Filomat 32:18 (2018), 6177–6187
https://doi.org/10.2298/FIL1818177C

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Iterative Approximation of Positive Solutions for Fractional Boundary Value Problem on the Half-line

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Abstract. In this paper, an iterative method is applied to solve some p-Laplacian boundary value problem involving Riemann-Liouville fractional derivative operator. More precisely, we establish the existence of two positive solutions. Moreover, we prove that these solutions are one maximal and the other is minimal. An example is presented to illustrate our main result. Finally, a numerical method to solve this problem is given.

1. Introduction

The subject of fractional calculus (that is, calculus of integrals and derivatives of any arbitrary real number) has gained considerable popularity and importance during the past four decades or so, due mainly to its demonstrated applications in numerous seemingly diverse and widespread fields of science, physics, finance, hydrology and engineering (see [1–6]). Fractional differential equations with p-laplacian operator arise in the modeling of different physical and natural phenomena, non-Newtonian mechanics, nonlinear flow laws and many other branches of engineering. The study of fractional differential equations was initiated by many authors, one may see [7–13] and references therein.

Let $J = (0, \infty)$. In [4], Zhao et al. discussed the existences of positive solutions for the following boundary value problems of fractional order differential equation

$$D^\alpha u(t) + f(t, u(t)) = 0, \quad t \in J, \quad 1 < \alpha \leq 2,$$

subject to the following boundary conditions:

$$u(0) = 0, \quad \lim_{t\to\infty} D^{\alpha-1} u(t) = \beta u(\xi),$$

where D^α is the standard Riemann-Liouville fractional derivative. In [12], Wang et al. used a fixed point Theorem to show existence and multiplicity of positive solutions to following boundary value problem of

2010 *Mathematics Subject Classification*. Primary 34A08; Secondary 30E25, 47J25
Keywords. Fractional differential equations, Half line, Iterative method, Numerical solution.
Received: 21 October 2017; Accepted: 24 September 2018
Communicated by Marko Petković
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nonlinear fractional differential equation with p-Laplacian operator

$$\begin{cases} D^{\gamma}\left(\Phi_p(D^{\alpha}u(t))\right) + f(t,u(t),D^{\rho}u(t)) = 0, \ \ 0<t<1,\ 0<\rho\leq 1,\ 2<\alpha\leq 3, \\ u(0)=u'(0)=u''(0)=0, \quad \lim\limits_{t\longrightarrow 0} D^{\gamma}u(t)=0, \end{cases}$$

where $0<\gamma<1$ and $\Phi_p(s)=|s|^{p-2}s, p>1$. Note that Φ_p is invertible and $\Phi_p^{-1}=\Phi_q$ with $\frac{1}{p}+\frac{1}{q}=1$.
Liang and Shi [17], considered the following boundary value problem

$$\begin{cases} D^{\gamma}\left(\Phi_p(D^{\alpha}u(t))\right) + a(t)f(t,u(t)) = 0, \ \ 2<\alpha\leq 3,\ t\in J, \\ u(0)=u'(0)=0,\ D^{\alpha-1}u(\infty)=\sum\limits_{i=1}^{m-2}\beta_i u(\xi_i), \ \ \lim\limits_{t\longrightarrow 0} D^{\alpha}u(t)=0 \end{cases} \tag{1}$$

where $0<\gamma\leq 1,\ 0<\xi_1<\xi_2<...<\xi_{m-2}<\infty,\ \beta_i\geq 0$ satisfies

$$0<\sum_{i=1}^{m-2}\beta_i\xi_i^{\alpha-1}<\Gamma(\alpha). \tag{2}$$

By using the properties of the Green function and some fixed point Theorems, authors gave sufficient conditions for the existence multiple positive solutions to the above boundary value problems.
Recently, the iterative method have been proposed by many authors, for example, we can cite the works [14–16].

