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Positive Solutions for a Multi-order Fractional Nonlinear System with Variable Delays

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Abstract. This paper is concerned with the existence and uniqueness of the positive solution for a multi-order fractional nonlinear system with variable delays. The fractional derivative will be in the Caputo sense. The obtained results are based on some fixed point theorems.

1. Introduction

Fractional Calculus is a three centuries-old mathematical topic, which generalizes the ordinary integration and differentiation to arbitrary non integer order. It is nowadays a growing interest area of mathematics. For a long time, the theory of Fractional Calculus developed only as a theoretical field of mathematics. However, in the last decades, it was shown that some fractional operators describe in a better way some complex physical phenomena, especially when dealing with memory processes. It has been proven that problems with fractional-order operators has more advantages than ordinary ones since fractional-order derivative can provide an excellent instrument for description of memory and hereditary properties of various materials and process.

Due to this, in the last three decades, a great interest has been devoted to the study of fractional differential equations. Thus, fractional order differential equations play today a very important role describing many practical problems such as in viscoelasticity, physics, chemistry, biology, medical science, etc ([1, 5, 6, 8, 9] and the references therein).

On the other hand, the accurate modeling of dynamics of many engineering, physics, economy system can be obtained using delayed equations. Many investigations have been done on the existence and uniqueness of solutions for delay fractional differential systems. Some recent results on these equations can be found in a series of papers [2, 7, 11, 12].

In [13], Ye et al. have studied the existence of a positive solution for a delay fractional differential equation

$$\begin{cases} D^{\alpha}\left[x(t)-x(0)\right]=x(t)\,f(t,x_t) & , \quad t\in(0,T], \\ x(t)=\phi(t)\geq 0 & , \quad t\in[-r,0], \end{cases}$$

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where $\alpha \in (0,1)$, D^{α} is the Riemann-Liouville fractional derivative, and ϕ and f are continuous. By using the sub- and super-solution method, they gave some sufficient conditions for the existence of positive solutions.

In this paper, motivated by the above-cited paper, we study the existence and uniqueness of a positive solution for the multi-order fractional differential system with variable delays

$$^{c}D^{\alpha_i} x_i(t) = f_i\left(t, x_1(t), ..., x_n(t), x_1(t-\tau_1(t)), ..., x_n(t-\tau_n(t))\right), \quad i \in [\![1,n]\!],\ t > 0. \tag{1}$$

$$x(t) = \Phi(t) \geq 0, \quad t \in [-\tau, 0], \tag{2}$$

where $^{c}D^{\alpha_i}$ is the Caputo fractional derivative of order $\alpha_i \in (0,1)$, for $i \in \{1,...,n\}$, $x(t) = (x_1(t), ..., x_n(t))'$, where $'$ denotes the transpose of the vector, $f_i : \mathbb{R}^+ \times \mathbb{R}^{2n} \to \mathbb{R}$ are continuous functions for $i \in [\![1,n]\!]$, $\Phi(t) = (\phi_1(t), ..., \phi_n(t))'$ is a given vector function defined on $[-\tau, 0]$ with values in $\mathbb{R}^n$, and τ_i are continuous real-valued functions defined on $\mathbb{R}^+$ such that $\tau = \max\{\sup_{t\in\mathbb{R}^+} \tau_i(t), i \in [\![1,n]\!]\} > 0$.

At first time, by using the sub- and super-solution method (on a cone), we establish the existence of a positive solution to the problem (1)-(2), and then, we show the uniqueness of a positive solution of the problem (1)-(2) by using the Banach fixed point theorem.

This paper is structured as follows. In Section 2, to make the paper self-contained, we give some preliminary results from functional analysis and fractional calculus, which will be used throughout this paper. In Section 3, after transforming the initial value problem (1)-(2) into an equivalent Volterra integral equation (Lemma 3.1), we show the existence of a positive solution to the problem (1)-(2) (Theorem 3.3). In the last section, we show the uniqueness of the positive solution to our considered problem (Theorem 4.1).

2. Preliminaries

In this section, we give some preliminary results from functional analysis and fractional calculus.

2.1. Fixed point theorem on a cone

In this part, we recall some definitions regarding cones in Banach spaces. More details may be found in [3, 4].

Let E be a real Banach space.

Definition 2.1. *A nonempty subset $K \subset E$ is called a cone if it is convex, closed, and satisfies the conditions:*

1. *$\lambda K \subset K$, for any real positive number λ,*
2. *$x, -x \in K$ imply $x = 0$.*

A cone K introduces a partial order $\leq$ in E in the following manner

$$x \leq y \text{ if } y - x \in K.$$

Definition 2.2. *A cone K is said to be normal if there exists a constant $c > 0$ such that for all $x, y \in E$:*

$$if\ 0 \leq x \leq y \ \ then\ \ \|x\| \leq c\left\|y\right\|.$$

Definition 2.3. *For $x, y \in E$ the order interval $\langle x, y \rangle$ is defined as*

$$\langle x, y \rangle = \{z \in E : \ \ x \leq z \leq y\}.$$

Definition 2.4. *Let $D \subset E$, an operator $F : D \to E$ is called compact if for any bounded set $S \subset D$, $F(S)$ is relatively compact. Moreover, F is said to be completely continuous if it is continuous and compact.*

Definition 2.5. *The functional $h(t, x_1, ..., x_n)$ is nondecreasing on $I \times E^n$, $I \subset \mathbb{R}$, if for any $\left(t, \phi_1, ..., \phi_n\right) \in I \times E^n$, $(t, \psi_1, ..., \psi_n) \in I \times E^n$, such that $\phi_i \leq \psi_i$, $i \in [\![1, n]\!]$, the inequality $h\left(t, \phi_1, ..., \phi_n\right) \leq h(t, \psi_1, ..., \psi_n)$ holds.*

Following is our main tool. For the proof, the reader is referred to [3, 4].

