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Identification of Points Sources via Time Fractional Diffusion Equation

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Abstract. This article investigates the source identification in the fractional diffusion equations, by performing a single measurement of the Cauchy data on the accessible boundary. The main results of this work consist in giving an identifiability result and establishing a local Lipschitz stability result. To solve the inverse problem of identifying fractional sources from such observations, a non-iterative algebraical method based on the Reciprocity Gap functional is proposed.

1. Introduction

The main purpose of this paper is the identification of source term F that represents the number, the positions and the intensities of monopolar sources located in an open bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, and with smooth regular boundary Σ. The corresponding forward problem is given by:

$$\begin{cases} {}_0^cD_t^\alpha u - \Delta u = F \text{ in } \Omega_T, \\ u(x,0) = 0 \;\; x \in \;\Omega, \\ u = f \text{ on } \Sigma_T, \end{cases} \tag{1.1}$$

where ${}_0^cD_t^\alpha$ represents the Caputo fractional derivatives of order α defined in definition 2.1, $f \in L^2(\Sigma_T)$ and $F(x,t)$ is the source term that have the following form:

$$F(x,t) = \sum_{j=1}^{m} \lambda_j(t)\, \delta_{S_j}(x), \tag{1.2}$$

$$\lambda_j(t) := \begin{cases} \beta_j > 0, \; t \in \; [0,T) \\ \\ 0, \;\; t \geq T \end{cases} \tag{1.3}$$

where $m \in \mathbb{N}$, $S_j \in \Omega$, and $\lambda_j(t)$, $j = 1, \ldots, m$, represent respectively the number, the locations, and the intensities of the monopolar sources inactive after the finite time $T > 0$ which represents the time of

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observation. We denote by $\Omega_T := \Omega \times (0, T)$ the space time domain, and $\Sigma_T := \Sigma \times (0, T)$ its lateral boundary. For $0 < \alpha < 1$, equation (1.1) is called a fractional diffusion equation, and it is called a fractional diffusion-wave equation in the case when $1 < \alpha < 2$. Note that if $\alpha = 0$, $\alpha = 1$ and $\alpha = 2$, the equation (1.1) represents the sources identification via the Helmholtz equation, the heat equation, and the wave equation which are studied by many authors [5, 15, 17, 27, 28]. In this paper, we are interested mainly in the fractional diffusion case (we restrict the order α to the case $0 < \alpha < 1$).

The fractional equation is one of tools for modeling several atypical phenomena in nature and in the theory of complex systems. The fractional diffusion equation has been introduced in physics to describe diffusions in media with fractal geometry see [32], to show anomalous diffusion in a highly heterogeneous aquifer see [2]. Metzler and Klafter [29] proved that a fractional diffusion equation governs a non-Markovian diffusion process with a memory. Ginoaet *et al* in [20] presented a fractional diffusion equation describing relaxation phenomena in complex viscoelastic materials.

The main motivation of this work concerns the inverse problem of identifying of contaminants sources in groundwater. There is a little work on inverse problems for fractional differential equations. Murio *et al* in [30] introduced a regularization technique for the approximate reconstruction of spatial and time varying source terms using the observed solutions of the forward time fractional diffusion problems on a discrete set of points. Nakagawa *et al* in [31] proposed that the solution can be uniquely determined by data in any small subdomain over time interval. Tuan [37] presented that by taking suitable initial distributions only finitely many measurements on the boundary are required to recover uniquely the diffusion coefficient of a one-dimensional fractional diffusion equation. Zhang and Xu [40] outlined that the unknown source term can also be uniquely determined by $u(0, t), 0 < t < T$. Wei and Zhang in [38] solve a nonlinear ill-posed problem for identifying a Robin coefficient in a time-fractional diffusion problem, they combine the integral equation method and the boundary element method to obtain a simple minimization problem with H^1 penalty terms. We remark that α involved in all the above articles was assumed to be in the interval $(0, 1)$, and most of the above fractional inverse problems are involved in one-dimensional spaces. Other recent results are obtained for the time-dependent source problem for multi-dimensional fractional diffusion equation. Wei et al in [39] studied the direct problem, showed that the inverse problem has a unique solution, and used the Tikhonov regularization method to solve the inverse source via an iterative method. Liu et al in [26] established multiple logarithmic stability and proposed a fixed point iteration for the numerical reconstruction. Wang et al in [34] gave a conditional stability for this inverse problem and proposed two regularization methods (an integral equation method and a standard Tikhonov regularization method) for the reconstruction of the time-dependent source term. Ruan et al. studied simultaneously in [35] the uniqueness of spacewise source term and the fractional order of 1D and 2D time fractional diffusion equations by using the Laplace transformation method and analytic continuation technique, and adopted an alternating minimization algorithm to solve the inverse problem.

In this work, equation (1.1) is supplemented by the boundary condition

$$\frac{\partial u}{\partial \nu}(x, t) = \varphi(x, t), \ (x, t) \in \ \Sigma_T \tag{1.4}$$

where ν represents the outward unit normal vector to Σ pointed outside Ω, $\varphi \in L^2(\Sigma_T)$, The inverse problem consists in identifying the source distribution F in the fractional problem (1.1) from the compatible boundary data $(f, \ \varphi)$.

This paper is organized as follows:

In section 2, we recall some results concerning the fractional diffusion equation and we discuss the question of existence and uniqueness of the direct problem. In section 3, an identifiability result is established. Local Lipschitz stability result is given in section 4. Finally, in section 5, an explicit non-iterative identification procedure is proposed.