Motivated by the above works, in addition to investigate the existence of two positive solutions for problem (1), we develop two computable explicit monotone iterative sequences for approximating the minimal and maximal positive solutions of (1). By applying monotone iterative techniques which used in [18, 19], we construct some successive iterative schemes to approximate the solutions in this paper.
Throughout this paper, we assume that $0<\gamma\leq 1$, $2<\alpha\leq 3$, ξ_i and β_i are positive constants satisfying (2).
Moreover, we assume the following conditions hold:
($\mathbf{H}_1$) $f\in C(J\times J,J)$, $\lim\limits_{s\rightarrow 0} f(t,s)\not\equiv 0$ on any subinterval of J. Moreover, $t\mapsto f(t,(1+t^{\alpha-1})u(t))$ is bounded on J when u is bounded.
($\mathbf{H}_2$) $a: J\longrightarrow J$ is not identical zero on any subinterval of J and

$$\int_0^{\infty}\Phi_q\left(\frac{1}{\Gamma(\gamma)}\int_0^s (s-\tau)^{\gamma-1}a(\tau)d\tau\right)ds<\infty.$$

($\mathbf{H}_3$) For all t the function $x\longmapsto f(t,x)$ is nondecreasing and there exists $b>0$ such that for all $(t,x)\in J\times[0,b]$, we have

$$f(t,(1+t^{\alpha-1})x)\leq \frac{b}{\left(Lb^{q-2}\int_0^{\infty}\Phi_q\left(\frac{1}{\Gamma(\gamma)}\int_0^s(s-\tau)^{\gamma-1}a(\tau)d\tau\right)ds\right)^{\frac{1}{q-1}}}.$$

The main result of this paper is the following.

Theorem 1.1. *Assume that the hypothesis ($\mathbf{H}_1$)-($\mathbf{H}_3$) hold. Then, the (1) has at least two positive solutions u_* and v_*. Moreover if ω is a solution to problem (1) such that $0<\omega\leq bt^{\alpha}$, then, $u_*\leq\omega\leq v_*$.*

This paper is organized as follows: In Section 2, we prepare some material needed to prove our main result. In Section 3, we prove existence results of positive solutions for (1) using iterative method. In Section 4, we give an example to illustrate our results.

2. Preliminaries

In this section, we introduce some necessary definitions and results which will used throughout this paper.

Definition 2.1. *(See [20]) Let $\delta > 0$. The Riemann-Liouville fractional derivative of order δ for a continuous function h is defined by*

$$D^{\delta}h(t) = (\frac{d}{dt})^{n} I^{n-\delta}h(t), \quad n = [\delta] + 1,$$

where $[\delta]$ is the integer part of δ and I^{δ} is the Riemann-Liouville fractional integral of order δ for a function h defined by

$$I^{\delta}h(t) = \frac{1}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-1} h(s)ds,$$

provided that such integral exists.

The following lemma can be found in [20, 21] which is crucial in finding an integral representation of fractional boundary value problem (1).

Lemma 2.1. *Let $\delta > 0$ and $u \in C(0,1) \cap L^1(0,1)$. Then, the fractional differential equation*

$$D^{\delta}u(t) = 0$$

has a unique solution given by

$$u(t) = C_1 t^{\delta-1} + C_2 t^{\delta-2} + ... + C_n t^{\delta-n}, \ C_i \in \mathbb{R}, \ i = 1, 2, ..., n, \ n = [\delta] + 1.$$

Moreover, if the fractional derivative of order δ belongs to $C(0,1) \cap L^1(0,1)$, then

$$I^{\delta}D^{\delta}u(t) = u(t) + c_1 t^{\delta-1} + c_2 t^{\delta-2} + ... + c_n t^{\delta-n}$$

for some $c_i \in \mathbb{R}$, $i = 1, 2, ..., n$, $n = [\delta] + 1$.