Theorem 2.6. *Let D be a subset of the cone K of a partially ordered space E, $F : D \to E$ be nondecreasing. Suppose that there exist $x_0, y_0 \in D$ such that $x_0 \leq y_0$, $\langle x_0, y_0\rangle \subset D$, satisfying $x_0 \leq Fx_0$ and $y_0 \geq Fy_0$. If P is normal and F is completely continuous, then the equation $x - F(x) = 0$ has a minimal solution and a maximal solution x^*, y^* in $\langle x_0, y_0\rangle$.*

2.2. Compactness criterion

Here and in the sequel we denote by I the interval $[-\tau, +\infty)$. Let $\theta : I \to (0, +\infty)$ be a continuous function, and let E be the linear space of all column $n-$vectors functions x, continuous on I and satisfying

$$\sum_{i=1}^{n} \sup_{t\in I} \{\theta(t) |x_i(t)|\} < \infty.$$

We check easily that E equipped with the Bielecki's type norm

$$||x||_\theta = \sum_{i=1}^{n} \sup_{t\in I} \{\theta(t) |x_i(t)|\},$$

is a Banach space.

In this paper, the functions that we are going to manipulate are defined on infinite intervals. Therefore, in our case, the theorem of Arzelà-Ascoli is not applicable, so we need the following local property

Definition 2.7. *Let X a family of functions defined on the infinite interval I. X is said to be almost equicontinuous on I, if it is equicontinuous on each interval $[-\tau, T]$, $-\tau < T < \infty$.*

The following compactness result is an adaptation of Zima's compactness theorem [14] to the case of systems.

Lemma 2.8. *Let $\Omega \subset E$. If Ω is almost equicontinuous on I and uniformly bounded in $(E, ||.||_{\theta^*})$, where the weight function θ^* is positive and continuous on I, and*

$$\theta(t) = o(\theta^*(t)), \text{ as } t \to +\infty,$$

then Ω is relatively compact in $(E, ||.||_\theta)$.

Proof. Let $\{x^k\}^{k\in\mathbb{N}} = \{x_i^k\}_{i=\overline{i,n}}^{k\in\mathbb{N}}$ a sequence of Ω. We will show that it has a convergent subsequence in $(E, ||.||_{\theta^*})$. We have, by assumption, Ω is uniformly bounded in $(E, ||.||_{\theta^*})$, that is, there exists a positive constant M such that, $\forall k \in \mathbb{N}$,

$$\left\|x^k\right\|_{\theta^*} = \sum_{i=1}^{n} \sup_{t\in I} \left\{\theta^*(t) \left|x_i^k(t)\right|\right\} \leq M.$$

This estimate helds on I, and in particularly on any subinterval $[-\tau, T]$ of I for $T > -\tau$. That is, $\forall T > -\tau$,

$$\sum_{i=1}^{n} \max_{t\in[-\tau,T]} \left\{\theta^*(t) \left|x_i^k(t)\right|\right\} \leq M. \tag{3}$$

Moreover, by assumption, Ω, and then $\{x^k\}$, is equicontinuous on each subinterval $[-\tau, T]$ of I. Therefore, by applying the Arzelà-Ascoli theorem and a diagonalization process, for each $i \in [\![1, n]\!]$, there exists a susbsequence $\{y_i^k\}$ defined on I, converging uniformly, on each interval $[-\tau, T]$ of I, to a some limit function y_i. Namely,

$$\max_{t\in[-\tau,T]} \left\{\theta^*(t)\left|y_i^k(t)-y_i(t)\right|\right\} \to 0, \text{ as } k \to +\infty. \tag{4}$$

To achieve the proof, we show that $\{y^k\}$ converges to y in $(E, \|.\|_\theta)$. We have

$$\begin{aligned}
\left\|y^k - y\right\|_\theta &= \{\sup_{t\in I} \frac{\theta(t)}{\theta^*(t)}\} \sum_{i=1}^n \sup_{t\in I} \left\{\theta^*(t)\left|y_i^k(t)-y_i(t)\right|\right\} \\
&\le \{\max_{t\in[-\tau,T]} \frac{\theta(t)}{\theta^*(t)}\} \sum_{i=1}^n \max_{t\in[-\tau,T]} \left\{\theta^*(t)\left|y_i^k(t)-y_i(t)\right|\right\} \\
&+ \{\sup_{t>T} \frac{\theta(t)}{\theta^*(t)}\} \sum_{i=1}^n \sup_{t>T} \left\{\theta^*(t)\left|y_i^k(t)-y_i(t)\right|\right\} \\
&\le \{\max_{t\in[-\tau,T]} \frac{\theta(t)}{\theta^*(t)}\} \sum_{i=1}^n \max_{t\in[-\tau,T]} \left\{\theta^*(t)\left|y_i^k(t)-y_i(t)\right|\right\} \\
&+ \{\sup_{t>T} \frac{\theta(t)}{\theta^*(t)}\} \sum_{i=1}^n \sup_{t>T} \left\{\theta^*(t)\left[\left|y_i^k(t)\right| + \left|y_i(t)\right|\right]\right\}.
\end{aligned}$$