2. Preliminaries

We start this section by giving some definitions and fundamental facts of fractional integrals and fractional derivatives, which can be found in [25, 33].

Let $\alpha > 0$ and n the integer satisfying $n-1 \leq \alpha < n$, $a, b \in \mathbb{R}$.

Definition 2.1. *Let $g : [a,\ b] \to \mathbb{R}$ be a function, and Γ the Euler gamma function.*

1. *The left and right Riemann-Liouville fractional integrals of order α are defined respectively by:*

$$ {}_aI_t^{\alpha} g(t) := \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} g(s)\ ds, \tag{2.1}$$

and

$$ {}_tI_b^{\alpha} g(t) := \frac{1}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} g(s)\ ds. \tag{2.2}$$

2. *The left and right Riemann-Liouville fractional derivatives of order α are defined respectively by*

$$ {}_aD_t^{\alpha} g(t) := \frac{d^n}{dt^n} {}_aI_t^{n-\alpha} g(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-s)^{n-\alpha-1} g(s)\ ds, \tag{2.3}$$

and

$$ {}_tD_b^{\alpha} g(t) := (-1)^n \frac{d^n}{dt^n} {}_tI_b^{n-\alpha} g(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_t^b (s-t)^{n-\alpha-1} g(s)\ ds \tag{2.4}$$

In particular, if $\alpha = 0$

$$ {}_aD_t^{0} g(t) = {}_tD_b^{0} g(t) = g(t)$$

and if $\alpha = k \in \mathbb{N}$

$$ {}_aD_t^{k} g(t) = {}_tD_b^{k} g(t) = g^{(k)}(t).$$

3. *The left and right Caputo fractional derivatives of order α are defined by*

$$ {}_a^cD_t^{\alpha} g(t) :={}_a I_t^{n-\alpha} g^{(n)}(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} g^{(n)}(s)\ ds, \tag{2.5}$$

and

$$ {}_t^cD_b^{\alpha} g(t) := (-1)^n\ {}_tI_b^{n-\alpha} g^{(n)}(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_t^b (s-t)^{n-\alpha-1} g^{(n)}(s)\ ds \tag{2.6}$$

In particular, if $0 < \alpha < 1$, we denote by

$$ \partial_t^{\alpha} g(t) :={}_0^c D_t^{\alpha} g(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} g'(s) ds.$$

Lemma 2.2. *[25] If $g(t) \in AC^n[a,b]$, then the Riemann-Liouville fraction derivative and the Caputo fractional derivative are connected with each other by the following relation*

$$ {}_aD_t^{\alpha} g(t) ={}_a^c D_t^{\alpha} g(t) + \sum_{k=0}^{n-1} \frac{g^{(k)}(a)}{\Gamma(1+k-\alpha)} (t-a)^{k-\alpha}. \tag{2.7}$$

and

$$ {}_tD_b^{\alpha} g(t) ={}_t^c D_b^{\alpha} g(t) + \sum_{k=0}^{n-1} \frac{(-1)^k g^{(k)}(b)}{\Gamma(1+k-\alpha)} (b-t)^{k-\alpha}. \tag{2.8}$$

Here

$$AC^n[a,b] = \left\{ g : [a,\ b] \to \mathbb{R} \text{ such that } \frac{d^{n-1}}{dx^{n-1}}(g) \in AC[a,b] \right\},$$

$$g \in AC[a,b] \Leftrightarrow \text{there exists } \varphi \in L(a,b) \text{ such that } g(x) = c + \int_a^b \varphi(t)\, dt, c \in \mathbb{R}$$

and $L(a,b)$ is the set of Lebesgue complex-valued measurable functions on $[a,b]$.

Remark 2.3. *In problem (1.1), $0 < \alpha < 1$, then*

$$_0D_t^\alpha u(x,t) = {}_0^c D_t^\alpha u(x,t) + \frac{u(x,0)}{\Gamma(1-\alpha)} t^{-\alpha}, \tag{2.9}$$

in addition $u(x,0) = 0$, then $_0D_t^\alpha u(x,t) = {}_0^c D_t^\alpha u(x,t)$.

In order to formulate the boundary integral equation corresponding to (1.1), we need to calculate the fundamental solution G with the initial condition $G(\cdot,\ 0) = 0$ in Ω. This function is constructed by taking the Laplace-transform in the time and the Fourier-transform in the spatial variable of the fractional diffusion equation

$$\partial_t^\alpha G(x,t) - \Delta G(x.t) = \delta(x,t) \text{ in } \mathbb{R}^d \times \mathbb{R},$$

where $\delta(x,t)$ is the Dirac's delta function [19, 23, 36].
The fundamental solution is given by:

$$G(x,t) = \begin{cases} \dfrac{H_{12}^{20}\left[\frac{1}{4}|x|^2 t^{-\alpha}\Big|^{(\alpha,\alpha)}_{(\frac{d}{2},1),(1,1)}\right]}{\pi^{\frac{d}{2}}|x|^d t^{1-\alpha}}, & t > 0. \\ 0, & t < 0. \end{cases} \tag{2.10}$$