Lemma 2.2. *(See [22]) The following problem*

$$\begin{cases} D^{\gamma}\left(\Phi_p(D^{\alpha}u(t))\right) + h(t) = 0, \ \ 2 < \alpha \le 3, \ t \in (0, \infty), \\ u(0) = u'(0) = 0, \ D^{\alpha-1}u(\infty) = \displaystyle\sum_{i=1}^{m-2} \beta_i u(\xi_i), \end{cases}$$

has a unique solution given by

$$u(t) = \int_0^{\infty} G(t,s)h(s)ds,$$

where $G(t,s) = G_1(t,s) + G_2(t,s)$ and

$$G_1(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1} - (t-s)^{\alpha-1} & if \ \ 0 \le s \le t < \infty \\ t^{\alpha-1} \ \ if \ \ 0 \le t \le s < \infty \end{cases}$$

$$G_2(t,s) = \frac{t^{\alpha-1} \displaystyle\sum_{i=1}^{m-2} \beta_i G_1(\xi_i, s)}{\Gamma(\alpha) - \displaystyle\sum_{i=1}^{m-2} \beta_i \xi_i^{\alpha-1}}.$$

Lemma 2.3. *(See [22]) The boundary value problem (1) is equivalent to the integral equation*

$$u(t) = \int_0^{\infty} G(t,s)\Phi_q\left(\frac{1}{\Gamma(\gamma)} \int_0^s (s-\tau)^{\gamma-1} a(\tau) f(\tau, u(\tau))d\tau\right)ds, \tag{3}$$

where $G(t,s)$ is given in Lemma 2.2

Lemma 2.4. *(See [22]) For $t, s \geq 0$, the Green's function G satisfies:*

$$0 \leq G(t,s) \leq Lt^{\alpha-1}, \tag{4}$$

where

$$L = \frac{1}{\Gamma(\alpha)} + \frac{\sum_{i=1}^{m-2} \beta_i \xi_{m-2}^{\alpha-1}}{\Gamma(\alpha)\left(\Gamma(\alpha) - \sum_{i=1}^{m-2} \beta_i \xi_i^{\alpha-1}\right)}.$$

We shall work in the Banach space

$$E := \left\{ u \in C(J, \mathbb{R}) : \sup_{t \in J} \frac{|u(t)|}{1 + t^{\alpha-1}} < \infty \right\},$$

endowed with the norm

$$\|u\| := \sup_{t \in J} \frac{|u(t)|}{1 + t^{\alpha-1}}.$$

Define the cone $K \subset E$ by

$$K = \{x \in E : x(t) \geq 0, \ \forall \ t \in J\}.$$

Associated to the problem (1), we define the operator $T : K \to E$ as follows

$$Tu(t) := \int_0^\infty G(t,s) \Phi_q \left(\frac{1}{\Gamma(\gamma)} \int_0^s (s-\tau)^{\gamma-1} a(\tau) f(\tau, u(\tau)) d\tau \right) ds. \tag{5}$$

Observe that boundary value problem (1) has a solution if and only if the integral operator T has a fixed point. Moreover, by a similar process used in [12], it is easy to show that $T : K \to K$ is completely continuous.

3. Proof of our main result

Let $B := \{u \in E : \|u\| \leq b\}$, where b is given in $(\mathbf{H}_3)$. Then, $T(B) \subset B$. Indeed, if $u \in B$, then by $(\mathbf{H}_3)$ and equations (4) and (5), one has

$$\begin{aligned}
\|Tu\| &= \sup_{t \in J} \int_0^\infty \frac{G(t,s)}{1 + t^{\alpha-1}} \Phi_q \left(\frac{1}{\Gamma(\gamma)} \int_0^s (s-\tau)^{\gamma-1} a(\tau) f(\tau, u(\tau)) d\tau \right) ds \\
&\leq L \int_0^\infty \Phi_q \left(\frac{1}{\Gamma(\gamma)} \int_0^s (s-\tau)^{\gamma-1} a(\tau) d\tau \right) ds \frac{b^{q-1}}{Lb^{q-2} \int_0^\infty \Phi_q \left(\frac{1}{\Gamma(\gamma)} \int_0^s (s-\tau)^{\gamma-1} a(\tau) d\tau \right) ds} \\
&= b
\end{aligned}$$

Now, we define the sequences $(u_k)_{k \in \mathbb{N}}$ and $(v_k)_{k \in \mathbb{N}}$ by

$$\begin{cases} v_0(t) = 0, \\ v_{k+1}(t) = Tv_k(t), \quad k \in \mathbb{N}, \end{cases} \text{ and } \begin{cases} u_0(t) = bt^{\alpha-1}, \\ u_{k+1}(t) = Tu_k(t), \quad k \in \mathbb{N}. \end{cases} \tag{6}$$