Thus, by using (3) and the fact that θ/θ^* is continuous on $[-\tau, T]$,

$$\left\|y^k - y\right\|_\theta \le C_T \sum_{i=1}^n \max_{t\in[-\tau,T]} \left\{\theta^*(t)\left|y_i^k(t)-y_i(t)\right|\right\} + 2nM\{\sup_{t>T} \frac{\theta(t)}{\theta^*(t)}\}.$$

Consequently, by using (4) and the fact that $\theta(t) = o(\theta^*(t))$, *as* $t \to +\infty$, we deduce that $\left\|y^k - y\right\|_\theta \to 0$, as$k \to +\infty$. □

2.3. Fractional calculus

In this part, we give the basic definitions and properties of fractional integral and Caputo fractional derivative, which will be used throughout this paper. The reader interested on more details about fractional calculus is referred to [5, 10].

Definition 2.9. *For all $T > 0$, the Riemann–Liouville fractional integral of order $\alpha \in \mathbb{R}^+$ of a function $f \in L^1[0, T]$ is given by:*

$$I^\alpha f(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\,ds, \ t \in [0, T]$$

where Γ is the gamma function. For $\alpha = 0$, we set $I^0 := Id$, the identity operator.

Definition 2.10. *The Caputo fractional derivative of order $\alpha \in \mathbb{R}^+$ of the function f with $D^n f \in L^1[0, T]$ is defined by*

$$^cD^\alpha f(t) = I^{n-\alpha} D^n f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)\,ds, \ t \ge 0,$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of number α and $D = \frac{d}{dt}$.

Obviously, the Caputo derivative of a constant is equal to zero.

Lemma 2.11. *Let $\alpha > 0$. Then*

$$^cD^\alpha I^\alpha f(t) = f(t). \tag{5}$$

3. Existence of positive solution

In this section, we give sufficient conditions for the existence of positive solutions.

Lemma 3.1. *The vector function* $x(t) := (x_1(t), ..., x_n(t))$ *is a solution of the problem* (1) − (2) *if and only if*

$$x_i(t) = \begin{cases} \phi_i(0) + I^{\alpha_i} f_i(t, x_1(t), ..., x_n(t), x_1(t - \tau_1(t)), ..., x_n(t - \tau_n(t))), & t > 0, \\ \phi_i(t), \quad t \in [-\tau, 0], \quad i \in [\![1, n]\!]. \end{cases}$$

Proof. For $t > 0$, the equation (1) can be written as

$$I^{1-\alpha_i} D x_i(t) = f_i(t, x_1(t), ..., x_n(t), x_1(t - \tau_1(t)), ..., x_n(t - \tau_n(t))).$$

Applying the operator I^{α_i} on both sides,

$$IDx_i(t) = I^{\alpha_i} f_i(t, x_1(t), ..., x_n(t), x_1(t - \tau_1(t)), ..., x_n(t - \tau_n(t))).$$
$$x_i(t) - x_i(0) = I^{\alpha_i} f_i(t, x_1(t), ..., x_n(t), x_1(t - \tau_1(t)), ..., x_n(t - \tau_n(t))).$$

Then

$$x_i(t) = \phi_i(0) + I^{\alpha_i} f_i(t, x_1(t), ..., x_n(t), x_1(t - \tau_1(t)), ..., x_n(t - \tau_n(t))). \tag{6}$$

The converse follows by direct computation. This ends the proof of Lemma. □

In the upcoming analysis, we set the appropriate functional framework. We denote by E the Banach space of all column vector-valued functions x, continuous on I and satisfying

$$\sum_{i=1}^{n} \sup_{t \in I} \left\{ e^{-\rho t} |x_i(t)| \right\} < \infty,$$

with the norm

$$\|x\|_\rho = \sum_{i=1}^{n} \sup_{t \in I} \left\{ e^{-\rho t} |x_i(t)| \right\}, \; x \in E,$$

where $\rho \in \mathbb{R}^+$ will be chosen later. In the space E, we define the set

$$K = \{x \in E : x_i(t) \geq 0, \; t \in I, \;\; i \in [\![1, n]\!]\}.$$

It is clear that K is a normal cone in E. Let

$$D = \{x \in K : x(t) = \Phi(t), \; -\tau \leq t \leq 0\} \subset K.$$

We define the integral operator F, for $i \in [\![1, n]\!]$, by

$$Fx_i = \begin{cases} \phi_i(t), & t \in [-\tau, 0] \\ \phi_i(0) + I^{\alpha_i} f_i(t, x_1(t), ..., x_n(t), x_1(t - \tau_1(t)), ..., x_n(t - \tau_n(t))), & t > 0. \end{cases} \tag{7}$$

In order to establish our results, we set the following hypotheses:

$(\mathcal{H}_1)$ There exist $g_{ij}, h_{ij} : \mathbb{R}^+ \to \mathbb{R}^+$ continuous and bounded, such that

$$f_i(t, u_1, ..., u_n, v_1, ..., v_n) \leq \sum_{j=1}^{n} g_{ij}\left(u_j\right) + \sum_{j=1}^{n} h_{ij}\left(v_j\right), \quad \text{for } \; i \in [\![1, n]\!].$$

($\mathcal{H}_2$) For $i \in [\![1,n]\!] : f_i : \mathbb{R}^+ \times \mathbb{R}_+^{2n} \to \mathbb{R}^+$ is continuous and nondecreasing function.