Here H_{12}^{20} is the Fox H-function, which is defined via Mellin-Barnes integral representation

$$H_{12}^{20}(z) := H_{12}^{20}\left[z\Big|^{(\alpha,\alpha)}_{(\frac{d}{2},1),(1,1)}\right] = \frac{1}{2\pi i}\int_C \frac{\Gamma(\frac{d}{2}+s)\Gamma(1+s)}{\Gamma(\alpha+\alpha s)} z^{-s}\, ds, \tag{2.11}$$

where C is an infinite contour on the complex plane circulating the negative real axis counterclockwise, $z \neq 0$ and $z^{-s} = \exp(-s[\ln|z| + i\arg z])$, $\ln|z|$ represents the natural logarithm of $|z|$ and $\arg z$ is not necessary the principal value.
We shall solve the problem (1.1) by using the Laplace transform and considering the equivalent problem on the space Laplace domain. For $s \in \mathbb{C}$ with $\mathcal{R}e(s) > 0$, and by applying the Laplace transform of the Riemann-Liouville fractional derivative to the problem (1.1) in time, we get

$$\begin{cases} s^\alpha \widetilde{u}(x,s) - \Delta\widetilde{u}(x,s) &= \displaystyle\sum_{j=1}^m \frac{\beta_j(1-e^{-sT})}{s}\,\delta_{S_j} \quad \text{in} \quad \Omega, \\ \widetilde{u}(x,s) &= \widetilde{f}(x,s) \quad \text{on} \quad \Sigma. \end{cases} \tag{2.12}$$

where $\widetilde{u}(x,s) = \mathcal{L}(u) = \displaystyle\int_0^\infty e^{-st}u(x,t)\, dt$ and $\tilde{f}(x,s) = \displaystyle\int_0^\infty e^{-st}f(x,t)\, dt$ represent respectively the Laplace transform of u and f.
We set $w(x,s) = \widetilde{u}(x,s) - \dfrac{1-e^{-sT}}{s}\displaystyle\sum_{j=1}^m \beta_j\, \widetilde{G}(x-S_j,s)$, where $\widetilde{G}$ represents the Laplace transform of G defined

by (2.10), which is the fundamental solution of the Helmholtz equation with wave number $k = is^{\frac{\alpha}{2}}$. Using the properties of the Fox H-functions ([24], Chapter 2), we write $\widetilde{G}$ in the following form

$$\widetilde{G}(x,s) = \begin{cases} \frac{i}{4} H_0^{(1)}(is^{\frac{\alpha}{2}}|x|), & d = 2 \\ \dfrac{e^{-s^{\frac{\alpha}{2}}|x|}}{4\pi|x|}, & d = 3. \end{cases} \tag{2.13}$$

where $H_0^{(1)}$ is the Hankel function of first kind and order 0 see [1].
Then, w is solution of the following problem

$$\begin{cases} \Delta w(x,s) - s^{\alpha} w(x,s) = 0 & \text{in} \quad \Omega, \\ w(x,s) = g(x,s) & \text{on} \quad \Sigma. \end{cases} \tag{2.14}$$

where $g(x,s) := \widetilde{f}(x,s) - \displaystyle\sum_{j=1}^{m} \frac{\beta_j(1 - e^{-sT})}{s} \widetilde{G}(x - S_j, s)$.

We notice that we realize (2.14) in the weak sense i.e. given $g \in H^{\frac{1}{2}}(\Sigma)$, find $w \in H^1(\Omega)$ such that for all $v \in H^1(\Omega)$

$$\int_{\Omega} \nabla w \cdot \nabla \bar{v} + s^{\alpha} w \bar{v} = \langle \Lambda(g), \bar{v} \rangle$$

where $\Lambda : H^{\frac{1}{2}}(\Sigma) \to H^{-\frac{1}{2}}(\Sigma)$ denotes the Dirichlet to Neumann map and $\langle \ , \ \rangle_{\Sigma}$ represents the duality paining on $H^{-\frac{1}{2}}(\Sigma) \times H^{\frac{1}{2}}(\Sigma)$.

Following the definition given in [22], we say that u is a weak solution of (1.1) if $\widetilde{u}$ is a weak solution of (2.12). To prove that (2.12) has a unique weak solution, it is sufficient to prove that (2.14) has a unique weak solution.

It is well known that the spectrum of $(-\Delta)$ with Dirichlet condition consists of a sequence of eigenvalues, counted according to their multiplicities, is a subset of $(0, \ \infty)$ see [10]. We remark that for $s \in \mathbb{C}$ with $\mathcal{R}e(s) > 0$, if $\arg(s)$ represents the principal argument of s, and $-\frac{\pi}{2} < \arg(s) < \frac{\pi}{2}$, since $0 < \alpha < 1$, then $-\frac{\pi}{2} < -\alpha\frac{\pi}{2} < \alpha \arg(s) < \alpha\frac{\pi}{2} < \frac{\pi}{2}$, then $-s^{\alpha} = -|s|^{\alpha} e^{i\alpha \arg(s)}$ is negative if $\arg(s) = 0$ or $-s^{\alpha}$ is a complex number, which implies that $-s^{\alpha}$ is not a Dirichlet eigenvalue of $-\Delta$. The existence and the uniqueness of the problem (2.14) are deduced from the following theorem given in [11]

Theorem 2.4. *[11] Let Ω be a bounded domain with C^2 boundary $\partial\Omega$ such that k^2 is not a Dirichlet eigenvalue of $-\Delta$. Then for every $g \in H^{\frac{1}{2}}(\partial\Omega)$, there exists a unique weak solution $u \in H^1(\Omega)$ of the Helmholtz equation in Ω such that $u = g$ on $\partial\Omega$ in the sense of the trace theorem.*