Since $T(B) \subset B$, then, $(u_k)_{k \in \mathbb{N}}$ and $(v_k)_{k \in \mathbb{N}}$ are well defined.
We intend to prove that there exist u_* and v_* such that $(u_k)_{k \in \mathbb{N}}$ and $(v_k)_{k \in \mathbb{N}}$ converge respectively to u_* and v_*. Moreover, for all $k \in \mathbb{N}$

$$v_0 \leq v_1 \leq ... \leq v_k \leq ... \leq v_* \leq ... \leq u_* \leq .... \leq u_k \leq ... \leq u_1 \leq u_0. \tag{7}$$

Let u and v such that $u \leq v$. Then, from ($\mathbf{H}_3$) and using the fact that Φ_q is nondecreasing on J, we obtain

$$\begin{aligned} Tu(t) &= \int_0^\infty G(t,s)\Phi_q\left(\frac{1}{\Gamma(\gamma)}\int_0^s (s-\tau)^{\gamma-1}a(\tau)f(\tau,u(\tau))d\tau\right)ds \\ &\geq \int_0^\infty G(t,s)\Phi_q\left(\frac{1}{\Gamma(\gamma)}\int_0^s (s-\tau)^{\gamma-1}a(\tau)f(\tau,v(\tau))d\tau\right)ds \\ &= Tv(t). \end{aligned}$$

This proves that T is a nondecreasing operator.
Now, let us prove by induction that $\{v_k\}$ is an increasing sequence. We have

$$0 = v_0 \leq \int_0^\infty G(t,s)\Phi_q\left(\frac{1}{\Gamma(\gamma)}\int_0^s (s-\tau)^{\gamma-1}a(\tau)f(\tau,v_0(\tau))d\tau\right)ds = v_1.$$

Let $n \in \mathbb{N}$. Assume that $v_n \leq v_{n+1}$. Since, T is nondecreasing, it follows that $v_{n+1} \leq v_{n+2}$.
On the other hand, we have

$$\begin{aligned} &u_1(t) = \int_0^\infty G(t,s)\Phi_q\left(\frac{1}{\Gamma(\gamma)}\int_0^s (s-\tau)^{\gamma-1}a(\tau)f(\tau,u_0(\tau))d\tau\right)ds \\ &\leq Lt^{\alpha-1}\int_0^\infty \Phi_q\left(\frac{1}{\Gamma(\gamma)}\int_0^s (s-\tau)^{\gamma-1}a(\tau)d\tau\right)ds \frac{b^{q-1}}{Lb^{q-2}\int_0^\infty \Phi_q\left(\frac{1}{\Gamma(\gamma)}\int_0^s (s-\tau)^{\gamma-1}a(\tau)d\tau\right)ds} \\ &= bt^{\alpha-1} = u_0(t). \end{aligned}$$

Like above and using the fact that T is nondecreasing, we can easily prove by induction that

$$u_{n+1} \leq u_n,\ \forall n \in \mathbb{N}.$$

Finally, from the fact that $v_0 = 0 \leq u_0 = bt^{\alpha-1}$, and the fact that T is nondecreasing, it follows that

$$v_n \leq u_n, \quad \forall n \in \mathbb{N}.$$

In view of the complete continuity of the operator T and the fact that $u_{k+1} = Tu_k$ and $v_{k+1} = Tv_k$, we deduce that $\{u_k\}$ and $\{v_k\}$ have a convergent subsequences. This with the monotonicity of the above sequences, we conclude that there exist u_* and v_* such that

$$\lim_{k\to\infty} u_k = u_* \quad \text{and} \quad \lim_{k\to\infty} v_k = v_*.$$