($\mathcal{H}_3$) There exist $u_0 = \left(u_0^1, ..., u_0^n\right)', v_0 = \left(v_0^1, ..., v_0^n\right)'$ which are respectively sub-solution and super-solution of the problem (1)-(2), satisfying $u_0(t) \le v_0(t)$, $t \in [0, +\infty)$.

Lemma 3.2. *Assume that ($\mathcal{H}_1$) holds. Then, the operator $F : D \to E$ is completely continuous.*

Proof. The operator $F : D \to E$ is continuous in view of the assumption of continuity of f. Next, if $G \subset D$ is bounded, then for each $x \in G$, we have, for $i \in [\![1,n]\!]$:

$$\begin{aligned}
|Fx_i(t)| &\le \left|\phi_i(0)\right| + \int_0^t \frac{(t-s)^{\alpha_i-1}}{\Gamma(\alpha_i)} \left|f_i(s, x_1(s), ..., x_n(s), x_1(s-\tau_1(s)), ..., x_n(s-\tau_n(s)))\right| ds \\
&\le \left|\phi_i(0)\right| + \sum_{j=1}^n \int_0^t \frac{(t-s)^{\alpha_i-1}}{\Gamma(\alpha_i)} g_{ij}\left(x_j(s)\right) ds \\
&+ \sum_{j=1}^n \int_0^t \frac{(t-s)^{\alpha_i-1}}{\Gamma(\alpha_i)} h_{ij}\left(x_j\left(s-\tau_j(s)\right)\right) ds.
\end{aligned}$$

Hence, for $\mu > 0$, we have

$$\begin{aligned}
e^{-\mu t}|Fx_i(t)| &\le e^{-\mu t}\left|\phi_i(0)\right| + \sum_{j=1}^n \int_0^t \frac{(t-s)^{\alpha_i-1}}{\Gamma(\alpha_i)} e^{-\mu(t-s)} e^{-\mu s} g_{ij}\left(x_j(s)\right) ds + \\
&+ \sum_{j=1}^n \int_0^t \frac{(t-s)^{\alpha_i-1}}{\Gamma(\alpha_i)} e^{-\mu\left(t-s+\tau_j(s)\right)} e^{-\mu\left(s-\tau_j(s)\right)} \left|h_{ij}\left(x_j\left(s-\tau_j(s)\right)\right)\right| ds \\
&\le e^{-\mu t}\left|\phi_i(0)\right| + \sum_{j=1}^n \sup_{\xi\in I}\{e^{-\mu\xi}\left|g_{ij}\left(x_j(\xi)\right)\right|\} \int_0^t \frac{(t-s)^{\alpha_i-1}}{\Gamma(\alpha_i)} e^{-\mu(t-s)} ds + \\
&+ \sum_{j=1}^n \int_0^t \frac{(t-s)^{\alpha_i-1}}{\Gamma(\alpha_i)} e^{-\mu\left(t-r_j(s)\right)} e^{-\mu r_j(s)} \left|h_{ij}\left(x_j\left(r_j(s)\right)\right)\right| ds, \quad r_j(s) = s - \tau_j(s) \\
&\le e^{-\mu t}\left|\phi_i(0)\right| + \sum_{j=1}^n \sup_{\xi\in I}\{e^{-\mu\xi}\left|g_{ij}\left(x_j(\xi)\right)\right|\} \int_0^t \frac{(t-s)^{\alpha_i-1}}{\Gamma(\alpha_i)} e^{-\mu(t-s)} ds + \\
&+ \sum_{j=1}^n \sup_{\xi\in I}\{e^{-\mu\xi}\left|h_{ij}\left(x_j(\xi)\right)\right|\} \int_0^{\mu t} \frac{u^{\alpha_i-1}}{\mu^{\alpha_i}\Gamma(\alpha_i)} e^{-u} e^{-\mu\tau_j\left(t-\frac{u}{\mu}\right)} du
\end{aligned}$$

From the hypothesis ($\mathcal{H}_1$), there exist L_j, L_j' such that $L_j = \sup_{t\in I}\{e^{-\mu t}\left|g_{ij}\left(x_j(t)\right)\right|\}$, $L_j' = \sup_{t\in I}\{e^{-\mu t}\left|h_{ij}\left(x_j(t)\right)\right|\}$, for $i, j \in [\![1,n]\!]$. Then

$$\begin{aligned}
e^{-\mu t}|Fx_i(t)| &\le e^{-\mu t}\left|\phi_i(0)\right| + \sum_{j=1}^n L_j \int_0^{\mu t} \frac{u^{\alpha_i-1}}{\mu^{\alpha_i}\Gamma(\alpha_i)} e^{-u} du \\
&+ \sum_{j=1}^n L_j' \int_0^{\mu t} \frac{u^{\alpha_i-1}}{\mu^{\alpha_i}\Gamma(\alpha_i)} e^{-u} e^{-\mu\tau_j\left(t-\frac{u}{\mu}\right)} du \\
&\le C + \frac{\sum_{j=1}^n \left(L_j + L_j'\right)}{\mu^{\alpha i}}, \quad \forall t > 0.
\end{aligned}$$