3. Identifiability

The first question we might ask for the study of this type of problem concerns the uniqueness of the solution F of the inverse problem from the measurements of u and $\dfrac{\partial u}{\partial \nu}$ on the boundary Σ_T. To prove Theorem 3.2, we need the following lemma and we recall its proof:

Lemma 3.1. *[21] Let B be a bounded domain in $\mathbb{R}^d$ and $v \in C^2(B) \cap C(\bar{B})$ satisfies*

$$\Delta v + k^2 v = 0 \text{ in } B, \tag{3.1}$$

and

$$v = 0 \text{ on } \partial B. \tag{3.2}$$

Suppose that $\mathcal{I}m(k) > 0$, where $\mathcal{I}m(k)$ represents the imaginary part of the complex wave number k. Then $v = 0$ in $\bar{B}$.

Proof. Multiplying both sides of (3.1) by $\bar{v}$ and integrating over B give

$$\int_B \Delta v\bar{v} + k^2 \int_B v\bar{v} = 0$$

Green's identity and the boundary conditions of v yield

$$-\int_B |\nabla v|^2 + k^2 \int_B |v|^2 = 0 \tag{3.3}$$

Now if $\mathcal{R}e(k) \neq 0$ ($\mathcal{R}e(k)$ represents the real part of k), the imaginary part of (3.3) gives $\int_B |v|^2 = 0$ hence $v = 0$. In the case where $\mathcal{R}e(k) = 0$, since $\mathcal{I}m(k) > 0$, we have

$$\int_B |\nabla v|^2 + \mathcal{I}m(k)^2 \int_B |v|^2 = 0,$$

therefore $v = 0$ in $\bar{B}$. □

In the following theorem, we give the uniqueness result of the inverse problem.

Theorem 3.2. *(***uniqueness***)*

Let $u_r,\ r = 1,\ 2$ be the solution of problem (1.1) with $F_r = \sum_{j=1}^{m^{(r)}} \lambda_j^{(r)}\ \delta_{S_j^{(r)}}$ as source terms, where

$$\lambda_j^{(r)}(t) := \begin{cases} \beta_j^{(r)} > 0,\ t \in\ [0, T), \\ \\ 0,\ t \geq T. \end{cases} \tag{3.4}$$

Assume that $u_{1|\Sigma_T} = u_{2|\Sigma_T}$ and $\dfrac{\partial u_1}{\partial \nu}_{|\Sigma_T} = \dfrac{\partial u_2}{\partial \nu}_{|\Sigma_T}$, then $F_1 = F_2$ up to a permutation.

Proof. Consider the difference $U = u_2 - u_1$, U satisfies the following problem:

$$\begin{cases} \partial_t^\alpha U(x,t) - \Delta U(x,t) &= F_2(x,t) - F_1(x,t) & \text{in} & \Omega_T, \\ U(\cdot,\ 0) &= 0 & \text{in} & \Omega, \\ U(x,t) &= 0 & \text{on} & \Sigma_T, \\ \dfrac{\partial U}{\partial \nu}(x,t) &= 0 & \text{on} & \Sigma_T. \end{cases} \tag{3.5}$$

Now, applying the Laplace transform of the Riemann-Liouville fractional derivative to the problem (3.5), and using the second condition of this problem, and for $s \in \mathbb{C}$ with $\mathcal{R}e(s) > 0$ and $\mathcal{I}m(is^{\frac{\alpha}{2}}) = \mathcal{R}e(s^{\frac{\alpha}{2}}) > 0$, we get

$$\begin{cases} s^\alpha \tilde{U}(x,s) - \Delta\tilde{U}(x,s) &= \tilde{F}_2(x,s) - \tilde{F}_1(x,s) & \text{in} & \Omega, \\ \tilde{U}(x,s) &= 0 & \text{on} & \Sigma, \\ \dfrac{\partial \tilde{U}}{\partial \nu}(x,s) &= 0 & \text{on} & \Sigma, \end{cases} \tag{3.6}$$

where

$$\tilde{U}(x,s) = \int_0^\infty e^{-st} U(x,t)\, dt \text{ and } \tilde{F}_r = \sum_{j=1}^{m^{(r)}} \frac{\beta_j^{(r)}(1 - e^{-sT})}{s}\ \delta_{S_j^{(r)}}, r = 1, 2.$$

We set $k = is^{\frac{\alpha}{2}}$ and $B = \Omega \setminus \{\cup S_j^{(r)}, r = 1, 2\}$, then we obtain

$$\begin{cases} \Delta\tilde{U} + k^2\tilde{U} &= 0 & \text{in} & B, \\ \tilde{U} &= 0 & \text{on} & \Sigma. \end{cases} \tag{3.7}$$

From Lemma 3.1, $\tilde{U} = 0$ in $\bar{B}$.
Extending $\tilde{U}$ out of Ω by 0, one gets

$$\Delta\tilde{U} + k^2\tilde{U} = \tilde{F}_1 - \tilde{F}_2 \text{ in } \mathbb{R}^d.$$