Moreover, for all $k \in \mathbb{N}$, we have

$$v_k \leq v_* \leq u_* \leq u_k.$$

Finally, let ω be a solution to the problem (1), such that for all $t \in J$, we have $0 < \omega \leq bt^{\alpha-1}$. Then, one gets

$$v_0 = 0 \leq \omega \leq bt^{\alpha-1} = u_0.$$

By the monotonicity of T, one has

$$v_k \leq \omega \leq u_k, \quad \forall k \in \mathbb{N}.$$

By letting k tends to infinity, we obtain

$$v_* \leq \omega \leq u_*. \tag{8}$$

Since $\lim_{s\to 0} f(t,s) \not\equiv 0$, then, 0 is not a solution of problem (1). Thus, by (8), we know that u_* and v_* are the maximal and minimal positive solution of (1) in $(0, bt^{\alpha-1}]$. The proof is completed

4. Numerical equation solving

In this section, we propose a numerical method to compute a solution of the problem (1). Indeed, we will consider the equivalent problem (3) expressed in a nonlinear integral equation form because it is more accessible. Unfortunately, the nonlinear integral equations remain considerably hard to be solved analytically. In us the case, it is probably impossible to have a solution in closed form. As a consequence, it is recommended to calculate a solution using the numeric techniques. In literature, we can find various numerical methods to resolve nonlinear integral equations. This includes, for example, a polynomial interpolation method combined by a predictor-corrector technique proposed by Cahlon in [23] to solving non-linear Volterra integrals. Another is the works of Masouri et al. and Babolian et al. in [24, 25]. Here, the solutions of the problem (1) must be approached using the nonlinear iterative method is given by the relations (6) while adding an adaptive quadrature rule for the improper integral because of the infinite domain. For that matter, we should not use a quadrature strategy based on the value of the integrand at the endpoints of the domain of integration, such as the trapezoidal and Simpson's methods. However, the Gauss rules use only interior points, so these are more qualified to inappropriate integral types. In addition, the Gaussian quadrature picks the best points in the interval of integration, it tends to be high-order accuracy. Motivated by the work of Atkinson et al. [26] and Pedas et al. [27], we propose an efficient numerical method based on a discrete collocation technique characterized by its superconvergence which guarantees a minimal cost. For more details, we can see Eslahchi et al. [28] and Pedas et al. [29]. We will treat the problem (3) in the interval $[0, M]$, where $M > 0$. We consider the approximate values of solution u_k at the mesh points of the segment $[0, M]$, defined as: $s_k = kh$ $(k = 0, 1, \cdots, N)$, with $h = M/N$. The approximate solution is denoted by u. We have:

$$u(s) = \sum_{k=0}^{k=N} u_k \, \varphi_k(s) \text{ for } p = 0, 1, \cdots, N,$$

where φ_k is an interpolation function.

To approximate the integral (3) numerically on the interval $[0, M]$, we consider for every subinterval $I_k = [t_k, t_{k+1}], (k = 0, ..., N-1)$, a P point of collocation $t_{k_1} < t_{k_2} < \cdots < t_{k_P}$ with $t_{k_i} = t_k + c_i h_i$ where $0 \leq c_1 < c_2 < \cdots < c_P \leq 1$ with the corresponding weight ω_i $(i = 1, \cdots, P)$.
To access to a numerical solution of u, we consider the following function

$$Z(s) = \Phi_q\left(\frac{1}{\Gamma(\gamma)} \int_0^s (s-\tau)^{\gamma-1} a(\tau) f(\tau, u(\tau)) d\tau\right).$$

The collocation technique is an efficient strategy. However, this technique could end up an expensive numerical problem because the nonlinearity of the function Φ_q and the double integrals in the equation (3). To overcome this, we propose a best approximations, in order to determine a solution as precisely as possible. In order to do so, we set up for some function an approximation technique based on the nodal values, only in the elementary intervals I_k.
We put

$$Z(s_k) := \Phi_q(A_k), \text{ where } A_k = \frac{1}{\Gamma(\gamma)} \int_0^{s_k} (s_k - \tau)^{\gamma-1} a(\tau) f(\tau, u(\tau)) d\tau. \tag{9}$$

Now, we will approximate A_k with piecewise linear function.