Hence, FG is uniformly bounded.
Next, we will show that FG is almost equicontinuous. There are three possible cases for $i \in [\![1, n]\!]$:

Case 1. For each $x \in G$, $\epsilon_i > 0, \forall T \in]0, +\infty)$, $t_1, t_2 \in [0, T]$, $t_1 < t_2$. Let $\delta_i = \left(\frac{\epsilon_i \Gamma(\alpha_i+1)}{2\sum_{j=1}^{n}\left(c_j+c'_j\right)}\right)^{\frac{1}{\alpha_i}}$, then when $t_2 - t_1 < \delta_i$, we have:

$$\begin{aligned}
&|Fx_i(t_1) - Fx_i(t_2)| \\
\leq\ & \int_0^{t_1} \frac{(t_1-s)^{\alpha_i-1} - (t_2-s)^{\alpha_i-1}}{\Gamma(\alpha_i)} \left| f_i(s, x_1(s), ..., x_n(s), x_1(s-\tau_1(s)), ..., x_n(s-\tau_n(s))) \right| ds \\
+\ & \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha_i-1}}{\Gamma(\alpha_i)} \left| f_i(s, x_1(s), ..., x_n(s), x_1(s-\tau_1(s)), ..., x_n(s-\tau_n(s))) \right| ds \\
\leq\ & \sum_{j=1}^{n} \frac{1}{\Gamma(\alpha_i)} \int_0^{t_1} \left((t_1-s)^{\alpha_i-1} - (t_2-s)^{\alpha_i-1}\right)\left(g_{ij}\left(x_j(s)\right) + h_{ij}\left(x_j\left(s-\tau_j(s)\right)\right)\right) ds + \\
+\ & \sum_{j=1}^{n} \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha_i-1}}{\Gamma(\alpha_i)} \left(g_{ij}\left(x_j(s)\right) + h_{ij}\left(x_j\left(s-\tau_j(s)\right)\right)\right) ds.
\end{aligned}$$

Since G is bounded, then, $\exists C > 0$ such that for $x \in G$, $\|x\|_\mu \leq C$. Hence, $\left|x_j(t)\right| \leq Ce^{\mu t} \leq Ce^{\mu T}$. Furthermore, the functions $g_{ij}(x_j)$ and $h_{ij}(x_j)$ bounded, so $\exists\, c_j, c'_j : c_j = \sup_{t\in[0,T]} \left|g_{ij}\left(x_j(t)\right)\right|$, $c'_j = \sup_{t\in[0,T]} \left|h_{ij}\left(x_j\left(t-\tau_j(t)\right)\right)\right|$, for $i, j \in [\![1, n]\!]$. Consequently,

$$\begin{aligned}
&|Fx_i(t_1) - Fx_i(t_2)| \\
\leq\ & \sum_{j=1}^{n} \frac{1}{\Gamma(\alpha_i)} \int_0^{t_1} \left((t_1-s)^{\alpha_i-1} - (t_2-s)^{\alpha_i-1}\right)\left(c_j + c'_j\right) ds + \\
+\ & \sum_{j=1}^{n} \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha_i-1}}{\Gamma(\alpha_i)} \left(c_j + c'_j\right) ds \\
\leq\ & \sum_{j=1}^{n} \frac{c_j + c'_j}{\Gamma(\alpha_i)} \left\{ \int_0^{t_1} \left((t_1-s)^{\alpha_i-1} - (t_2-s)^{\alpha_i-1}\right) ds + \int_{t_1}^{t_2} (t_2-s)^{\alpha_i-1} ds \right\} \\
\leq\ & \sum_{j=1}^{n} \frac{c_j + c'_j}{\alpha_i \Gamma(\alpha_i)} \left\{ (t_2-t_1)^{\alpha_i} + t_1^{\alpha_i} - t_2^{\alpha_i} + (t_2-t_1)^{\alpha_i} \right\} \\
<\ & 2\sum_{j=1}^{n} \frac{c_j + c'_j}{\Gamma(\alpha_i+1)} (t_2-t_1)^{\alpha_i} \\
<\ & 2\sum_{j=1}^{n} \frac{c_j + c'_j}{\Gamma(\alpha_i+1)} \delta_i^{\alpha_i} = \epsilon_i.
\end{aligned}$$