We can then obtain its explicit expression by a convolution with the fundamental solution Φ_s of the Helmholtz equation with the wave number $k = is^{\frac{\alpha}{2}}$

$$\tilde{U}(x) = \sum_{j=1}^{m^{(1)}} \frac{\beta_j^{(1)}(1-e^{-sT})}{s} \Phi_s(x - S_j^{(1)}) - \sum_{j=1}^{m^{(2)}} \frac{\beta_j^{(2)}(1-e^{-sT})}{s} \Phi_s(x - S_j^{(2)}),$$

where

$$\Phi_s(x) = \frac{1}{(2\pi)^{\frac{d}{2}}} \left(\frac{|x|}{s^{\frac{\alpha}{2}}}\right)^{1-\frac{d}{2}} K_{\frac{d}{2}-1}(s^{\frac{\alpha}{2}}|x|),\ d = 2, 3,$$

$K_{\frac{d}{2}-1}$ is the modified bessel function of the third kind or Macdonald function [1].
Since $\tilde{U}$ is analytic in the connected domain $\mathbb{R}^d \setminus \{\cup S_j^{(r)}, r = 1, 2\}$ and null outside of Ω, it is null also in $\mathbb{R}^d \setminus \{\cup S_j^{(r)}, r = 1, 2\}$. Therefore for all x in $\mathbb{R}^d \setminus \{\cup S_j^{(r)}, r = 1, 2\}$, one has:

$$\sum_{j=1}^{m^{(1)}} \beta_j^{(1)} \Phi_s(x - S_j^{(1)}) - \sum_{j=1}^{m^{(2)}} \beta_j^{(2)} \Phi_s(x - S_j^{(2)}) = 0. \tag{3.8}$$

Now suppose that it exists $j_0 \in \{1, \dots, m^{(2)}\}$ such that $S_{j_0}^{(2)} \neq S_k^{(1)}$, for all $k \in \{1, \dots, m^{(1)}\}$. From (3.8) we have

$$\sum_{j=1}^{m^{(1)}} \beta_j^{(1)} \Phi_s(x - S_j^{(1)}) - \sum_{j=1,\ j\neq j_0}^{m^{(2)}} \beta_j^{(2)} \Phi_s(x - S_j^{(2)}) = -\beta_{j_0}^{(2)} \Phi_s(x - S_{j_0}^{(2)}) \tag{3.9}$$

Since for small arguments $0 < |z| \ll \sqrt{n+1}$, we have:[1]

$$K_n(z) \sim \begin{cases} -\ln(\frac{z}{2}) - \gamma, & n = 0 \\ \frac{\Gamma(n)}{2}\left(\frac{2}{z}\right)^n & n > 0. \end{cases} \tag{3.10}$$

where γ is the Euler-Mascheroni constant. Then,

$$\lim_{x \to S_{j_0}^{(2)}} |\beta_{j_0}^{(2)} \Phi_s(x - S_{j_0}^{(2)})| = \infty$$

and

$$\lim_{x \to S_{j_0}^{(2)}} |\sum_{j=1}^{m^{(1)}} \beta_j^{(1)} \Phi_s(x - S_j^{(1)}) - \sum_{j=1,\ j\neq j_0}^{m^{(2)}} \beta_j^{(2)} \Phi_s(x - S_j^{(2)})| < \infty$$

Then, by letting x tends to $S_{j_0}^{(2)}$ in equation (3.9) we obtain a contradiction. Thus, the sets $\{S_j^{(r)}, 1 \leq j \leq m^{(r)}\}$, $(r = 1,\ 2)$, must be identical. Then, one can write $S_j^{(1)} = S_j^{(2)}$ after renumbering of (S_j) if necessary and the same argument yields $\beta_j^{(1)} = \beta_j^{(2)}$. The proof is completed. □

Remark 3.3. *The proof of theorem 3.2 is also valid for the problem (1.1) with a more general source term of the following form* $F(x,t) = \sum_{j=1}^{m} \lambda_j(t)\, \delta_{S_j}(x)$ *with*

$$\lambda_j(t) := \begin{cases} \beta_j > 0, \ t \in \ [0, T_j) \\ 0, \ \ t \geq T_j \end{cases} \tag{3.11}$$

where T_j *is the time of activity of the source* S_j*, from the measurements of* u *and* $\frac{\partial u}{\partial \nu}$ *on the boundary* Σ_T*. Indeed, following the line of the prove of Theorem 3.2, if* u_r*,* $r = 1,\ 2$ *are the solutions of problem (1.1) with* $F_r = \sum_{j=1}^{m^{(r)}} \lambda_j^{(r)}\, \delta_{S_j^{(r)}}$ *as source terms, where*

$$\lambda_j^{(r)}(t) := \begin{cases} \beta_j^{(r)} > 0, \ t \in \ (0, T_j^{(r)}), \\ 0, \ t \geq T_j^{(r)}. \end{cases} \tag{3.12}$$

we show that $S_j^{(1)} = S_j^{(2)}$ *and* $\beta_j^{(1)}(1 - e^{-sT_j^{(1)}}) = \beta_j^{(2)}(1 - e^{-sT_j^{(2)}})$*. If we take* $s > 0$ *sufficiently large, we conclude that* $\beta_j^{(1)} = \beta_j^{(2)}$ *and* $T_j^{(1)} = T_j^{(2)}$*. We will see in section 5 that the proposed method for the identification of the source term* F *does not separately give the intensities* β_j *and the times* T_j*, which justifies the choice (1.3) of* F*.*

4. Stability Result

In this section, we study the continuous dependence of the unknown source term on the measured data on the boundary Σ_T, which is the crucial issue for numerical application. The question of stability has been the concern of several authors in different contexts. Alessandrini *et al* [3, 4], and Bellout *et al* [9] have dealt with stability for an inverse conductivity problem. The notion of local Lipschitz stability which has been used by several authors [6, 12, 13]. In many works, local Lipschitz stability results was obtained, derived from algebraic relations, for elliptic sources identification problems [7, 16, 18, 28].