Proposition 4.1. *We have the following approximation:*

$$A_k \approx \frac{1}{\Gamma(\gamma)} \sum_{l=0}^{l=k-1} \frac{h}{2} (s_k - \bar{s}_l)^{\gamma-1} a(\bar{s}_l) \left[f(\bar{s}_l, \frac{1}{2} u_l) + f(\bar{s}_l, \frac{1}{2} u_{l+1}) \right], \tag{10}$$

with $\bar{s}_l = \dfrac{s_l + s_{l+1}}{2} = h(l + \frac{1}{2})$.

Proof.
To begin with, we decompose A_k as a sum of k-integrals, for each subinterval I_l

$$A_k = \frac{1}{\Gamma(\gamma)} \sum_{l=0}^{l=k-1} \int_{s_l}^{s_{l+1}} (s_k - \tau)^{\gamma-1} a(\tau) f(\tau, u(\tau)) d\tau.$$

With a piecewise linear approximation, A_k becomes

$$A_k := \frac{1}{\Gamma(\gamma)} \sum_{l=0}^{l=k-1} \int_{s_l}^{s_{l+1}} (s_k - \tau)^{\gamma-1} a(\tau) \left[f(\tau, u_l \varphi_l(\tau)) + f(\tau, u_{l+1} \varphi_{l+1}(\tau)) \right] d\tau,$$

with $\varphi_l(t) = \dfrac{t - t_{l-1}}{t_l - t_{l-1}}$.
The last integral domain can be changed to the reference interval $[-1, 1]$ by the following way: If $\tau \in [s_l, s_{l+1}]$, we consider $\tau = \dfrac{h}{2}\theta + \bar{s}_l$, then

$$A_k := \frac{1}{\Gamma(\gamma)} \sum_{l=0}^{l=k-1} \frac{h}{2} \int_{-1}^{1} \left\{ (s_k - \frac{h}{2}\theta - \bar{s}_l)^{\gamma-1} a(\frac{h}{2}\theta + \bar{s}_l) \left[f\Big(\frac{h}{2}\theta + \bar{s}_l, u_l \ \hat{\varphi}_{-1}(\theta)\Big) \right.\right.$$
$$\left.\left. + f\Big(\frac{h}{2}\theta + \bar{s}_l, u_{l+1} \ \hat{\varphi}_1(\theta)\Big) \right] \right\} d\theta,$$

with $\hat{\varphi}$ is piecewise linear approximation in the reference interval $[-1, 1]$.
Putting $\theta = 0$ (mid-point rule), we have $\hat{\varphi}_1(0) = \hat{\varphi}_{-1}(0) = \frac{1}{2}$. Then, we obtain

$$A_k := \frac{1}{\Gamma(\gamma)} \sum_{l=0}^{l=k-1} \frac{h}{2} \left\{ (s_k - \bar{s}_l)^{\gamma-1} a(\bar{s}_l) \left[f\Big(\bar{s}_l, \frac{1}{2}u_l\Big) + f\Big(\bar{s}_l, \frac{1}{2}u_{l+1}\Big) \right] \right\}.$$

□

The aim now is to calculate the solution u by using (3) and the approximation equation (9)
For all $m \in \{0, \cdots, N\}$, we have

$$u(t_m) = \sum_{n=0}^{n=N} \int_{s_n}^{s_{n+1}} G(s, t_m) \Big[\Phi_q(A_{n+1}) \varphi_{n+1}(s) + \Phi_q(A_n) \varphi_n(s)) \Big] ds. \tag{11}$$

By applying to (11) the same previous change of variables, we obtain

$$u(t_m) = \sum_{n=0}^{n=N} \int_{-1}^{1} \frac{h}{2} G(\frac{h}{2}\xi + \bar{s}_n, t_m) \Big[\Phi_q(A_{n+1}) \hat{\varphi}_{-1}(\xi) + \Phi_q(A_n) \hat{\varphi}_1(\xi)) \Big] d\xi. \tag{12}$$

Thus,

$$u(t_m) = \sum_{n=0}^{n=N} \frac{h}{4} G(\bar{s}_n, t_m) \Big[\Phi_q(A_{n+1}) + \Phi_q(A_n) \Big]. \tag{13}$$