Case 2. For each $x \in G, \epsilon_i > 0, t_1 \in [-\tau, 0], t_2 \in [0, T], \forall T \in]0, +\infty)$. Since $\phi_i \in C[-\tau, 0], \exists \delta' : \left|\phi_i(t_1) - \phi_i(0)\right| < \frac{\epsilon_i}{2}$

when $0- t_1 < \delta'$. When $t_2 - t_1 < \delta_i$, $\delta_i = \min\left(\delta', \left(\frac{\epsilon_i \Gamma(\alpha_i+1)}{2\sum_{j=1}^{n}\left(c_j+c_j'\right)}\right)^{\frac{1}{\alpha_i}}\right)$, we have:

$$
\begin{aligned}
& |Fx_i(t_1) - Fx_i(t_2)| \\
\leq \;& \left|\phi_i(t_1) - \phi_i(0)\right| + \int_0^{t_2} \frac{(t_2 - s)^{\alpha_i - 1}}{\Gamma(\alpha_i)} \left|f_i\left(s, x_1(s), ..., x_n(s), x_1(s - \tau_1(s)), ..., x_n(s - \tau_n(s))\right)\right| ds \\
\leq \;& \frac{\epsilon_i}{2} + \int_0^{t_2} \frac{(t_2 - s)^{\alpha_i - 1}}{\Gamma(\alpha_i)} \sum_{j=1}^{n} \left\{ g_{ij}\left(x_j(s)\right) + \left|h_{ij}\left(x_j\left(s - \tau_j(s)\right)\right)\right| \right\} ds \\
< \;& \frac{\epsilon i}{2} + \sum_{j=1}^{n} \frac{c_j + c_j'}{\alpha_i \Gamma(\alpha_i)} t_2^{\alpha_i} \\
< \;& \frac{\epsilon_i}{2} + \sum_{j=1}^{n} \frac{c_j + c_j'}{\Gamma(\alpha_i + 1)} \delta_i^{\alpha_i} \\
< \;& \frac{\epsilon_i}{2} + \frac{\epsilon_i}{2} = \epsilon_i
\end{aligned}
$$

Case 3. For each $x \in G$, $\epsilon_i > 0$, $t_1, t_2 \in [-\tau, 0]$. In fact by continuity of ϕ_i, when $t_2 - t_1 < \delta_i$, we have

$$|Fx_i(t_1) - Fx_i(t_2)| = \left|\phi_i(t_1) - \phi_i(t_2)\right| < \epsilon_i.$$

Therefore, FG is equicontinuous in each bounded interval. Taking $\theta(t) = e^{-\rho t}$ and $\theta^*(t) = e^{-\mu t}$, where $0 < \mu < \rho$, in Lemma 2.8, we conclude that FG is relatively compact. Hence the operator F is completely continuous. This completes the proof. □

The main result of this paper is the following theorem, which provides sufficient conditions for the problem (1.1)-(1.2) to have at least one solution that is positive on $(0, +\infty)$.

Theorem 3.3. *Assume that* $(\mathcal{H}_1)$-$(\mathcal{H}_3)$ *hold. Then the problem* (1)-(2) *has at least a positive solution.*

Proof. By Lemma 3.1 the problem (1)-(2) is equivalent to the fixed point problem $x = Fx$. Furthermore, by Lemma (3.2), we have $F : D \to E$ is completely continuous. By $(\mathcal{H}_2)$, if $x, y \in D$, $x \leq y$, we have for $i \in [\![1, n]\!]$:

$$
\begin{aligned}
Fx_i(t) &= x_i(0) + I^{\alpha_i} f_i\left(t, x_1(t), ..., x_n(t), x_1(t - \tau_1(t)), ..., x_n(t - \tau_n(t))\right) \\
&\leq y_i(0) + I^{\alpha_i} f_i\left(t, y_1(t), ..., y_n(t), y_1(t - \tau_1(t)), ..., y_n(t - \tau_n(t))\right) \\
&\leq Fy_i(t).
\end{aligned}
$$

Hence F is a nondecreasing operator. Clearly, for $j \in [\![1, n]\!]$: $u_0^j \leq Fu_0^j$, $v_0^j \geq Fv_0^j$. Hence, by Theorem 2.6, $x = Fx$ has a fixed point $x \in \langle u_0, v_0 \rangle$. □

Corollary 3.4. *Assume that* $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ *are satisfied, and*

$(\mathcal{H}_4)$ *There exists a positive function* $\Psi(t) = (\psi_1(t), ..., \psi_n(t))'$, $t \geq 0$, *such that*

$$f_i\left(t, x_1(t), ..., x_n(t), x_1(t - \tau_1(t)), ..., x_n(t - \tau_n(t))\right) \leq \psi_i(t), \quad t \geq 0, \quad i \in [\![1, n]\!],$$

provided that for all $t \geq 0$: $\int_0^t (t - s)^{\alpha_i - 1} \psi_i(s)\, ds < \infty$. *Then, problem* (1)-(2) *has at least a positive solution.*

Proof. Since $f_i \leq \psi_i$, the solutions of the following problems, for $i \in [\![1, n]\!]$,

$$\begin{cases} {}^cD^{\alpha_i} u_0^i(t) = 0, & t \geq 0 \\ u_0^i(t) = \phi_i(t), & -\tau \leq t \leq 0, \end{cases} \quad \text{and} \quad \begin{cases} {}^cD^{\alpha_i} u_0^i(t) = \psi_i(t), & t \geq 0 \\ u_0^i(t) = \phi_i(t), & -\tau \leq t \leq 0, \end{cases}$$

are, respectively, the sub-solutions and super-solutions of the problem (1)-(2). A simple computation shows that they are given by

$$u_0^i(t) = \begin{cases} \phi_i(0), & t \geq 0, \\ \phi_i(t), & -\tau \leq t \leq 0 \end{cases} \quad \text{and} \quad v_0^i(t) = \begin{cases} \phi_i(0) + (\Gamma(\alpha_i))^{-1} \int_0^t (t-s)^{\alpha_i - 1} \psi_i(s)\, ds, & t \geq 0, \\ \phi_i(t), & -\tau \leq t \leq 0. \end{cases}$$