In this section, we give a local Lipschitz stability result inspired from the stability result given in [28] for the problem of identification of sources via the Helmholtz equation, which is derived from the Gâteaux differentiability, by establishing that the Gâteaux derivative is not zero.

We suppose that Ω contains m monopolar sources located at S_j with respectively intensities τ_j, $j = 1, \ldots, m$. We define the perturbed source term F^h by:

$$F^h = -\sum_{j=1}^{m} \tau_j^h\, \delta_{S_j^h},$$

where

$$(\tau_j^h,\ S_j^h) := (\tau_j + h\, \mu_j, S_j + h\, R_j),\ 1 \leq j \leq m,$$
$$\{(\mu_j,\ R_j),\ 1 \leq j \leq m\} \subset \mathbb{R} \times \mathbb{R}^2,$$

h being sufficiently small to insure that $S_j + h\, R_j$ remain in Ω. We denote by u_0 and u_h the solutions of (4.1) with respectively source terms $F = F^0$ and $F = F^h$.

$$\begin{cases} \Delta u + k^2 u &= F \quad \text{in} \quad \Omega \\ \dfrac{\partial u}{\partial \nu} &= \varphi \quad \text{on} \quad \Sigma, \end{cases} \tag{4.1}$$

$\varphi \in H^{-\frac{1}{2}}(\partial\Omega)$ being the flux on $\partial\Omega$ ($\varphi \neq 0$ on $\partial\Omega$), k is the wave number on Ω. We set $u_{0|\partial\Omega} = f$, $u_{h|\partial\Omega} = f^h$.

Theorem 4.1. *[28] (***Local Lipschitz stability***). Assume that k^2 is not an eigenvalue of $-\Delta$ with Neumann condition in the boundary. Then, $\lim_{h\to 0} \frac{|f^h - f|_{L^2(\partial\Omega)}}{|h|}$ exists and is strictly positive.*

Now, we are ready to give the main result of this section. Assuming that the domain Ω contains m monopolar sources $S_1, \dots, S_m$ with respectively intensities $\lambda_1(t), \dots, \lambda_m(t)$ where

$$\lambda_j(t) := \begin{cases} \beta_j > 0 & \text{if} \quad t \in (0, T) \\ 0 & \text{if} \quad t \geq T \end{cases}$$

We denote by $\tilde{\mu}_j$ the piecewise function defined by

$$\tilde{\mu}_j := \begin{cases} \mu_j & \text{if} \quad t \in (0, T) \\ 0 & \text{if} \quad t \geq T \end{cases}$$

where $\mu_j \in \mathbb{R}$, and let $\tau_j \in \mathbb{R}^2$ such that $\|\tau_j\| \leq 1$ for $j = 1, \dots, m$.
We set

$$\Phi := (\lambda_j, S_j), \ \Phi^h := (\lambda_j^h, S_j^h) = (\lambda_j + h\mu_j, S_j + h\tau_j),$$

and

$$F^h := \sum_{j=1}^{m} \lambda_j^h \delta_{S_j^h},$$

$h \neq 0$ being sufficiently small to insure that S_j^h remains in Ω. Let u_0 and u_h be the solutions of problems (1.1)-(1.4) with respectively sources F^0 and F^h, we set $u_0 = f$ and $u_h = f^h$ on Σ_T. Then, our main result of stability is given in the following theorem

Theorem 4.2. *(Local Lipschitz stability)*
If $\mu_j \neq 0$, then

$$\lim_{h\to 0} \frac{|f^h - f|_{L^2(\Sigma_T)}}{h} \neq 0.$$

Proof. Extending the function φ and f by 0 outside the interval $[0,\ T]$, consider the time-integrated quantities

$$\Theta^h(x) := \int_0^\infty e^{-st} u_h(x,t)\, dt, \text{ and } \Theta^0(x) := \int_0^\infty e^{-st} u_0(x,t)\, dt, \ \mathcal{R}e(s) > 0$$

which are well-defined since all sources are assumed inactive for $t \geq T$. Applying the Laplace transform to the problems (1.1)-(1.4) corresponding respectively to the sources F^h and F^0, for $s \in \mathbb{C}$ with $\mathcal{R}e(s) > 0$ and $\mathcal{I}m(is^{\frac{\alpha}{2}}) > 0$, the function Θ^h is solution of the Helmholz equation with the wave number $k = is^{\frac{\alpha}{2}}$:

$$\begin{cases} \Delta\Theta^h + k^2\Theta^h = -\widetilde{F}^h & \text{in} \quad \Omega \\ \partial_\nu \Theta^h = \tilde{\varphi} & \text{on} \quad \Sigma \\ \Theta^h = \tilde{f}^h & \text{on} \quad \Sigma \end{cases} \tag{4.2}$$

where

$$\tilde{f}^h(x,s) = \int_0^\infty e^{-st} f^h(x,t)\, dt, \ \tilde{f}(x) = \int_0^\infty e^{-st} f(x,t)\, dt,$$