5. Numerical example

In this section, a numerical example is presented with result verification have been proposed. For that purpose, we have implemented the following example in Matlab.
Take $\alpha = \frac{5}{2}$, $\gamma = 1$, $\beta_1 = \beta_2 = \frac{1}{4}$, $\chi_1 = \frac{1}{4}$ and $\chi_2 = 1$. Let a be a function such that

$$\int_0^{\infty} \int_0^{s} a(\tau) d\tau ds < \infty. \tag{14}$$

Note that, we can take cut functions.

Consider the following fractional boundary value problem

$$\begin{cases} D^{\frac{7}{2}} u(t) + a(t) f(t, u(t)) = 0, \ 0 < t < \infty \\ u(0) = u'(0) = 0, \ D^{\frac{3}{2}} u(\infty) = \frac{1}{4} u(\frac{1}{4}) + \frac{1}{4} u(1), \ \lim\limits_{t \longrightarrow 0} D^{\frac{5}{2}} u(t) = 0, \end{cases} \tag{15}$$

where

$$f(t, u) = \frac{1}{10(1 + t^4)} + \left(\frac{u \wedge 1}{1 + t^{\frac{3}{2}}} \right)^5.$$

It is easy to see that $f \in C(J \times J, J)$, $\lim\limits_{s \to 0} f(t, s) \not\equiv 0$. Moreover

$$f(t, (1 + t^{\frac{3}{2}} u)) = \begin{cases} \frac{1}{10(1+t^4)} + \frac{1}{10} u^5, & 0 \leq u \leq 1, \\ \frac{1}{10(1+t^4)} + \frac{1}{10} (\frac{1}{1+t^{\frac{3}{2}}})^5, & u > 1. \end{cases}$$

Then, it follows that

$$f(t, (1 + t^{\frac{3}{2}} u)) \leq \frac{1}{5}. \tag{16}$$

That is $(\mathbf{H}_1)$ is satisfied.
Now, since $\gamma = 1$ and $p = 2$, then from (14) we obtain

$$\int_0^{\infty} \Phi_q \left(\frac{1}{\Gamma(\gamma)} \int_0^{s} (s - \tau)^{\gamma - 1} a(\tau) d\tau \right) ds = \int_0^{\infty} \int_0^{s} a(\tau) d\tau ds < \infty, \tag{17}$$

that is $(\mathbf{H}_2)$ is satisfied.
It is easy see by calculating that

$$L = \frac{96 \sqrt{\pi} + 28}{3 \sqrt{\pi}(24 \sqrt{\pi} - 9)} \approx 1.11136.$$

For $b > 0$, then, from (16) and (17), we obtain

$$f(t, (1 + t^{\frac{3}{2}} u)) \leq \frac{1}{5} \leq \frac{b}{\left(L b^{q-2} \int_0^{\infty} \Phi_q \left(\frac{1}{\Gamma(\gamma)} \int_0^{s} (s - \tau)^{\gamma - 1} a(\tau) d\tau \right) ds \right)^{\frac{1}{q-1}}} < \infty.$$

That is $(\mathbf{H}_3)$ is satisfied.
Hence, all condition of Theorem 1.1 are satisfied. Therefore problem admits a minimal and maximal positive solutions in $[0, b\, t^{\frac{3}{2}}]$.