Thus, $(\mathcal{H}_3)$ is hold, and then Theorem 3.3 gives the desired result. □

Here is an illustrative example:

Example 3.5. *Consider the problem*

$$(\mathcal{P}) \quad \begin{cases} D^{\frac{1}{2}} x_1(t) = e^{-t} \left[\pi + \arctan\left(x_1\left(t - \frac{1}{t+1} - 2\right)\right)\right]^{\frac{1}{2}} \frac{x_2(t)}{(t+1)\sqrt{1+x_2^2(t)}}, & t > 0, \\ D^{\frac{1}{3}} x_2(t) = (2t+1)^{-1} \left[\frac{\pi}{2} + \arctan(x_1(t))\right] + e^{-3t} \frac{x_2^2(t - \cos^2(t))}{2 + x_2^2(t - \cos^2(t))}, & t > 0, \\ x_1(t) = 4 + t, \ t \in [-3, 0] \\ x_2(t) = t^2, \ t \in [-3, 0]. \end{cases}$$

We have

$$\alpha_1 = \frac{1}{2}, \alpha_2 = \frac{1}{3}, \tau_1(t) = 2 + \frac{1}{t+1}, \tau_2(t) = \cos^2(t), \tau = 3,$$

and

$$f_1(t, u_1, u_2, v_1, v_2) = e^{-t} [\pi + \arctan v_1]^{\frac{1}{2}} \frac{u_2}{(t+1)\sqrt{1+u_2^2}},$$

$$f_2(t, u_1, u_2, v_1, v_2) = (2t+1)^{-1} \left[\frac{\pi}{2} + \arctan u_1\right] + e^{-3t} \frac{v_2^2}{2 + v_2^2},$$

which can be estimated as follows:

$$\begin{aligned} f_1(t, u_1, u_2, v_1, v_2) &\leq e^{-2t} [\pi + \arctan v_1] + \frac{u_2^2}{(t+1)^2 \left(1 + u_2^2\right)} \\ &\leq \pi + \arctan v_1 + \frac{u_2^2}{1 + u_2^2}, \end{aligned}$$

and

$$f_2(t, u_1, u_2, v_1, v_2) \leq \frac{\pi}{2} + \arctan u_1 + \frac{v_2^2}{2 + v_2^2}.$$

Thus,

$$g_{11}(u_1) = 0, g_{12}(u_2) = \frac{u_2^2}{1 + u_2^2}, \ h_{11}(v_1) = \pi + \arctan v_1, h_{12}(v_2) = 0,$$

$$g_{21}(u_1) = \frac{\pi}{2} + \arctan u_1, g_{22}(u_2) = 0, \ h_{21}(v_1) = 0, h_{22}(v_2) = \frac{v_2^2}{2 + v_2^2}.$$

Moreover, there exist ψ_1 and ψ_2 such that

$$f_1(t,u_1,u_2,v_1,v_2) \le \frac{3\pi}{2}e^{-t} + \frac{1}{(t+1)^2} = \psi_1(t),$$
$$f_2(t,u_1,u_2,v_1,v_2) \le \frac{\pi}{2t+1} + e^{-3t} = \psi_2(t).$$

We check easily that $\int_0^{+\infty}(t-s)^{-1/2}\psi_1(s)\,ds$ and $\int_0^{+\infty}(t-s)^{-2/3}\psi_2(s)\,ds$ are finite. Thus, all hypotheses of Theorem 3.3 are fulfilled and so the problem $(\mathcal{P})$ has at least positive solution.

4. Uniqueness of solution

In this section, we discuss the uniqueness of solution.

Theorem 4.1. *Let $f_i : \mathbb{R}^+ \times \mathbb{R}^{2n} \to \mathbb{R}^+$ be continuous and satisfy the Lipschitz condition for $i \in [\![1,n]\!]$:*

$$\left|f_i(t,u_1,...,u_n,U_1,...,U_n) - f_i(t,v_1,...,v_n,V_1,...,V_n)\right| \le \sum\nolimits_{j=1}^{n} l_{ij}\left|u_j - v_j\right| + \sum\nolimits_{j=1}^{n} k_{ij}\left|U_j - V_j\right|.$$

Then the problem (1)-(2) *has a unique positive solution.*

Proof. Let $u,v \in D$, then:

$$|Fu_i(t) - Fv_i(t)| \le \sum_{j=1}^{n}\int_0^t \frac{(t-s)^{\alpha_i-1}}{\Gamma(\alpha_i)}\left\{l_{ij}\left|u_j(s)-v_j(s)\right| + k_{ij}\left|u_j\left(s-\tau_j(s)\right)-v_j\left(s-\tau_j(s)\right)\right|\right\}ds$$