$$\widetilde{F}^h(x,s) = \sum_{j=1}^{m} \widetilde{\lambda}_j^h(s)\delta_{S_j^h}(x), \ \widetilde{F}^0(x) = \sum_{j=1}^{m} \widetilde{\lambda}_j(s)\delta_{S_j}(x),$$

$$\widetilde{\lambda}_j^h(s) = \frac{\beta_j(1 - e^{-sT})}{s} + h\frac{\mu_j(1 - e^{-sT})}{s}$$

and

$$\widetilde{\lambda}_j(s) = \frac{\beta_j(1-e^{-sT})}{s}$$

The source $\widetilde{F}^h$ represents the linear perturbation of the source $\widetilde{F}^0$ in the direction $\Psi = \{(\frac{\mu_j(1-e^{-sT})}{s},\ \tau_j)_{1\leq j\leq m}\}$, having the same number m of sources as $\{(\frac{\beta_j(1-e^{-sT})}{s}, S_j)_{1\leq j\leq m}\}$ for the problem of identifying monopolar sources S_j, located in Ω with respectively intensities $\frac{\beta_j(1-e^{-sT})}{s},\ j = 1,\dots,m$ via the Helmholtz equation with wave number $k = is^{\frac{\alpha}{2}}$, which is not an eigenvalue of $-\Delta$ with Neumann condition in the boundary, from the given Cauchy data $\tilde{\varphi}$ and $\tilde{f}$ on Σ. We have

$$(\widetilde{\lambda}^h_j(s), S^h_j) = (\widetilde{\lambda}_j(s) + h\ \frac{\mu_j(1-e^{-sT})}{s}, S_j + h\ \tau_j)$$

From Theorem 4.1, which its proof is also valid for the wave number $k = is^{\frac{\alpha}{2}}$ where $s \in \mathbb{C}$ with $\mathcal{R}e(s) > 0$ and $\mathcal{I}m(is^{\frac{\alpha}{2}}) > 0$, we deduce the following result:

$$\lim_{h\to 0} \frac{|\tilde{f}^h - \tilde{f}|_{L^2(\Sigma)}}{h} \neq 0.$$

From Cauchy-Lipschitz inequality one has

$$|\tilde{f}^h - \tilde{f}|_{L^2(\Sigma)} \leq \frac{1}{\sqrt{2\mathcal{R}e(s)}} |f^h - f|_{L^2(\Sigma_T)},$$

then, we obtain the following local Lipschitz result:

$$\lim_{h\to 0} \frac{|f^h - f|_{L^2(\Sigma_T)}}{h} \neq 0.$$

□

Remark 4.3. *If* $\lim_{h\to 0} \frac{|f^h - f|_{L^2(\Sigma_T)}}{|h|} = \ell \in \mathbb{R}^*_+$ *or if* $\lim_{h\to 0} \frac{|f^h - f|_{L^2(\Sigma_T)}}{h} = \infty$, *then there exists* $\delta > 0$ *and* $c > 0$ *such that if* $|h| < \delta$, *then* $|h| < c\ |f^h - f|_{L^2(\Sigma_T)}$, *which implies that there exists* $\tilde{c} > 0$ *such that for* $|h| < \delta$

$$\sum_{j=1}^{m} ||S^h_j - S_j|| + ||\lambda^h_j - \lambda_j||_{L^2(0,\,T)} \leq \tilde{c}\ |f^h - f|_{L^2(\Sigma_T)}$$

which gives the local Lipschitz stability result for the identification of monopolar sources problem. The result of the Theorem 4.2 means that one can distinguish between Φ^h *and* Φ *by measurements of the trace of* u *on* Σ_T, *provided that the error in measurements is* $o(h)$.

5. Identification Process

We present in this section a quasi-explicit method to recover the point sources (1.2) from the lateral observations $\frac{\partial u}{\partial \nu}$ and u on Σ_T. This method is inspired from the algorithm given in [14, 16] for the monopolar source identification via the Laplace equation in 2D case. This algorithm is based on the reciprocity gap functional defined by (5.2) which has been introduced by Bellout *et al* in [9] and has been formalized by Andrieux *et al* in [6], who used it in numerical reconstruction procedure for the inverse planar crack problem. To develop this algorithm we need the following result concerning integration by parts formulas. For $\alpha > 0$ and $n \in \mathbb{N}$ such that $n-1 \le \alpha < n$, we have [8]:

$$\int_a^b g(t)\,{}_a^cD_t^\alpha f(t)\,dt = \int_a^b f(t)\,{}_tD_b^\alpha g(t)\,dt + \sum_{j=0}^{n-1}\left[{}_tD_b^{\alpha+j-n} g(t)\cdot {}_tD_b^{n-1-j} f(t)\right]_a^b \tag{5.1}$$

We begin by considering the subset $\mathcal{H}_0$ defined by:

$$\mathcal{H}_0 = \{v:\ ({}_tD_T^\alpha - \Delta)v = 0, \text{in } \Omega_T\}$$

Let $v \in \mathcal{H}_0$, multiplying equation (1.1) by v and integrating on Ω_T, by applying (5.1) in time and the second Green's identity in the spatial variable, and using boundary condition, the problem (1.1)-(1.4) admits the following variational formulation:

$$\sum_{j=1}^{m}\beta_j\int_0^T v(S_j,t)\,dt = \mathcal{R}(u,v), \tag{5.2}$$

where

$$\mathcal{R}(u,v) = \int_{\Sigma_T}\left(u\frac{\partial v}{\partial \nu} - \frac{\partial u}{\partial \nu}v\right)d\Sigma_T + \int_\Omega\left[{}_tI_T^{1-\alpha}v(x,t)u(x,t)\right]_0^T\,dx \tag{5.3}$$