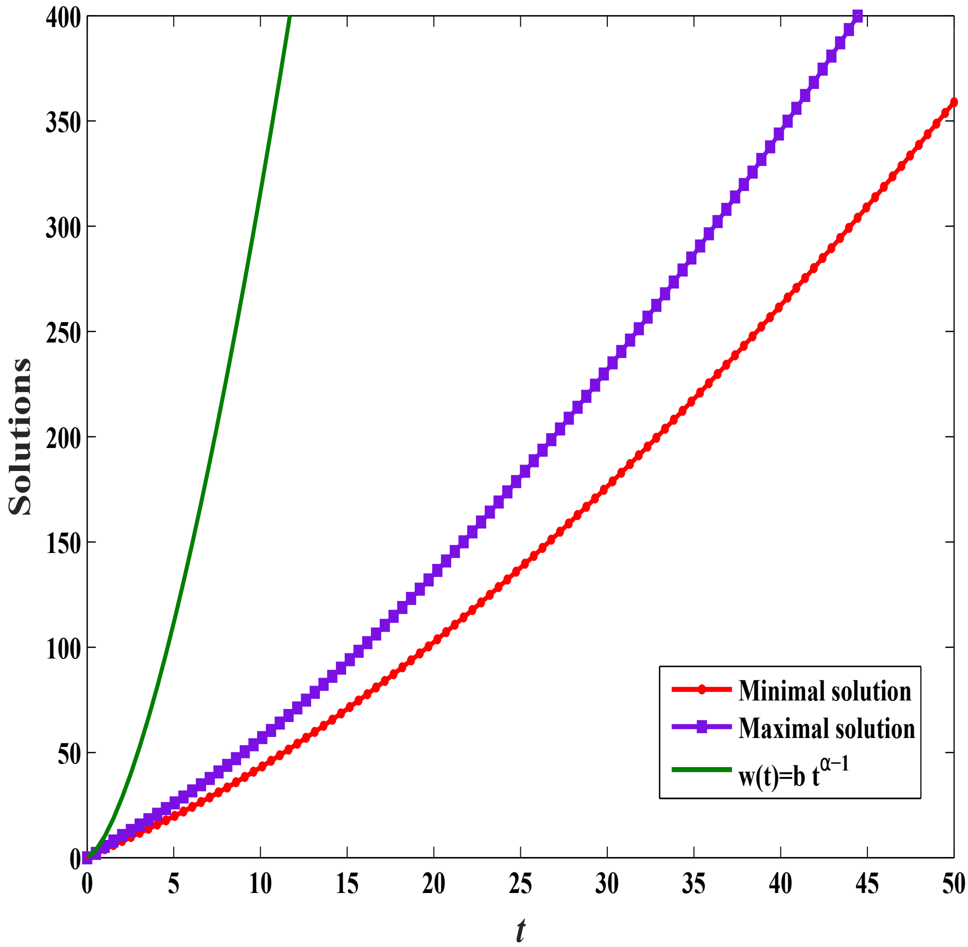

Figure 1: The green solid line shown the function $w(t) = bt^{\frac{3}{2}}$, the while the others are the maximal and the minimal solutions.

Discussions

To approximate the improper integral, we have used the quadrature rules in (13). But we still need to overcome the difficulties to implement an integral on the infinite domain. To do so, we have transformed the infinite interval to the interval $[0,1]$ by making the change of variables $x = \dfrac{1}{1+t}$. After few iterations, the proposed iterative algorithm has rapidly converged to the maximal and minimal solutions. True, both selected solutions should satisfy an upper bound of the relative error. But, the choice of precision for this problem type, must not be very restrictive in order to achieve a convergence with a reasonable number of iterations. In this example, the error tolerance is fixed at 10^{-6}. The solutions have successfully calculated on the interval $[0,50]$ using 100 nodes and a step-size $h = 0.01$ to discretize the interval $[0,1]$, for around 22 iterations. Fig.1 shows a comparison of these calculated solutions. We noted that the both solution (maximal and minimal) have respected the limited area described in the analytical part of this paper.

Within a framework of the purely numeric, we will now discuss the uniqueness of the solution. In so doing, we have attempted to adjust numerically the parameter b to ensure when the $\infty-$norm of the difference between the maximal solution v and minimal one u is very small. If one varies the value of b, we discovered that if $b > 5.15$, on the one hand, the norm $\|v-u\|_\infty$ is maximal and maintains a fixed value of 114.0978. We are still far from the uniqueness. On the other hand, the value of the norm $\|v-u\|_\infty$ to be about 0.0683 if the value of b is beyond of 1. If $b < 1$ the solutions start coming out of the area limited by the functions $u_0 = 0$ and $v_0 = b\,t^{\alpha-1}$.

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