Let $l = \sum\limits_{i=1}^{n}|l_i| = \sum\limits_{j=1}^{n}\max_{\forall i}\left|l_{ij}\right|$, $k = \sum\limits_{i=1}^{n}|k_i| = \sum\limits_{j=1}^{n}\max_{\forall i}\left|k_{ij}\right|$. Then

$$\begin{aligned}
& e^{-\rho t}|Fu_i(t) - Fv_i(t)| \\
\le\ & l_i\sum_{j=1}^{n}\int_0^t \frac{(t-s)^{\alpha_i-1}}{\Gamma(\alpha_i)}e^{-\rho(t-s)}e^{-\rho s}\left|u_j(s)-v_j(s)\right|ds + \\
+\ & k_i\sum_{j=1}^{n}\int_0^t \frac{(t-s)^{\alpha_i-1}}{\Gamma(\alpha_i)}e^{-\rho\left(t-s+\tau_j(s)\right)}e^{-\rho\left(s-\tau_j(s)\right)}\left|u_j\left(s-\tau_j(s)\right)-v_j\left(s-\tau_j(s)\right)\right|ds. \\
\le\ & l_i\sum_{j=1}^{n}\int_0^t \frac{(t-s)^{\alpha_i-1}}{\Gamma(\alpha_i)}e^{-\rho(t-s)}\left|u_j(s)-v_j(s)\right|ds + \\
+\ & k_i\sum_{j=1}^{n}\int_0^t \frac{(t-s)^{\alpha_i-1}}{\Gamma(\alpha_i)}e^{-\rho\left(t-r_j(s)\right)}e^{-\rho r_j(s)}\left|u_j\left(r_j(s)\right)-v_j\left(r_j(s)\right)\right|ds, \quad r_j(s) = s-\tau_j(s) \\
\le\ & l_i\sum_{j=1}^{n}\sup_{\xi\in\mathbb{R}^+}\left\{e^{-\rho\xi}\left|u_j(\xi)-v_j(\xi)\right|\right\}\int_0^t \frac{(t-s)^{\alpha_i-1}}{\Gamma(\alpha_i)}e^{-\rho(t-s)}ds + \\
+\ & k_i\sum_{j=1}^{n}\sup_{\xi\in\mathbb{R}^+}\left\{e^{-\rho\xi}\left|u_j(\xi)-v_j(\xi)\right|\right\}\int_0^t \frac{(t-s)^{\alpha_i-1}}{\Gamma(\alpha_i)}e^{-\rho\left(t-r_j(s)\right)}ds \\
\le\ & l_i\sum_{j=1}^{n}\sup_{\xi\in\mathbb{R}^+}\left\{e^{-\rho\xi}\left|u_j(\xi)-v_j(\xi)\right|\right\}\frac{1}{\rho^{\alpha_i}}\int_0^{\rho t}\frac{u^{\alpha_i-1}e^{-u}}{\Gamma(\alpha_i)}du +
\end{aligned}$$

$$+ \quad k_i \sum_{j=1}^{n} \sup_{\xi \in \mathbb{R}^+} \left\{ e^{-\rho \xi} \left| u_j(\xi) - v_j(\xi) \right| \right\} \int_0^{\rho t} \frac{u^{\alpha_i - 1} e^{-u}}{\rho^{\alpha_i} \Gamma(\alpha_i)} e^{-\rho \tau_j (t - \frac{u}{\rho})} du.$$

Consequently,

$$\begin{aligned}
& e^{-\rho t} \left| F u_i(t) - F v_i(t) \right| \\
\leq \quad & l_i \sum_{j=1}^{n} \sup_{\xi \in \mathbb{R}^+} \left\{ e^{-\rho \xi} \left| u_j(\xi) - v_j(\xi) \right| \right\} \frac{1}{\rho^{\alpha_i}} \int_0^{\rho t} \frac{u^{\alpha_i - 1} e^{-u}}{\Gamma(\alpha_i)} du + \\
+ \quad & k_i \sum_{j=1}^{n} \sup_{\xi \in \mathbb{R}^+} \left\{ e^{-\rho \xi} \left| u_j(\xi) - v_j(\xi) \right| \right\} \int_0^{\rho t} \frac{u^{\alpha_i - 1} e^{-u}}{\rho^{\alpha_i} \Gamma(\alpha_i)} du \\
\leq \quad & \frac{n l_i}{\rho^{\alpha_i}} \|u - v\|_\rho + \frac{n k_i}{\rho^{\alpha_i}} \|u - v\|_\rho \\
\leq \quad & n \frac{l_i + k_i}{\rho^{\alpha_i}} \|u - v\|_\rho .
\end{aligned}$$

Hence

$$\begin{aligned}
\|Fu - Fv\|_\rho &= \sum_{i=1}^{n} \sup_{t \in I} e^{-\rho t} \left| F u_i(t) - F v_i(t) \right| \\
&\leq \sum_{i=1}^{n} n \frac{l_i + k_i}{\rho^{\alpha_i}} \|u - v\|_\rho \\
&\leq n \frac{l + k}{\rho^{\alpha}} \|u - v\|_\rho ,
\end{aligned}$$

where $\alpha = \max_{1 \leq i \leq n} \alpha_i$. Therefore,

$$\|Fu - Fv\|_\rho \leq n \frac{l + k}{\rho^{\alpha}} \|u - v\|_\rho .$$

We choose ρ large enough such that $n \frac{l+k}{\rho^{\alpha}} < 1$, then, by Banach fixed point theorem, F has a unique fixed point in D, which is the unique positive solution. □

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