Now, with the observation $u(\cdot,T)$ made on Ω the reciprocity gap functional (5.3) is known (if v is). The reciprocity gap (RG) in the equation (5.2) links the causes hidden in Ω to their measurable consequences. The inverse problem consists to find the number, the locations and the intensities of the sources from equation (5.2). In the following along the lines followed in papers [14], we will show how an appropriate choice of test functions unveils these information. The problem is reduced to the problem of determining the parameters (m, S_j, β_j) by the knowledge of the right hand side of (5.2). From now on, a spatially two-dimensional setting is assumed, with complex polynomials used for adjoint fields. Associating $\mathbb{R}^2$ with $\mathbb{C}$ through $x_1 + ix_2 = z$, the following family of test functions defined by:

$$v_k(z,t) = (T-t)^{\alpha-1}z^k \in \mathcal{H}_0,\ k \in \mathbb{N}$$

In fact, the functions v_k are holomorphic, have harmonic real and imaginary parts in spatial variable:

$$\Delta v_k(\cdot,t) = 0$$

and, since ${}_tD_T^\alpha(T-t)^{\alpha-1} = 0$ ([25],p73), then

$${}_tD_T^\alpha v_k(z,\cdot) = 0$$

Since ${}_tI_T^{1-\alpha}(T-t)^{\alpha-1} = \Gamma(\alpha)$ see ([25],p88), then the components of the equality (5.2) are then given by:

$$\mathcal{R}(u,v_k) = \frac{T^\alpha}{\alpha}\sum_{j=1}^{m}\beta_j\,\sigma_j^k,\ k \in \mathbb{N} \tag{5.4}$$

where

$$\mathcal{R}(u,v_k) = \int_{\Sigma_T}\left(u\frac{\partial v_k}{\partial \nu} - \frac{\partial u}{\partial \nu}v_k\right)d\Sigma_T + \Gamma(\alpha)\int_\Omega u(x,T)\,z^k dx,$$

and σ_j denotes the affix of the j-th source location S_j. The source reconstruction thus consists in finding the number of sources m, the locations σ_j, the intensities β_j, and the extinction times T_j of the sources S_j verifying the equality (5.4).

Let M be an upper bound of the exact number m of the unknown monopolar sources ($M \ge m$), let:

$$\alpha_k := \frac{\alpha\,\mathcal{R}(u,v_k)}{T^\alpha},\ k = 0,\ldots.2M-1,$$

$$\mu_n = \begin{pmatrix} \alpha_n \\ \alpha_{n+1} \\ \vdots \\ \alpha_{M+n-1} \end{pmatrix} \in \mathbb{C}^M, \ \Lambda_m = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_m \end{pmatrix} \in \mathbb{R}^m, \tag{5.5}$$

and the matrix

$$A_{n,M} = \begin{pmatrix} \sigma_1^n & \sigma_2^n & \dots & \sigma_m^n \\ \sigma_1^{n+1} & \sigma_2^{n+1} & \dots & \sigma_m^{n+1} \\ \vdots & \vdots & \vdots & \vdots \\ \sigma_1^{M+n-1} & \sigma_2^{M+n-1} & \dots & \sigma_m^{M+n-1} \end{pmatrix} \in \mathcal{M}_{M\times m}(\mathbb{C}).$$

Following the line of the algorithm given in [14], the unknown m, σ_j, and β_j can then be deduced from the following lemma:

Lemma 5.1. *[14]*

1. *The rank of the family $(\mu_0, \mu_1, \dots, \mu_{M-1})$ is $r = m$, and the vectors $(\mu_0, \mu_1, \dots, \mu_{m-1})$ are independent.*
2. *The affixes σ_j of the monopolar sources S_j are the eigenvalues of the matrix $\mathcal{T}$ which is defined by $\mathcal{T}\mu_j = \mu_{j+1}$, for $j = 0, \dots, m-1$.*
3. *$\beta_1, \dots, \beta_m$ are solutions of the linear system $A_{0,m}\Lambda_m = \mu_0$ where $A_{0,m}$ is the Vandermonde matrix of σ_j.*

Remark 5.2.

1. *In the case where Ω contains a unique monopolar source S_1, then:*

$$\beta_1 = \alpha_0 \ \text{and} \ \sigma_1 = \frac{\alpha_1}{\alpha_0}.$$

2. *In the case where Ω contains two monopolar sources S_1, S_2, and if (a,b) are the components of the vector μ_2 in the basis (μ_0, μ_1), then:*

$$\sigma_1 = \frac{b + \sqrt{b^2 + 4a}}{2}, \ \sigma_2 = \frac{b - \sqrt{b^2 + 4a}}{2},$$
$$\beta_1 = \frac{\alpha_1 - \alpha_0\sigma_2}{\sigma_1 - \sigma_2} \ \text{and} \ \beta_2 = \frac{\alpha_1 - \alpha_0\sigma_1}{\sigma_2 - \sigma_1}.$$

3. *For $\alpha = 1$, we find the family of test functions used in [5, 15] for monopolar source identification problem via the heat equation. For the numerical experiments of this algorithm, we refer the reader to [5, 7, 27].*

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