaller $C$ implies that the distance from the point $(0, 0)$ to the line $p_3$ increses. Let $\mathcal{W}^s$ denotes the global stable manifold of the positive equlibrium of Eq.(4). Now, if we choose constants $a$ and $b$ such that

$$\frac{1-c}{AB\sqrt{2}}(B-A) < \varepsilon$$

where $\varepsilon > 0$ is a small enough, then $\left|p_3(x) - \mathcal{W}^s(x)\right| < \left|p_1(x) - p_2(x)\right| < \varepsilon$ for all $x \geq 0$, where $p_1(x)$, $p_2(x)$, $p_3(x)$ and $\mathcal{W}^s(x)$ denote the values of corresponding functions evaluated at the point $x$, which completely determine the global behavior of this equation. That means the line $p_3$ is approximation of stable manifold(s) $\mathcal{W}^s$. All this leads to the following theorem:

**Theorem 2.5.** *Consider Eq. (4) where parameters $a, b, c$ satisfy conditions $a, b > 0$ and $0 \leq c < 1$. Let $A = min\{a, b\}$ and $B = max\{a, b\}$ and assume that*

$$\frac{1-c}{AB\sqrt{2}}(B-A) < \varepsilon$$

*where $\varepsilon > 0$ is a small enough real number. Then the line $p_3$ given by (9) is approximation of global stable manifold(s) of Eq. (4) corresponding to the positive equilibrium of Eq.(4) or period-two solutions.*

It is easy to show that the area between lines $p_1$ and $p_2$ in the first quadrant is given by

$$P = \frac{(1-c)^2}{2}\frac{B^2-A^2}{B^2A^2}.$$

Hence, if we assume that

$$\frac{1-c}{AB}(B+A) < \varepsilon$$

where $\varepsilon > 0$ is small enough real number, then the inequality below holds

$$\frac{1-c}{AB}B - A < \varepsilon.$$

In this case we get $P < \frac{\varepsilon^2}{2}$ and so both lines $p_1$ and $p_2$ are approximations of global stable manifold(s) of Eq. (4) corresponding to the positive equilibrium of Eq. (4) or period-two solutions. This proves the following Theorem:

**Theorem 2.6.** *Consider Eq. (4) where parameters $a, b, c$ satisfy conditions $a, b > 0$ and $0 \leq c < 1$. Let $A = min\{a, b\}$ and $B = max\{a, b\}$ and assume that*

$$\frac{1-c}{AB}(B+A) < \varepsilon$$

*where $\varepsilon > 0$ is a small enough real number. Then the lines $p_1$ and $p_2$ given by (5) and (6) are pproximations of global stable manifold(s) of Eq. (4) corresponding to the positive equilibrium of Eq. (4) or period-two solutions.*

Next Figures support and illustrate Theorems (2.3) and (2.5) with wide dynamics. All figures are generated by Mathematica 6.0 and Dynamica 3.0

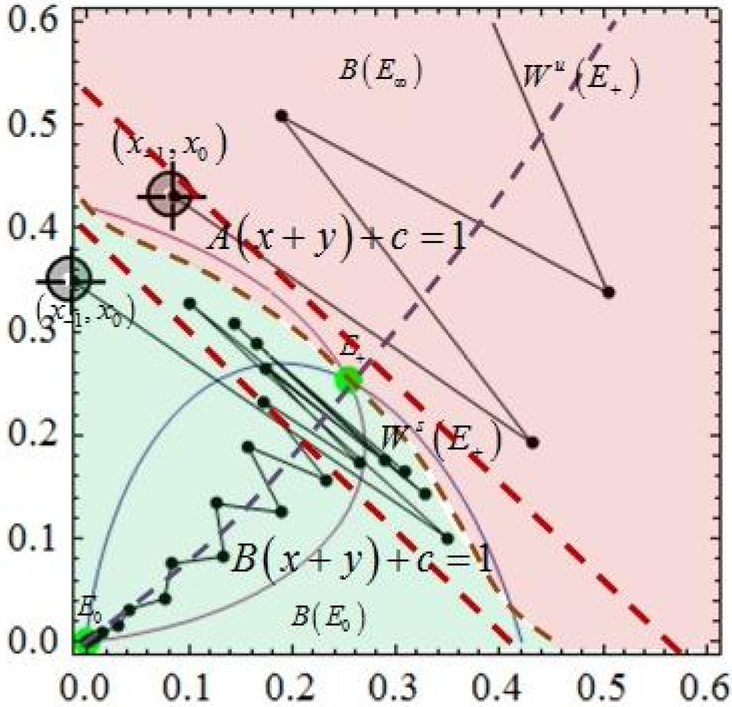


Figure 1: Case when Eq.(4) has no period-two solutions

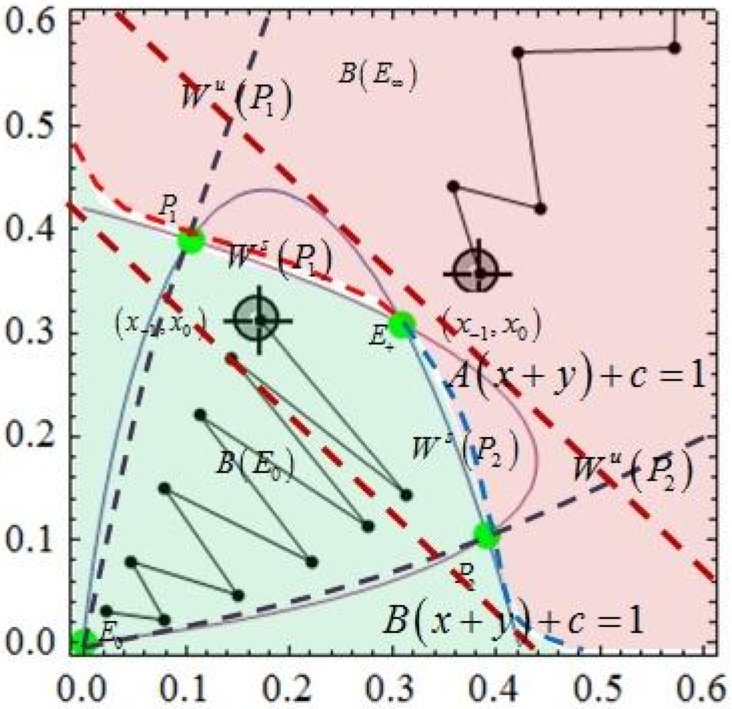


Figure 2: Case when Eq.(2) has period-two solutions

## 3. Conclusion

In this paper we restrict our attention to certain polynomial quadratic second order difference equation Eq. (2). It is important to mention that we have accurately determined the Julia set of Eq. (2) and the basins of attractions for the zero equilibrium and the positive equilibrium point. In general, all theoretical concepts which are very useful in proving the results of global attractivity of equilibrium points and period-two solutions only give us existence of global stable manifold(s) whose computation leads to very uncomfortable calculus. Also, the last part of this paper is devoted to the approximation of the global stable manifold(s) for a class of quadratic difference equations given by (4). The given results are more special but very applicable.

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